§14. Differential operators

§14.1. Operators on subsets of $\mathbb{R}^n$.

Assume that $U \subset \mathbb{R}^n$ is open.

Defn. A *differential operator* of order $m \in \mathbb{N}_0$ on U has the form

$$P = \sum_{|\alpha| \le m} p_\alpha(x) D_x^\alpha \quad \text{where } p_\alpha \in C^\infty(U), \quad D_x^\alpha = (-i)^{|\alpha|} \partial_{x^\alpha}.$$

Denote by $\mathrm{Diff}^m(U)$ the space of all such operators.

Each $P \in \mathrm{Diff}^m(U)$ maps each of the spaces $C^\infty(U)$, $C_c^\infty(U)$, $\mathcal{D}'(U)$, $\mathcal{E}'(U)$ into themselves. It also gives (sequentially continuous) operators on Sobolev spaces,

$$P: H^s_{loc}(U) \to H^{s-m}_{loc}(U)$$

$$P: H^s_c(U) \to H^{s-m}_c(U) \quad \forall s \in \mathbb{R}$$

(Since multiplication by C_c^∞ functions maps $H^s(\mathbb{R}^n) \circlearrowleft$ and $\partial^\alpha: H^s(\mathbb{R}^n) \to H^{s-|\alpha|}(\mathbb{R}^n)$)

In all this $Pu = \sum_{|\alpha| \le m} p_\alpha(x) D_x^\alpha u(x)$

- Each $P \in \mathrm{Diff}^m(U)$ is <u>local</u>:
  $\forall u \in \mathcal{D}'(U)$ we have

$$(*) \quad \mathrm{supp}(Pu) \subset \mathrm{supp}\, u.$$

In fact, the converse is true: if $P : C^\infty(U) \to C_c^\infty(U)$ is a sequentially continuous operator satisfying $(*)$ then $P \in \mathrm{Diff}^m(U)$ for some m. (Peetre's Theorem)

We won't prove it (see Friedlander - Joshi Exercise 6.3) but the scheme of proof is:

1. If $Q \in \mathcal{D}'(U \times U)$ is the Schwartz kernel of P, show that $\mathrm{supp}\, Q$ is compact and lies in the diagonal
2. Adapt the classification of distributions supported at a single point (see §4.3) to distributions supported on a submanifold.

We now define the principal symbol of a differential operator:

Defn. Let $P \in \mathrm{Diff}^m(U)$.

Define the principal symbol $\sigma_m(P)$ as the following C^∞ function on $U \times \mathbb{R}^n$:

$$\sigma_m(P)(x,\xi) = \sum_{|\alpha|=m} p_\alpha(x)\,\xi^\alpha, \quad x \in U,\ \xi \in \mathbb{R}^n$$

where $P = \sum_{|\alpha|\le m} p_\alpha(x) D_x^\alpha$, $D = -i\partial$, $p_\alpha \in C^\infty(U)$.

- $\sigma_m(P)(x,\xi)$ is a homogeneous polynomial of degree m in ξ with coefficients smooth in x.
- We often write $\sigma(P) := \sigma_m(P)$ when the order m is clear

Example: Laplacian w.r.t. a Riemannian metric.

Assume that

$$g = \sum_{j,k=1}^{n} g_{jk}(x)\,dx_j\,dx_k$$

is a Riemannian metric on $U \subset \mathbb{R}^n$, where $g_{jk}(x) \in C^\infty(U;\mathbb{R})$ $\forall j,k$ and the matrix $G(x) = (g_{jk}(x))_{j,k=1}^{n}$ is symmetric & positive definite $\forall x$

(Here $\forall x \in U$, $\forall \vec{v},\vec{w} \in T_x U = \mathbb{R}^n$ we have $g(x)(\vec{v},\vec{w}) = \sum_{j,k=1}^{n} g_{jk}(x) v_j w_k = (G(x)\vec{v})\cdot\vec{w}$.)

For $u \in C^\infty(U)$, define its gradient $\nabla_g u \in C^\infty(U;\mathbb{R}^n)$ by

$$g(x)(\nabla_g u(x), w) = du(x)(w)$$
$$\forall x \in U,\ w \in \mathbb{R}^n$$

That is, $(\nabla_g u(x))_j = \sum_{k=1}^{n} g^{jk}(x)\,\partial_{x_k} u(x)$

where $G^{-1}(x) = (g^{jk}(x))_{j,k=1}^{n}$

I.e. $\nabla_g u(x) = G^{-1}(x)\,du(x)$

where $du(x) = (\partial_{x_1} u(x), \dots, \partial_{x_n} u(x))$

We now define the Laplace-Beltrami operator $\Delta_g \in \mathrm{Diff}^2(\bar{U})$ as the unique operator such that

$$-\int_U (\Delta_g u(x)) v(x)\, d\mathrm{Vol}_g(x) =$$
$$= \int_U \langle \nabla_g u(x), \nabla_g v(x)\rangle_{g(x)}\, d\mathrm{Vol}_g(x)$$

$\forall u \in C_c^\infty(\bar{U}),\ v \in C^\infty(\bar{U})$

(or $u \in C^\infty(\bar{U}),\ v \in C_c^\infty(\bar{U})$)

where $d\mathrm{Vol}_g(x) = \sqrt{\det G(x)}\, dx$,

$$\langle \nabla_g u(x), \nabla_g v(x)\rangle_{g(x)}$$
$$= (G(x)\nabla_g u(x)) \cdot \nabla_g v(x)$$
$$= (G^{-1}(x) du(x)) \cdot (dv(x)).$$

To compute Δ_g, we integrate by parts:

$$-\int_U (\Delta_g u(x)) v(x)\, d\mathrm{Vol}_g(x) =$$
$$= -\int_U (\Delta_g u(x)) v(x) \sqrt{\det G(x)}\, dx =$$

$$= \int_U \langle \nabla_g u(x), \nabla_g v(x) \rangle_{g(x)} \, d\mathrm{Vol}_g(x)$$

$$= \int_U (G^{-1}(x)\, du(x)) \cdot dv(x) \cdot \sqrt{\det G(x)}\, dx$$

$$= \int_U \sum_{j,k=1}^n g^{jk}(x)\, \partial_{x_j} u(x)\, \partial_{x_k} v(x) \sqrt{\det G(x)}\, dx$$

$$\overset{\mathrm{IBP}}{=} - \int_U \sum_{j,k=1}^n \partial_{x_k} \left(\sqrt{\det G(x)}\, g^{jk}(x)\, \partial_{x_j} u(x) \right) \cdot v(x)\, dx.$$

So

$$\Delta_g u(x) = \sum_{j,k=1}^n \frac{1}{\sqrt{\det G(x)}}\, \partial_{x_k} \left(\sqrt{\det G(x)}\, g^{jk}(x)\, \partial_{x_j} u(x) \right)$$

So indeed, $\Delta_g \in \mathrm{Diff}^2(U)$

and the principal symbol is

$$\boxed{\sigma_2(\Delta_g)(x,\xi) = - \sum_{j,k=1}^n g^{jk}(x)\, \xi_j \xi_k .}$$

Here is a very useful statement featuring the principal symbol:

<u>Thm</u> Assume that $U \subset \mathbb{R}^n$ open, $P \in \mathrm{Diff}^m(U)$, and $\varphi \in C^\infty(U; \mathbb{R})$ (phase), $a \in C^\infty(U; \mathbb{C})$ (amplitude). Then we have as $\lambda \to \infty$

$$P\left(e^{i\lambda\varphi(x)} a(x)\right) =$$

$$= e^{i\lambda\varphi(x)}\left(\sigma_m(P)(x, d\varphi(x))\, a(x) \cdot \lambda^m + \right.$$

$$\left. + O(\lambda^{m-1})_{C^\infty(U)}\right).$$

Here $d\varphi(x) = (\partial_{x_1}\varphi(x), \dots, \partial_{x_n}\varphi(x))$

<u>Remark</u> Special case: $\varphi(x) = x \cdot \xi$, $\xi \in \mathbb{R}^n$, $a \equiv 1$

Can check $P(e^{i\lambda x\cdot\xi}) = \lambda^m p(x, \xi)$

where $p(x,\xi) = \sum_{|\alpha| \le m} P_\alpha(x)\xi^\alpha$ is the full symbol of P.

Proof We look at the conjugated operator

$$e^{-i\lambda\varphi(x)} P e^{i\lambda\varphi(x)} \in \mathrm{Diff}^m(\bar U),$$

mapping $u \in C^\infty(\bar U)$ to

$$e^{-i\lambda\varphi(x)} P\left(e^{i\lambda\varphi(x)} u(x)\right).$$

We have $P = \sum_{|\alpha|\le m} p_\alpha(x) D_x^\alpha$ and

$e^{-i\lambda\varphi(x)} p_\alpha(x) e^{i\lambda\varphi(x)} = p_\alpha(x)$; $D = -i\partial$, so

$$e^{-i\lambda\varphi(x)} D_{x_j} e^{i\lambda\varphi(x)} = D_{x_j} + \lambda(\partial_{x_j}\varphi(x))$$

Thus $e^{-i\lambda\varphi(x)} P e^{i\lambda\varphi(x)} a(x) =$

$$= \sum_{|\alpha|\le m} p_\alpha(x) (D_{x_1} + \lambda\partial_{x_1}\varphi)^{\alpha_1} \cdots (D_{x_n} + \lambda\partial_{x_n}\varphi)^{\alpha_n} a(x).$$

Now write it all out in powers of λ: the coefficient of λ^m is

$$\sum_{|\alpha|=m} p_\alpha(x)(\partial_{x_1}\varphi)^{\alpha_1}\cdots(\partial_{x_n}\varphi)^{\alpha_n} a(x) =$$

$$= \sigma_m(P)(x, d\varphi(x))\, a(x). \qquad \square$$

An application of the Thm above is the behavior of the principal symbol under pullback:

Thm Assume that $U, V \subset \mathbb{R}^n$ are open and $\Phi : U \to V$ is a diffeomorphism. Let $P \in \mathrm{Diff}^m(V)$ and define its pullback $\Phi^* P : C^\infty(U) \to C^\infty(U)$

$$\boxed{\Phi^*(Pv) = (\Phi^* P)\, \Phi^* v}$$

$\forall v \in C^\infty(V)$. Then $\Phi^* P \in \mathrm{Diff}^m(U)$ and

$$\sigma_m(\Phi^* P)(x, \xi) = \sigma_m(P)(\Phi(x), d\Phi(x)^{-T}\xi).$$

Here $d\Phi(x)^{-T}$ is the inverse of the transpose of $d\Phi(x) : \mathbb{R}^n \to \mathbb{R}^n$.

Proof 1. To see that

$\Phi^* P \in \mathrm{Diff}^m(U)$, use

the Chain Rule:

take any $u \in C^\infty(U)$ and put

$v := u \circ \Phi^{-1}$, so that

$$\Phi^*(Pv) = (\Phi^* P)\, \Phi^* v \quad \text{becomes}$$

$$(\Phi^* P) u = \left(P(u \circ \Phi^{-1})\right) \circ \Phi.$$

Now by the Chain Rule, $\forall \alpha$

$\partial^\alpha (u \circ \Phi^{-1})$ = linear combination

(with C^∞ coefficients) of

$(\partial^\beta u) \circ \Phi^{-1}$ where $\beta \leq \alpha$.

So indeed $\Phi^* P \in \mathrm{Diff}^m(U)$.

2) To compute the principal symbol, take any $\varphi \in C^\infty(U;\mathbb{R})$.

Then $(\Phi^* P) e^{i\lambda\varphi(x)}$ (as $\lambda \to \infty$)

$$= e^{i\lambda\varphi(x)}\left(\sigma_m(\Phi^* P)(x, d\varphi(x))\lambda^m + O(\lambda^{m-1})\right)$$

But $(\Phi^* P) e^{i\lambda\varphi(x)} = \left(P e^{i\lambda\varphi(\Phi^{-1}(x))}\right)\circ\Phi$.

$= \left(Pe^{i\lambda\tilde{\varphi}(y)}\right)\circ\Phi$ where $\tilde{\varphi} = \varphi\circ\Phi^{-1}$
$\tilde{\varphi} \in C^\infty(V;\mathbb{R})$

$$= \left(e^{i\lambda\tilde{\varphi}(y)}\left(\sigma_m(P)(y, d\tilde{\varphi}(y))\lambda^m + O(\lambda^{m-1})\right)\right)\circ\Phi$$

$$= e^{i\lambda\varphi(x)}\left(\sigma_m(P)(\Phi(x), d\tilde{\varphi}(\Phi(x)))\lambda^m + O(\lambda^{m-1})\right).$$ So

$$\sigma_m(\Phi^* P)(x, d\varphi(x)) =$$
$$= \sigma_m(P)(\Phi(x), d\tilde{\varphi}(\Phi(x)))$$
$$= \sigma_m(P)(\Phi(x), d\varphi(x)\circ d\Phi^{-1}(\Phi(x)))$$
(thinking of $d\varphi$ as a linear map $\mathbb{R}^n \to \mathbb{R}$)
$$= \sigma_m(P)(\Phi(x), d\Phi(x)^{-T} d\varphi(x)). \qquad \square$$

§14.2. Differential operators on manifolds

Let M be an n-dim. manifold.

We say that an operator $P : C^\infty(M) \to C^\infty(M)$ lies in $\mathrm{Diff}^m(M)$, if:

1. P is **local**, i.e. $\forall u \in C^\infty(M)$,

$$\operatorname{supp}(Pu) \subset \operatorname{supp} u$$

2. $\forall$ coordinate system $\varkappa : \underset{\underset{M}{\cap}}{U_0} \to \underset{\underset{\mathbb{R}^n}{\cap}}{V_0}$, the pullback

$(\varkappa^{-1})^* P : C^\infty(V_0) \to C^\infty(V_0)$ lies in $\mathrm{Diff}^m(V_0)$

(Here $\forall v \in C^\infty(V_0)$, we define

$$((\varkappa^{-1})^* P)v := (P(v \circ \varkappa)) \circ \varkappa^{-1}.$$

- Each $P \in \mathrm{Diff}^m(M)$ maps $C^\infty(M)$, $C_c^\infty(M)$, $\mathcal{D}'(M)$, $\mathcal{E}'(M)$ into themselves & maps $(\forall s \in \mathbb{R})$

$$H^s_{loc}(M) \to H^{s-m}_{loc}(M)$$

$$H^s_c(M) \to H^{s-m}_c(M).$$

- Principal symbol of $P \in \mathrm{Diff}^m(M)$: a function on the <u>cotangent bundle</u>

$$\sigma_m(P) \in C^\infty(T^*M):$$

$\forall$ coordinate system $\varkappa: U_0 \to V_0$, $\forall x \in U_0$, $\forall \xi \in T_x^*M$, put

$$\sigma_m(P)(x, \xi) =$$

$$= \sigma_m((\varkappa^{-1})^* P)(\varkappa(x), d\varkappa(x)^{-T}\xi).$$

Here $(\varkappa^{-1})^* P \in \mathrm{Diff}^m(V_0)$ where $V_0 \subset \mathbb{R}^n$, so $(\varkappa^{-1})^* P \in C^\infty(V_0 \times \mathbb{R}^n)$ is defined in §14.1.

And $d\varkappa(x)^{-T} : T_x^* M \to \mathbb{R}^n$ is the inverse-transpose of

$$d\varkappa(x) : T_x M \to \mathbb{R}^n,$$

namely $\forall\, \xi \in T_x M$, $\forall\, w \in \mathbb{R}^n$,

$$(d\varkappa(x)^{-T} \xi) \cdot w =$$

$$= \xi(\underbrace{d\varkappa(x)^{-1} \cdot w}_{\in\, T_x M}).$$

This is independent of the choice of the coordinate $\varkappa$, see Pset 9

Examples:

1. Let $X \in C^\infty(M; TM)$ be a vector field. Define

$$P = -iX \in \mathrm{Diff}^1(M) \text{ by}$$

$$Pu(x) = -i\, du(x) \cdot X(x).$$

Then $\sigma_1(P)(x, \xi) = \xi(X(x))$.

2. Assume g is a Riemannian metric on M. Then define $\Delta_g \in \mathrm{Diff}^2(M)$ by

$$-\int (\Delta_g u) \cdot v \, dVol_g = \int g(\nabla_g u, \nabla_g v)\, dVol_g$$

$$\forall u \in C_c^\infty(M), \ v \in C^\infty(M).$$

We have $\sigma_2(-\Delta_g)(x, \xi) =$

$= \langle \xi, \xi \rangle_{g(x)}$ where

$\langle \cdot, \cdot \rangle_{g(x)}$ is the inner product on $T_x^* M$ induced by g.