§7. Tensor products

§7.1. Test functions depending on parameter

Here we prove

Lemma Assume that $U \subset \mathbb{R}^n$, $V \subset \mathbb{R}^m$ are open and $v \in \mathcal{D}'(V)$, $\varphi \in C_c^\infty(U \times V)$.

Define $\psi(x) := (v(y), \varphi(x,y))$

$$= (v, \varphi(x,\cdot)), \quad x \in U.$$

where $\varphi(x,\cdot) \in C_c^\infty(V)$,

$$\varphi(x,\cdot)(y) = \varphi(x,y).$$

Then $\psi \in C_c^\infty(U)$.

Proof We have $\operatorname{supp} \varphi \subset K_U \times K_V$ for some compact $K_U \subset U$, $K_V \subset V$.

Then $\forall x \in U$, $\operatorname{supp} \varphi(x,\cdot) \subset K_V$ and if $x \notin K_U$ then $\varphi(x,\cdot) = 0$

So $\psi(x) = 0$

So ψ is supported in K_U.

- **Continuity of ψ:**

Since $v \in \mathcal{D}'(V)$ and

$$\operatorname{supp}(\varphi(x,\cdot)) \subset K_V \quad \forall x$$

we see that $\exists\, C, N : \ \forall x, x' \in U$

$$|\psi(x) - \psi(x')| = |(v, \varphi(x,\cdot) - \varphi(x',\cdot))| \le$$

$$\le C \|\varphi(x,\cdot) - \varphi(x',\cdot)\|_{C^N}$$

$$\le C \max_{|\alpha| \le N} \sup_y |\partial^\alpha \varphi(x,y) - \partial^\alpha \varphi(x',y)|$$

$$\to 0 \quad \text{as } x' \to x.$$

- **Differentiability of ψ:**

$$\frac{\psi(x+te_j) - \psi(x)}{t} = \left(v, \frac{\varphi(x+te_j,\cdot) - \varphi(x,\cdot)}{t}\right)$$

$$\xrightarrow{t \to 0} (v, \partial_{x_j}\varphi(x,\cdot))$$

since $\dfrac{\varphi(x+te_j,y) - \varphi(x,y)}{t} \to \partial_{x_j}\varphi(x,y)$

in $C_c^\infty(V)$ (in y)

- Iterating, we get

$$(v, \varphi(x, \cdot)) \in C_c^\infty(U) \quad \text{and} \quad \forall \alpha,$$

$$\partial^\alpha (v, \varphi(x, \cdot)) = (v, \partial_x^\alpha \varphi(x, \cdot)).$$

$\square$

<u>Note</u>: we could alternatively take

$v \in \mathcal{E}'(V)$, $\varphi \in C^\infty(U \times V)$

in which case

$$(v, \varphi(x, \cdot)) \in C^\infty(U)$$

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§7.2. Tensor product of distributions

Let $u \in \mathcal{D}'(U)$, $v \in \mathcal{D}'(V)$ where

$U \subset \mathbb{R}^n$, $V \subset \mathbb{R}^m$ open.

We want to define

$$u \otimes v \in \mathcal{D}'(U \times V) \quad \text{such that}$$

if $u, v \in L^1_{loc}$ then

$$(u \otimes v)(x, y) = u(x)\, v(y).$$

How to express this distributionally

Let first $u \in L^1_{loc}(U)$, $v \in L^1_{loc}(V)$ and take $\varphi \in C_c^\infty(U)$, $\psi \in C_c^\infty(V)$.

Define $\varphi \otimes \psi \in C_c^\infty(U \times V)$ by

$$(\varphi \otimes \psi)(x,y) = \varphi(x)\psi(y).$$

Then $(u \otimes v, \varphi \otimes \psi)$

$$= \int_{U\times V} (u(x)v(y))(\varphi(x)\psi(y))\,dx\,dy \overset{\text{(Fubini)}}{=}$$

$$= \int_U u(x)\varphi(x)\,dx \int_V v(y)\psi(y)\,dy$$

$$= (u,\varphi)(v,\psi).$$

<u>Defn.</u> Let $u \in \mathcal{D}'(U)$, $v \in \mathcal{D}'(V)$.

We say that $w \in \mathcal{D}'(U \times V)$ is the tensor product $u \otimes v$ if

$$(*) \quad (w, \varphi \otimes \psi) = (u,\varphi)(v,\psi)$$

$$\forall \varphi \in C_c^\infty(U),\ \psi \in C_c^\infty(V).$$

<u>Thm.</u> $\forall u, v$ there exists unique w s.t. $(*)$ holds.

<u>Proof</u> <u>Existence</u>: if u, v were functions then $\forall \beta \in C_c^\infty(U \times V)$ we compute by Fubini's Thm

$$(u \otimes v, \beta) = \int_{U \times V} u(x) v(y) \beta(x,y)\, dx\, dy$$

$$= \int_U u(x) \left(\int_V v(y) \beta(x,y)\, dy \right) dx.$$

So for $u, v \in \mathcal{D}'$ and $\beta \in C_c^\infty(U \times V)$ define

$$(w, \beta) := (u(x), (v(y), \beta(x,y))).$$

By Lemma in §4.1,

$$(v(y), \beta(x,y)) \in C_c^\infty(U)$$

So the pairing is well-defined. Also, from the proof of that Lemma,

$\beta_k \to 0$ in $C_c^\infty(U \times V) \Rightarrow$

$\Rightarrow (v(y), \beta_k(x,y)) \to 0$ in $C_c^\infty(U)$.

$\Rightarrow (w, \beta_k) \to 0$. So $w \in \mathcal{D}'(U \times V)$.

And if $\beta = \varphi \otimes \psi$ then

$$(w,\beta) = (u(x), (v(y), \varphi(x)\psi(y)))$$

$$= (u(x), \varphi(x)(v,\psi)) = (u,\varphi)(v,\psi)$$

So $(*)$ holds.

Note: we could have alternatively defined

$$(w,\beta) := (v(y), (u(x), \beta(x,y)))$$

but it is not yet clear why this would give the same β.

Uniqueness Assume that $w \in \mathcal{D}'(U \times V)$ and

$$(w, \varphi \otimes \psi) = 0 \quad \forall \varphi \in C_c^\infty(U), \psi \in C_c^\infty(V).$$

We need to show that $w = 0$.

One approach is to approximate any $\beta \in C_c^\infty(U \times V)$ by partial sums of its Fourier series, which are linear combinations of "$\varphi \otimes \psi$", (see Friedlander-Joshi)

Here we instead use convolutions:

1. Enough to show $(\chi_1 \otimes \chi_2) w = 0$ for all $\chi_1 \in C_c^\infty(U)$, $\chi_2 \in C_c^\infty(V)$.

So, may assume that $w \in \mathcal{E}'(U \times V)$.

Extending w by 0 to $\mathbb{R}^{n+m}$, reduce to the following statement:

$$\boxed{\begin{array}{l} \text{if } \tilde{w} \in \mathcal{E}'(\mathbb{R}^{n+m}) \text{ and} \\ (\tilde{w}, \varphi \otimes \psi) = 0 \quad \forall \varphi \in C_c^\infty(\mathbb{R}^n), \psi \in C_c^\infty(\mathbb{R}^m) \\ \text{then } \tilde{w} = 0. \end{array}}$$

(Here $\tilde{w}$ = extension of $(\chi_1 \otimes \chi_2) w$ by 0 to $\mathbb{R}^{n+m}$.)

Now, let $\chi_1 \in C_c^\infty(\mathbb{R}^n)$, $\chi_2 \in C_c^\infty(\mathbb{R}^m)$, $\int \chi_1 = \int \chi_2 = 1$, define

$$\chi_{1,\varepsilon}(x) = \varepsilon^{-n} \chi_1\left(\frac{x}{\varepsilon}\right), \quad \chi_{2,\varepsilon}(y) = \varepsilon^{-m} \chi_2\left(\frac{y}{\varepsilon}\right).$$

Then by Proposition from §6.2 (with $\chi := \chi_1 \otimes \chi_2$) we have

$$\tilde{w} * (\chi_{1,\varepsilon} \otimes \chi_{2,\varepsilon}) \xrightarrow[\varepsilon \to 0+]{} \tilde{w}$$

in $\mathcal{D}'(\mathbb{R}^{n+m})$.

But for any $x \in \mathbb{R}^n$, $y \in \mathbb{R}^m$

$$\tilde{w} * (\chi_{1,\varepsilon} \otimes \chi_{2,\varepsilon})(x,y)$$

$$= (\tilde{w}(x',y'), \chi_{1,\varepsilon}(x-x')\chi_{2,\varepsilon}(y-y'))$$

$$= (\tilde{w}, \underbrace{\chi_{1,\varepsilon}(x-\cdot)}_{\varphi} \otimes \underbrace{\chi_{2,\varepsilon}(y-\cdot)}_{\psi})$$

$= 0$. So $\tilde{w} = 0$. $\square$

Example: if $u = \delta_0 \in \mathcal{D}'(\mathbb{R}^n)$
$v = \delta_0 \in \mathcal{D}'(\mathbb{R}^m)$

then $u \otimes v = \delta_0 \in \mathcal{D}'(\mathbb{R}^{n+m})$.

Indeed, $\forall \varphi \in C_c^\infty(\mathbb{R}^n)$, $\psi \in C_c^\infty(\mathbb{R}^m)$

$$(\delta_0, \varphi \otimes \psi) = (\varphi \otimes \psi)(0) = \varphi(0)\psi(0)$$
$$= (\delta_0, \varphi)(\delta_0, \psi)$$

§7.3. Distributional kernels

Integral operators: if $U \subset \mathbb{R}^n$, $V \subset \mathbb{R}^m$ open and $Q \in L^1_{loc}(U \times V)$ then can define the operator

$A : C_c^\infty(V) \to L^1_{loc}(U)$ by

$$A\varphi(x) = \int_V Q(x,y)\varphi(y)\,dy, \quad \varphi \in C_c^\infty(V).$$

What does $A\varphi$ do as a distribution?

If $\psi \in C_c^\infty(U)$ then

$$(A\varphi, \psi) = \int_U \left(\int_V Q(x,y)\varphi(y)\,dy \right) \psi(x)\,dx$$

$$= \int_{U \times V} Q(x,y)\,\psi(x)\varphi(y)\,dx\,dy$$

$$= (Q, \psi \otimes \varphi).$$

But this makes sense when $Q \in \mathcal{D}'(U \times V)$:

Defn. Let $Q \in \mathcal{D}'(U \times V)$.
Define the operator
$$A : C_c^\infty(V) \to \mathcal{D}'(U) \text{ by}$$
$$(A\varphi, \psi) = (Q, \psi \otimes \varphi) \qquad (*)$$
$\forall \varphi \in C_c^\infty(V), \psi \in C_c^\infty(U)$.
We say Q is the <u>Schwartz kernel of A</u>.

Remarks:

① A determines Q: if $A = 0$ then
$(Q, \psi \otimes \varphi) = 0 \quad \forall \varphi, \psi \Rightarrow Q = 0$

② For $\varphi \in C_c^\infty(V)$, $A\varphi$ is a distribution:
if $\psi_k \to 0$ in $C_c^\infty(U)$ then
$(A\varphi, \psi_k) = (Q, \psi_k \otimes \varphi) \to 0$
since $\psi_k \otimes \varphi \to 0$ in $C_c^\infty(U \times V)$

③ The operator A is sequentially continuous:
if $\varphi_k \to 0$ in $C_c^\infty(V)$ then
$A\varphi_k \to 0$ in $\mathcal{D}'(U)$. Indeed, $\forall \psi \in C_c^\infty(U)$
we have $(A\varphi_k, \psi) = (Q, \psi \otimes \varphi_k) \to 0$
since $\psi \otimes \varphi_k \to 0$ in $C_c^\infty(U \times V)$.

Schwartz Kernel Theorem

Assume $A : C_c^\infty(V) \to \mathcal{D}'(U)$ is sequentially continuous. Then A has the form $(*)$ above for some (unique) $Q \in \mathcal{D}'(U \times V)$ (called the Schwartz kernel of A).

We will skip the proof: for those interested see Friedlander–Joshi or Hörmander.

Interpretation:

$$\boxed{\begin{array}{l}\text{Operators} \\ C_c^\infty(V) \to \mathcal{D}'(U)\end{array}} = \boxed{\begin{array}{l}\text{Distributions} \\ \text{in } \mathcal{D}'(U\times V)\end{array}}$$

Example:

Let $U = V$, $A = I$ the identity operator.

What is Q? Need: $\forall \varphi, \psi \in C_c^\infty(U)$

$$(Q, \psi \otimes \varphi) = (I\varphi, \psi) = (\varphi, \psi) = \int_U \varphi(x)\psi(x)\,dx$$

We take $Q(x,y) := \delta(x-y)$, defined by

$$(\delta(x-y), \beta(x,y)) := \int_U \beta(x,x)\,dx, \quad \beta \in C_c^\infty(U\times U)$$

Note: $\text{supp}\,\delta(x-y) =$ the diagonal $\{(x,x) \mid x \in U\}$

§7.4. Transpose operators

Defn. Let $A : C_c^\infty(V) \to \mathcal{D}'(U)$ be sequentially continuous. We define its **transpose** $A^t : C_c^\infty(U) \to \mathcal{D}'(V)$ by the identity

$$\boxed{(A^t\psi, \varphi) = (A\varphi, \psi)}$$

$\forall\ \varphi \in C_c^\infty(V),\ \psi \in C_c^\infty(U)$

Note: such A^t is again sequentially continuous and $(A^t)^t = A$.

In terms of Schwartz kernels
$Q \in \mathcal{D}'(U \times V)$ of A,
$Q^t \in \mathcal{D}'(V \times U)$ of A^t, we have

$$Q^t(y,x) = Q(x,y).$$

More precisely,

$$(Q^t(y,x), \beta(y,x)) = (Q(x,y), \tilde\beta(x,y))$$

$$\forall\ \beta \in C_c^\infty(V \times U),$$

$$\tilde\beta(x,y) := \beta(y,x),\quad \tilde\beta \in C_c^\infty(U \times V).$$

<u>Example</u>: if $A = \partial_{x_j}$ $(U = V)$ then $A^t = -\partial_{x_j}$: $\forall \varphi, \psi \in C_c^\infty(U)$,

$$-(\partial_{x_j}\psi, \varphi) = (\partial_{x_j}\varphi, \psi)$$

<u>Thm</u> Assume $A : C_c^\infty(V) \to \mathcal{D}'(U)$ is sequentially continuous and $A^t : C_c^\infty(U) \to C^\infty(V)$ is sequentially continuous, i.e.

$\varphi_k \to 0$ in $C_c^\infty(U) \Rightarrow A^t\varphi_k \to 0$ in $C^\infty(V)$.

!!

Then A extends uniquely to a sequentially continuous $\tilde{A} : \mathcal{E}'(V) \to \mathcal{D}'(U)$.

<u>Proof</u> Uniqueness: if $\tilde{A} : \mathcal{E}'(V) \to \mathcal{D}'(U)$ is sequentially continuous and $A|_{C_c^\infty(V)} = 0$ then $A = 0$. Indeed, any $v \in \mathcal{E}'(V)$ is the limit in $\mathcal{E}'(V)$ of some sequence of $v_k \in C_c^\infty(V)$ [proved very similarly to the density of C_c^∞ in $\mathcal{D}'$].

But $Av_k = 0$ and $Av_k \to Av$ in $\mathcal{D}'(U)$, so $Av = 0$.

<u>Existence:</u> just take $v \in \mathcal{E}'(V)$, $\psi \in C_c^\infty(U)$ and define

$$(\tilde{A}v, \psi) := (v, A^t\psi)$$

where $A^t\psi \in C^\infty(V)$.

- Such $\tilde{A}v$ lies in $\mathcal{D}'(U)$: if $\psi_k \to 0$ in $C_c^\infty(U)$ then $A^t\psi_k \to 0$ in $C^\infty(V)$, so

$$(\tilde{A}v, \psi_k) = (v, A^t\psi_k) \longrightarrow 0.$$

- $\tilde{A} : \mathcal{E}'(V) \longrightarrow \mathcal{D}'(U)$ is sequentially continuous: if $v_k \to 0$ in $\mathcal{E}'(V)$ then $\forall \psi \in C_c^\infty(U)$, $A^t\psi \in C^\infty(V)$, so

$$(\tilde{A}v_k, \psi) = (v_k, A^t\psi) \longrightarrow 0$$

- $\tilde{A}$ extends A: if $v \in C_c^\infty(V)$ then $\forall \psi \in C_c^\infty(U)$,

$$(\tilde{A}v, \psi) = (v, A^t\psi) = (A^t\psi, v) = (Av, \psi).$$

$\square$

Remarks: 1 if we have instead

$$A^t : C_c^\infty(U) \to C_c^\infty(V)$$

sequentially continuous then we can extend A to

$$\tilde{A} : \mathcal{D}'(V) \to \mathcal{D}'(U).$$

We henceforth write $A := \tilde{A}$.

2 We used this technique before

e.g. $(\partial_{x_j})^t = -\partial_{x_j} : C_c^\infty(U) \circlearrowleft$

gave us $\partial_{x_j} : \mathcal{D}'(U) \circlearrowleft$