§2. Distributions

§2.1. Defintion of distributions

Let $U \subset \mathbb{R}^n$ be an open set.

We define distributions as continuous linear functionals on $C_c^\infty(U)$.

More precisely:

Defn. Let $u : C_c^\infty(U) \to \mathbb{C}$ be linear.

We say that u is a distribution on U, if $\forall$ compact set $K \subset U$

$\exists$ constants C, N such that

$\forall \varphi \in C_c^\infty(U)$ with $\operatorname{supp} \varphi \subset K$

we have $|u(\varphi)| \leq C \|\varphi\|_{C^N}$

where $\|\varphi\|_{C^N} := \max_{|\alpha| \leq N} \sup |\partial^\alpha \varphi|$.

Denote by $\mathcal{D}'(U)$ the vector space of all distributions on U

Fundamental example of distribution:

If $f \in L^1_{loc}(U)$ then f defines a distribution $\tilde{f}$ on U by

$$\tilde{f}(\varphi) = \int_U f(x)\varphi(x)\,dx \quad \forall \varphi \in C_c^\infty(U)$$

Indeed, if $\operatorname{supp}\varphi \subset K$ then

$$|\tilde{f}(\varphi)| \leq \|f\|_{L^1(K)} \cdot \|\varphi\|_{C^0}.$$

<u>Note:</u> by the lemma at the end of Lecture 1

$\tilde{f} = 0 \Rightarrow f = 0$ (almost everywhere)

So $f \mapsto \tilde{f}$ gives a linear embedding (i.e. injective map) $L^1_{loc}(U) \hookrightarrow \mathcal{D}'(U)$

Important notation:

- For $f \in L^1_{loc}(U)$, we view it as an element of $\mathcal{D}'(U)$ by identifying it with $\tilde{f}$

- For $u \in \mathcal{D}'(U)$, $\varphi \in C_c^\infty(U)$ we denote $(u,\varphi) := u(\varphi)$

If u, φ are functions on U then $(u,\varphi) := \int_U u(x)\varphi(x)\,dx$

Note: not the same as
$$\langle u,\varphi\rangle_{L^2} := \int u(x)\overline{\varphi(x)}\,dx$$

Not all distributions are functions.
An example is the delta function:
$$\delta \in \mathcal{D}'(\mathbb{R}^n), \quad (\delta,\varphi) := \varphi(0)$$
There is no $f \in L^1_{loc}(\mathbb{R}^n)$ s.t.
$$\forall \varphi \in C_c^\infty(\mathbb{R}^n), \quad \int_{\mathbb{R}^n} f(x)\varphi(x)\,dx = \varphi(0)$$
(e.g. replace φ with $\varphi(\frac{x}{\varepsilon})$, then $LHS \xrightarrow{\varepsilon \to 0} 0$, $RHS = \varphi(0)$)
by D.C.T

More generally, if $y \in U$, then $\delta_y \in \mathcal{D}'(U)$ where $\delta_y(\varphi) := \varphi(y)$.

Philosophical discussion:

- To specify a function f on U need to specify $f(x)$ $\forall x \in U$
- To specify a distribution u on U need to specify "$\int u(x)\varphi(x)dx$" $\forall \varphi \in C_c^\infty(U)$

weaker collection of data: some distributions don't have values at points e.g. $\delta_0(0)$ is undefined.

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§2.2 All things convergence

We won't introduce a topology on $C_c^\infty(U)$ (too painful and not really needed) but...

Defn. Let $\varphi_k, \varphi \in C_c^\infty(U)$. We say $\varphi_k \xrightarrow[k\to\infty]{} \varphi$ (in $C_c^\infty(U)$) if

1. $\exists K \subset U$ compact s.t. $\forall k$, $\operatorname{supp}\varphi_k \subset K$
2. $\forall N$, $\|\varphi_k\|_{C^N} \to 0$

Thm. Let $u : C_c^\infty(U) \to \mathbb{C}$ be linear. Then

$$u \in \mathcal{D}'(U) \Leftrightarrow \forall \{\varphi_k \in C_c^\infty(U)\},$$

$$\varphi_k \to 0 \text{ in } C_c^\infty \Rightarrow (u, \varphi_k) \to 0$$

("called "sequential continuity")

Proof $\Rightarrow$ Easy: if $\operatorname{supp} \varphi_k \subset K$ for all k then

$$\exists\, C, N \quad \forall k \quad |(u, \varphi_k)| \leq C \|\varphi_k\|_{C^N} \xrightarrow{k \to \infty} 0$$

$\Leftarrow$ Assume that $u \notin \mathcal{D}'(U)$.

Then $\exists\, K \subset U$ cpct s.t.

$\forall\, C, N \quad \exists\, \varphi \in C_c^\infty(U)$ s.t.

$\operatorname{supp} \varphi \subset K$ and $|(u, \varphi)| > C \|\varphi\|_{C^N}$

Thus $\exists$ a sequence $\varphi_N \in C_c^\infty(U)$

s.t. $\operatorname{supp} \varphi_N \subset K$, $(u, \varphi_N) = 1$,

and $\|\varphi_N\|_{C^N} \leq \frac{1}{N}$

Then $\varphi_N \to 0$ in $C_c^\infty(U)$ but

$(u, \varphi_N) = 1 \not\to 0$. $\square$

Defn. Let $u_k, u \in \mathcal{D}'(U)$.
We say $u_k \to u$ in $\mathcal{D}'(U)$
if $\forall \varphi \in C_c^\infty(U)$, $(u_k, \varphi) \to (u, \varphi)$.

This is a very <u>week</u> notion of convergence: easier to get a limit.
On the other hand, convergence in C_c^∞ is very <u>strong</u> ...

<u>Example 1</u>

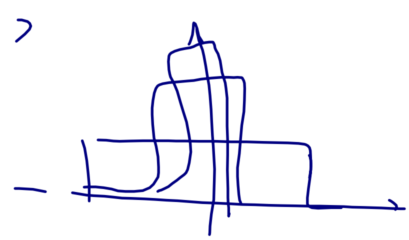
if $u_k, u \in L^1_{loc}(U)$ and

- $u_k(x) \to u(x)$ for a.e. $x \in U$,
- $|u_k(x)| \leq g(x)$ for some $g \in L^1_{loc}(U)$ and all k

then $u_k \to u$ in $\mathcal{D}'(U)$
as follows from the Dominated Convergence Thm.

<u>Example 2</u> $u_k \in L^1_{loc}(\mathbb{R})$,

$$u_k(x) = \begin{cases} k, & \text{if } |x| \leq \frac{1}{k} \\ 0 & \text{otherwise} \end{cases}$$

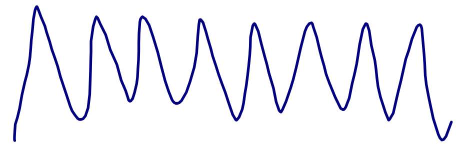


Then $u_k \to 2\delta_0$ in $\mathcal{D}'(\mathbb{R})$

Example 3

Take $u_k \in L^1_{loc}(\mathbb{R})$,

$$u_k(x) = e^{ikx}.$$



Then $u_k \xrightarrow[k\to\infty]{} 0$ in $\mathcal{D}'(\mathbb{R})$.

Indeed, $\forall \varphi \in C_c^\infty(\mathbb{R})$,

$$(u_k, \varphi) = \int_{\mathbb{R}} \varphi(x)\, e^{ikx}\, dx = \hat{\varphi}(-k) \xrightarrow{k\to\infty} 0$$

Fourier transform (more later)

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§2.3. Localization

How does $\mathcal{D}'(U)$ depend on the set U?

Assume first $V \subset U \subset \mathbb{R}^n$ open.

We have the restriction operator

$$L^1_{loc}(U) \to L^1_{loc}(V)$$
$$f \mapsto f|_V$$

(<u>note</u>: this map is not onto)

This can be generalized to distributions:

<u>Defn.</u> If $u \in \mathcal{D}'(U)$ then the <u>restriction</u> of u to $V \subset U$ is the distribution $u|_V \in \mathcal{D}'(V)$ defined by

$$(u|_V, \varphi) = (u, \varphi) \quad \forall \varphi \in C_c^\infty(V)$$

Here we use that $C_c^\infty(V) \hookrightarrow C_c^\infty(U)$ (extend by 0 on $V \setminus U$)

Distributions form a <u>sheaf</u>:

<u>Thm.</u> Assume that $(U_j)_{j \in J}$ (any set) are open sets and $U = \bigcup_{j \in J} U_j$. Assume next that we are given $u_j \in \mathcal{D}'(U_j)$ for each $j \in J$ s.t.

$$u_j|_{U_j \cap U_{j'}} = u_{j'}|_{U_j \cap U_{j'}} \quad \forall j, j' \in J.$$

Then there exists unique $u \in \mathcal{D}'(U)$ such that $u|_{U_j} = u_j \ \forall j \in J$

Informally,
u is determined by its restrictions to all U_j's

Proof Uniqueness: assume that
$u \in \mathcal{D}'(U)$ and $u|_{U_j} = 0 \quad \forall j \in J$.
We want to show that $u = 0$, i.e.
$(u, \varphi) = 0 \quad \forall \varphi \in C_c^\infty(U)$.
If $\varphi \in C_c^\infty(U_j)$ for some j, then we are done: $(u,\varphi) = (u_j, \varphi) = 0$.
In general, since $\operatorname{supp} \varphi$ is compact, there exists finite $J' \subset J$ s.t. $\operatorname{supp} \varphi \subset \bigcup_{j \in J'} U_j$
Take a partition of unity:
$\chi_j \in C_c^\infty(U_j), \quad j \in J'$,
$1 = \sum_{j \in J'} \chi_j$ on $\operatorname{supp} \varphi$
Then $\varphi = \sum_{j \in J'} \chi_j \varphi$, $\chi_j \varphi \in C_c^\infty(U_j)$
$(u, \varphi) = \sum_{j \in J'} (u, \chi_j \varphi) = 0$.

Existence: we need to define (u,φ) for every $\varphi \in C_c^\infty(U)$.

Fix such φ and take a finite partition of unity as before:

$$\chi_j \in C_c^\infty(U_j), \text{ all but finitely many } \chi_j \text{ are } 0,$$

$$\sum_{j\in J} \chi_j = 1 \text{ on } \operatorname{supp}\varphi$$

Define $(u,\varphi) := \sum_{j\in J} (u_j, \chi_j\varphi)$

where $\chi_j\varphi \in C_c^\infty(U_j)$

· Independence of the choice of χ_j:

imagine that $\tilde{\chi}_j$ is another partition of unity.

Then $\sum_{j\in J}(u_j,\chi_j\varphi) \overset{①}{=} \sum_{j,j'\in J}(u_j, \chi_j\tilde{\chi}_{j'}\varphi)$

$\overset{②}{=} \sum_{j,j'\in J}(u_{j'}, \chi_j\tilde{\chi}_{j'}\varphi)$

$\overset{③}{=} \sum_{j'\in J}(u_{j'}, \tilde{\chi}_{j'}\varphi)$

①, ③ true because

$\chi_j\varphi = \sum_{j'} \chi_j\tilde{\chi}_{j'}\varphi$

$\tilde{\chi}_{j'}\varphi = \sum_j \chi_j\tilde{\chi}_{j'}\varphi$

② true because $\chi_j\tilde{\chi}_{j'}\varphi \in C_c^\infty(U_j\cap U_{j'})$, and $u_j|_{U_j\cap U_{j'}} = u_{j'}|_{U_j\cap U_{j'}}$.

- The resulting u is in $\mathcal{D}'(U)$: easy to check linearity & continuity from the definition of u
- Restrictions: if $\varphi \in C_c^\infty(U_{j_0})$, can take a 1-element partition of unity: $\chi_{j_0} \in C_c^\infty(U_{j_0})$, $\chi_{j_0} = 1$ on $\operatorname{supp}\varphi$, $\chi_j = 0$ for $j \neq j_0$. Then

$$(u, \varphi) = (u_{j_0}, \chi_{j_0}\varphi) = (u_{j_0}, \varphi)$$

Since $\chi_{j_0}\varphi = \varphi$, so

$$u|_{U_{j_0}} = u_{j_0}. \qquad \square$$