§3. Differential operations on distributions

§3.1. Differentiation

We first define the differentiation linear operators $\tilde{\partial}_{x_j} : \mathcal{D}'(U) \to \mathcal{D}'(U)$ (every distribution can be differentiated!) with the following properties:

1. if $u \in C^1(U)$ then $\tilde{\partial}_{x_j} u$ as a distribution $=$ the usual $\partial_{x_j} u$
2. $\tilde{\partial}_{x_j}$ is sequentially continuous $\mathcal{D}'(U)$s: if $u_k \in \mathcal{D}'(U)$ and $u_k \to 0$ in $\mathcal{D}'(U)$ then $\tilde{\partial}_{x_j} u_k \longrightarrow 0$ in $\mathcal{D}'(U)$

<u>Remark</u>. It turns out that the operator $\tilde{\partial}_{x_j}$ is uniquely determined by 1 & 2.
Indeed, we will show later that $C_c^\infty(U)$ is dense in $\mathcal{D}'(U)$ w.r.t. $\mathcal{D}'$-convergence.
So $\forall u \in \mathcal{D}'(U)$ $\exists\, u_k \in C_c^\infty(U) : u_k \to u$ in $\mathcal{D}'$
But then $\partial_{x_j} u_k = \tilde{\partial}_{x_j} u_k \longrightarrow \tilde{\partial}_{x_j} u$ in $\mathcal{D}'(U)$
which determines $\tilde{\partial}_{x_j} u$.

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How to construct the operator $\tilde{\partial}_{x_j}$?
We follow the following general strategy of duality:

Step 1: take $u \in C^1(U)$ and
"nice" fns on which the operation is already defined

an arbitrary $\varphi \in C_c^\infty(U)$. "test function"

Express $(\partial_{x_j} u, \varphi) = (u, \text{something}(\varphi))$

We do it here using

Lemma. [Integration by parts]
Let $U \subset \mathbb{R}^n$ be open and $u \in C^1(U)$, $\varphi \in C_c^1(U)$. Then
$$\int_U (\partial_{x_j} u, \varphi)\, dx = -\int_U (u, \partial_{x_j}\varphi)\, dx$$

Proof Apply the Divergence Thm to the vector field $X = u \cdot \varphi \cdot \vec{e}_j$ $\quad \vec{e}_j = (0, .., \overset{j}{1}, .., 0)$

We should set $\int_{\partial U} X \cdot \vec{n}\, dS = \int_U \operatorname{div} \underline{X}\, dx$

But $X = 0$ on ∂U
because $u \cdot \varphi = 0$ on ∂U
as φ is compactly supported inside U,
so $\int_U \operatorname{div} X = 0$.

(This only works for general X when U has a nice boundary. But here X is a <u>compactly supported</u> vector field, so $\int_U \operatorname{div} X = 0$ for rough ∂U as well (exercise...))

Now $\operatorname{div} X = \partial_{x_j}(u \cdot \varphi) = \partial_{x_j} u \cdot \varphi + u \cdot \partial_{x_j} \varphi$

So $0 = \int_U \operatorname{div} X = \int_U \partial_{x_j} u \cdot \varphi \, dx + \int_U u \cdot \partial_{x_j} \varphi \, dx$. $\square$

<u>So</u>: we got $\forall u \in C^1(U), \varphi \in C_c^\infty(U)$

$$\boxed{(\partial_{x_j} u, \varphi) = -(u, \partial_{x_j} \varphi)} \qquad (*)$$

<u>Step 2</u>: use $(*)$ to define $\tilde{\partial}_{x_j} u \in \mathcal{D}'(U)$ for any $u \in \mathcal{D}'(U)$. In other words,

if $u \in \mathcal{D}'(U)$ and $\varphi \in C_c^\infty(U)$, put $\boxed{(\tilde{\partial}_{x_j} u)(\varphi) := -u(\partial_{x_j}\varphi).}$

- Well-defined: $\varphi \in C_c^\infty(U) \Rightarrow \partial_{x_j}\varphi \in C_c^\infty(U)$
- $\tilde{\partial}_{x_j} u : C_c^\infty(U) \to \mathbb{C}$ is linear because ∂_{x_j} is linear
- $\tilde{\partial}_{x_j} u \in \mathcal{D}'(U)$: let $K \subset U$ cpct

Then $\exists\, C, N$: $\forall \psi \in C_c^\infty(U)$, $\operatorname{supp}\psi \subset K$ we have $|u(\psi)| \le C\|\psi\|_{C^N}$.

So if $\varphi \in C_c^\infty(U)$, $\operatorname{supp}\varphi \subset K$ then $\operatorname{supp}\partial_{x_j}\varphi \in K$ and

$$|(\tilde{\partial}_{x_j} u)(\varphi)| = |u(\partial_{x_j}\varphi)| \le C\|\partial_{x_j}\varphi\|_{C^N}$$
$$\le C\|\varphi\|_{C^{N+1}}.$$

- if $u \in C^1(U)$ then $\tilde{\partial}_{x_j} u = \partial_{x_j} u$ as follows from Step 1.
- Sequential continuity: if $u_k \to 0$ in $\mathcal{D}'(U)$ then $\forall \varphi \in C_c^\infty(U)$,

$$(\partial_{x_j} u_k)(\varphi) = -u_k(\partial_{x_j}\varphi) \longrightarrow 0 \quad \text{as } \partial_{x_j}\varphi \in C_c^\infty(U).$$

!!! WE NOW WRITE $\partial_{x_j} := \tilde{\partial}_{x_j}$!!!

Examples: $(U = \mathbb{R})$

1. $u(x) = |x|$. We have $\forall \varphi \in C_c^\infty(\mathbb{R})$

$$-\int_{\mathbb{R}} u(x)\,\varphi'(x)\,dx = \int_{-\infty}^{0} x\,\varphi'(x)\,dx - \int_{0}^{\infty} x\varphi'(x)\,dx$$

$$\underset{\text{IBP}}{=} -\int_{-\infty}^{0} \varphi(x)\,dx + \int_{0}^{\infty} \varphi(x)\,dx$$

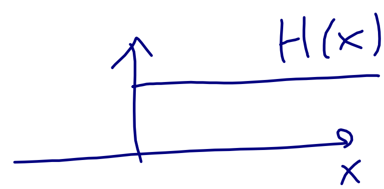
$$= \int_{\mathbb{R}} \operatorname{sgn} x \cdot \varphi(x)\,dx$$

where the bdary terms in IBP vanish since $x=0$ at 0

So $\partial_x |x| = \operatorname{sgn} x \leftarrow$ in $L^1_{loc}(\mathbb{R})$

2. Heaviside function:

$$H(x) = \begin{cases} 1, & x > 0 \\ 0, & x < 0 \end{cases}$$



Compute $\forall \varphi \in C_c^\infty(\mathbb{R})$,

$$-\int_{\mathbb{R}} H(x)\,\varphi'(x)\,dx = -\int_{0}^{\infty} \varphi'(x) = \varphi(0)$$

So $H'(x) = \delta_0$.

3 $u(x) = \delta_0(x)$.

$$-(u,\varphi') = -\varphi'(0) = (u',\varphi)$$

So $(\delta_0', \varphi) = -\varphi'(0)$

Here's our first differential equation:

$u' = 0$ where $u \in \mathcal{D}'(U)$, $U \subset \mathbb{R}$ open interval

Thm. If $u' = 0$ then $u \equiv c$ for some $c \in \mathbb{C}$.

Proof $u' = 0$ means that

$(u, \psi') = 0$ for all $\psi \in C_c^\infty(U)$.

Fix $\chi_0 \in C_c^\infty(U)$, $\int_U \chi_0 = 1$.

Then each $\varphi \in C_c^\infty(U)$ can be written as

$$\boxed{\varphi = \left(\int_U \varphi\right)\chi_0 + \psi'}$$ for some

$\psi \in C_c^\infty(U)$: it is enough to consider the case $\int_U \varphi = 0$

where we put $\psi(x) = \int_a^x \varphi(t)\,dt$,

$a \in U$, $a < \operatorname{supp}\varphi$.

So $(u,\varphi) = (u, (\int\varphi)\chi_0 + \psi')$
$= (\int\varphi)\cdot(u,\chi_0)$. Putting
$c := (u,\chi_0)$ we see that $u = c$,
i.e. $\forall \varphi \in C_c^\infty(U)$, $(u,\varphi) = c\int_U \varphi$. $\square$

<u>Note</u>: $\partial_{x_j}\partial_{x_k} = \partial_{x_k}\partial_{x_j}$ in $\mathcal{D}'(U)$
so can define $\partial_x^\alpha : \mathcal{D}'(U) \circlearrowleft$
for any multiindex α.

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§3.2. Multiplication by smooth functions

Let $u \in \mathcal{D}'(U)$, $a \in C^\infty(U)$.
We define $a\cdot u \in \mathcal{D}'(U)$
by the formula

$$(a\cdot u, \varphi) = (u, a\varphi) \quad \forall \varphi \in C_c^\infty(U)$$

This agrees with pointwise multiplication
when $u \in L^1_{loc}(U)$:

$$\int_U a u \varphi\, dx = (u, a\varphi)$$

We leave as an exercise to check:

- $u \in \mathcal{D}'(U) \Rightarrow au \in \mathcal{D}'(U)$ (i.e. it's continuous)
- $u \in \mathcal{D}'(U) \mapsto au \in \mathcal{D}'(U)$ is sequentially continuous

The above uses the fact that

$\varphi \in C_c^\infty(U), a \in C^\infty(U) \Rightarrow a\varphi \in C_c^\infty(U)$.

In general, we cannot define $a \cdot u$ for $a \notin C^\infty$, $u \in \mathcal{D}'$

And we cannot define $u \cdot v$ for $u, v \in \mathcal{D}'$ (cannot multiply distributions)

e.g. $\delta_0 \cdot \delta_0$ does not make sense

<u>Thm</u>. [Leibniz formula]

If $u \in \mathcal{D}'(U)$ and $a \in C^\infty(U)$ then

$$\partial_{x_j}(a \cdot u) = (\partial_{x_j} a) \cdot u + a \cdot (\partial_{x_j} u)$$

<u>Proof 1</u> (direct): we need to show that $\forall \varphi \in C_c^\infty(U)$,

$$(\partial_{x_j}(au), \varphi) = ((\partial_{x_j} a) u, \varphi) + (a \cdot (\partial_{x_j} u), \varphi)$$

That is,

$$-(au, \partial_{x_j}\varphi) = (u, (\partial_{x_j} a)\varphi) + (\partial_{x_j} u, a\varphi)$$

which becomes

$$-(u, a\partial_{x_j}\varphi) = (u, (\partial_{x_j} a)\varphi) - (u, \partial_{x_j}(a\varphi))$$

It remains to use that

$$-a\partial_{x_j}\varphi = (\partial_{x_j} a)\varphi - \partial_{x_j}(a\varphi)$$

which is the usual Leibniz rule since here a, φ are smooth. $\square$

<u>Proof 2</u> (faster but uses density, not proved yet)

Since $C_c^\infty(U)$ is dense in $\mathcal{D}'(U)$, can take $u_k \in C_c^\infty(U)$, $u_k \to u$ in $\mathcal{D}'(U)$

Then $\partial_{x_j}(a \cdot u_k) = (\partial_{x_j} a) u_k + a \cdot (\partial_{x_j} u_k)$

$\forall k$ (classical derivatives)

and we take the limit as $k \to \infty$ to get the needed identity for u (using sequential continuity of $u \mapsto \partial_{x_j} u$ and $u \mapsto au$). $\square$

<u>Basic example:</u> $(U = \mathbb{R})$

$a \in C^\infty(\mathbb{R}) \Rightarrow a \cdot \delta_0 = a(0)\delta_0$

Indeed, $\forall \varphi \in C_c^\infty(\mathbb{R})$

$$(a\delta_0, \varphi) = (\delta_0, a\varphi) = (a\varphi)(0)$$
$$= a(0)\varphi(0) = (a(0)\delta_0, \varphi)$$

Here is another differential equation:

$$xu = 0, \quad u \in \mathcal{D}'(\mathbb{R})$$

<u>Thm.</u> We have $xu = 0 \Leftrightarrow u = c \cdot \delta_0$ for some $c \in \mathbb{C}$.

<u>Proof</u> $\Leftarrow$ follows from the example above

$\Rightarrow$ will use the following

<u>Division Lemma</u> If $f \in C^\infty(\mathbb{R})$ and $f(0) = 0$ then $f(x) = xg(x)$ for some $g \in C^\infty(\mathbb{R})$

In fact, $g(x) = \begin{cases} f(x)/x, & x \neq 0 \\ f'(0), & x = 0. \end{cases}$

Proof of the division lemma:

Let $x \in \mathbb{R}$. Applying F.T.C. to the function $\tilde{f}(t) = f(tx)$ on the interval $[0,1]$, get

$$\int_0^1 \tilde{f}'(t)\,dt = \tilde{f}(1) - \tilde{f}(0) = f(x)$$

(since $f(0) = 0$)

But $\tilde{f}'(t) = x f'(tx)$. So:

$$f(x) = \int_0^1 x f'(tx)\,dt = x g(x)$$

where $g(x) := \int_0^1 f'(tx)\,dt$

and differentiating under the $\int$ we see that $g \in C^\infty$. $\square$

End of the proof of Thm:

fix $\chi_0 \in C_c^\infty(\mathbb{R})$, $\chi_0(0) = 1$.

Then $xu = 0 \Rightarrow u(x\psi) = 0 \quad \forall \psi \in C_c^\infty(\mathbb{R})$

For $\varphi \in C_c^\infty(\mathbb{R})$, write

$$\underbrace{\varphi(x) - \varphi(0)\chi_0}_{\in\, C_c^\infty(\mathbb{R})} = x\psi(x)$$

for some $\psi \in C_c^\infty(\mathbb{R})$ using the Lemma

Now $u(\varphi) - \varphi(0)\, u(\chi_0)$

$= u(x\psi) = 0$, so

$u(\varphi) = \varphi(0)\, u(\chi_0) = c\delta_0(\varphi)$

where $c := u(\chi_0)$. $\square$