§12. Fourier Transform II

§12.1. Sobolev spaces

- For $\xi \in \mathbb{R}^n$, denote $\langle \xi \rangle := \sqrt{1 + |\xi|^2}$.
- For $s \in \mathbb{R}$, denote

$$L^2_s = \langle \xi \rangle^{-s} L^2(\mathbb{R}^n), \text{ i.e.}$$

$f : \mathbb{R}^n \to \mathbb{C}$ (measurable) lies in L^2_s iff

$$\|f\|_{L^2_s} := \sqrt{\int_{\mathbb{R}^n} \langle \xi \rangle^{2s} |f(\xi)|^2 d\xi} < \infty.$$

Note: L^2_s is a Hilbert space and $\mathcal{S}(\mathbb{R}^n) \subset L^2_s \subset \mathcal{S}'(\mathbb{R}^n)$; $\mathcal{S}$ is dense in L^2_s

Defn. The Sobolev space $H^s(\mathbb{R}^n)$ consists of $u \in \mathcal{S}'(\mathbb{R}^n)$ such that

$$\hat{u} \in L^2_s.$$

Note: $\mathcal{S}(\mathbb{R}^n) \subset H^s(\mathbb{R}^n) \subset \mathcal{S}'(\mathbb{R}^n)$ and $H^s(\mathbb{R}^n)$ is a Hilbert space with norm $\|u\|_{H^s} := (2\pi)^{-\frac{n}{2}} \|\hat{u}\|_{L^2_s}$

Basic properties:

- $H^0(\mathbb{R}^n) = L^2(\mathbb{R}^n)$ since
  $\forall u \in S'(\mathbb{R}^n)$, $u \in L^2 \Leftrightarrow \hat{u} \in L^2$
  and $\|u\|_{L^2} = (2\pi)^{-\frac{n}{2}} \|\hat{u}\|_{L^2}$.
- $S(\mathbb{R}^n)$ is dense in $H^s(\mathbb{R}^n)$ $\forall s$.
  In fact, $C_c^\infty(\mathbb{R}^n)$ is dense in $H^s(\mathbb{R}^n)$
  (Since any Schwartz fn. is approximated by compactly supported ones in H^s norm)
- $\partial_{x_j} : H^s(\mathbb{R}^n) \to H^{s-1}(\mathbb{R}^n)$ is bounded.
  This follows from the definition of $H^s(\mathbb{R}^n)$ & the fact that $\widehat{\partial_{x_j} u}(\xi) = i\xi_j \hat{u}(\xi)$ $\forall u \in S'(\mathbb{R}^n)$.
- If s is a nonnegative integer then
  $$H^s(\mathbb{R}^n) = \{u \in L^2(\mathbb{R}^n) \mid \forall \alpha, |\alpha| \leq s, \text{ the distributional derivative } \partial^\alpha u \text{ lies in } L^2(\mathbb{R}^n)\}.$$
  And $\|u\|_{H^s}$ is equivalent to $\sqrt{\sum_{|\alpha| \leq s} \|\partial^\alpha u\|_{L^2}^2}$.

Proof $\subseteq$: if $u \in H^s(\mathbb{R}^n)$, $|\alpha| \leq s$
then $\partial^\alpha u \in H^{s-|\alpha|} \subset H^0 \subset L^2$.

$\supseteq$: assume that $u \in L^2(\mathbb{R}^n)$ and $\forall \alpha$, $|\alpha| \le s$, we have $\partial^\alpha u \in L^2(\mathbb{R}^n)$.
Then $\widehat{\partial^\alpha u} \in L^2$, so $\xi^\alpha \hat{u}(\xi) \in L^2(\mathbb{R}^n)$.

Now, $\langle \xi \rangle^s \le C \sum_{|\alpha| \le s} |\xi^\alpha|$, so

$\langle \xi \rangle^s \hat{u}(\xi) \in L^2(\mathbb{R}^n)$; thus $u \in H^s(\mathbb{R}^n)$.

- Multiplication by cutoffs:

if $a \in \mathcal{S}(\mathbb{R}^n)$, then $\forall s \in \mathbb{R}$ the operator

$$M_a : \mathcal{D}'(\mathbb{R}^n) \to \mathcal{D}'(\mathbb{R}^n), \quad M_a u := au,$$

is a bounded operator $H^s(\mathbb{R}^n) \to H^s(\mathbb{R}^n)$.

Proof 1. Assume that $u \in H^s(\mathbb{R}^n) \subset \mathcal{S}'(\mathbb{R}^n)$.
By a version of the Convolution Theorem + Fourier Inversion Formula we have

$$\boxed{\widehat{au} = (2\pi)^{-n} \hat{a} * \hat{u}}.$$

Indeed, $\forall \xi \in \mathbb{R}^n$,

$\hat{a} * \hat{u}(\xi) = (\hat{u}(\eta), b_\xi(\eta))$ where

$b_\xi(\eta) = \hat{a}(\xi - \eta)$; $b_\xi \in \mathcal{S}(\mathbb{R}^n)$ $\forall \xi$

Now $\hat{a} * \hat{u}(\xi) = (u(x), \hat{b}_\xi(x))$

and $\hat{b}_\xi(x) = \int_{\mathbb{R}^n} e^{-ix\cdot\eta} \hat{a}(\xi - \eta)\, d\eta \underset{(\eta \to \xi - \eta)}{=}$

$= \int_{\mathbb{R}^n} e^{ix\cdot(\eta - \xi)} \hat{a}(\eta)\, d\eta \overset{\text{Fourier Inversion}}{=} e^{-ix\cdot\xi} (2\pi)^n a(x).$

So $(2\pi)^{-n} \hat{a} * \hat{u}(\xi) = (u(x), \underbrace{e^{-ix\cdot\xi} a(x)}_{\in\, \mathcal{S}(\mathbb{R}^n)}).$

But this is equal to $\widehat{au}(\xi)$:

indeed, $\forall \varphi \in \mathcal{S}(\mathbb{R}^n)$, we have

$(\widehat{au}, \varphi) = (au, \hat{\varphi}) = (u, a\hat{\varphi})$

$= (u(x), \int_{\mathbb{R}^n} e^{-ix\cdot\xi} a(x) \varphi(\xi)\, d\xi) \overset{\text{(Riemann sums etc.)}}{=}$

$= \int_{\mathbb{R}^n} (u(x), e^{-ix\cdot\xi} a(x)) \varphi(\xi)\, d\xi.$

2. Having proved that $\widehat{au} = (2\pi)^{-n} \hat{a} * \hat{u}$,

we now need to show that

$$\hat{u} \in L^2_s \Rightarrow \hat{a} * \hat{u} \in L^2_s.$$

We write

$$\hat{a} * \hat{u}(\xi) = \int_{\mathbb{R}^n} \hat{a}(\xi - \eta)\,\hat{u}(\eta)\,d\eta$$

Since $\hat{u} \in L^2_s$, can write $\hat{u}(\eta) = \langle\eta\rangle^{-s} v(\eta)$ for some $v \in L^2(\mathbb{R}^n)$, so

$$\langle\xi\rangle^s (\hat{a} * \hat{u})(\xi) = \int_{\mathbb{R}^n} \hat{a}(\xi-\eta)\frac{\langle\xi\rangle^s}{\langle\eta\rangle^s} v(\eta)\,d\eta$$

We need to show this is in L^2 in ξ, using that $v \in L^2$, $\hat{a} \in \mathcal{S}$.

We also use <u>Young's convolution inequality</u>:

$$\|f * g\|_{L^2} \le \|f\|_{L^2} \cdot \|g\|_{L^1}. \qquad (\star)$$

<u>Proof of $(\star)$</u>: $\forall \xi$, estimate by Cauchy-Schwartz

$$|f*g(\xi)|^2 = \left|\int_{\mathbb{R}^n} f(\xi-\eta)g(\eta)\,d\eta\right|^2 \le \left(\int_{\mathbb{R}^n} |f(\xi-\eta)|\cdot|g(\eta)|^{\frac{1}{2}}\cdot|g(\eta)|^{\frac{1}{2}}d\eta\right)^2$$

$$\overset{(C-S)}{\le} \int_{\mathbb{R}^n} |f(\xi-\eta)|^2\cdot|g(\eta)|\,d\eta \cdot \int_{\mathbb{R}^n} |g(\eta)|\,d\eta, \text{ so}$$

$$\|f*g\|^2_{L^2} \le \|g\|_{L^1}\cdot\int_{\mathbb{R}^{2n}} |f(\xi-\eta)|^2\cdot|g(\eta)|\,d\eta\,d\xi$$

$$= \|g\|^2_{L^1}\cdot\|f\|^2_{L^2}. \qquad \square$$

Now, if $\xi - \eta$ is large then $\hat{a}(\xi-\eta)$ is small and if $\xi-\eta$ is small then $\frac{\langle\xi\rangle^s}{\langle\eta\rangle^s} \sim 1$.

More precisely, we have

$$\langle\xi\rangle^2 \leq 2\langle\eta\rangle^2 \cdot \langle\xi-\eta\rangle^2$$

(recall $\langle\xi\rangle^2 = 1 + |\xi|^2$)

and, switching ξ and η,

$$\langle\eta\rangle^2 \leq 2\langle\xi\rangle^2 \cdot \langle\xi-\eta\rangle^2,$$ so

$$\frac{\langle\xi\rangle^s}{\langle\eta\rangle^s} \leq C_s \langle\xi-\eta\rangle^{|s|}.$$

Thus $|\langle\xi\rangle^s(\hat{a}*\hat{u})(\xi)| \leq$

$$\leq C_s \int_{\mathbb{R}^n} |\hat{a}(\xi-\eta)|\,\langle\xi-\eta\rangle^{|s|} \cdot |v(\eta)|\,d\eta.$$

The latter is a convolution, and by Young's inequality

$$\|\langle\xi\rangle^s(\hat{a}*\hat{u})(\xi)\|_{L^2} \leq C_s \|\langle\xi\rangle^{|s|}\hat{a}(\xi)\|_{L^1(\mathbb{R}^n)} \cdot \|v\|_{L^2(\mathbb{R}^n)}$$

$< \infty$, since $v \in L^2$ and $\langle\xi\rangle^{|s|}\hat{a}(\xi) \in L^1$. $\square$

• A double integral characterization:

if $u \in L^2(\mathbb{R}^n)$ and $0<s<1$

then $u \in H^s(\mathbb{R}^n)$

$\Updownarrow$

$$I_s(u) := \int_{\mathbb{R}^{2n}} \frac{|u(x)-u(y)|^2}{|x-y|^{n+2s}}\, dx\,dy < \infty.$$

Proof Write

$$I_s(u) = \int_{\mathbb{R}^{2n}} \frac{|u(x+w)-u(x)|^2}{|w|^{n+2s}}\, dx\,dw. \quad (w = y-x)$$

Now, what is $\int_{\mathbb{R}^n} |u(x+w)-u(x)|^2 dx$?

This is the L^2 norm squared of

$\tau_{-w}u - u$ where $\tau_{-w}u(x) = u(x+w)$.

And (Pset 7, Problem 3)

$$\widehat{\tau_{-w}u}(\xi) = e^{iw\cdot\xi}\hat{u}(\xi), \text{ so}$$

$$\widehat{\tau_{-w}u-u}(\xi) = (e^{iw\cdot\xi}-1)\hat{u}(\xi)$$

Thus $\int_{\mathbb{R}^n} |u(x+w)-u(x)|^2 dx$

$$= (2\pi)^{-n}\int_{\mathbb{R}^n} |e^{iw\cdot\xi}-1|^2 \cdot |\hat{u}(\xi)|^2 d\xi$$

So $$I_s(u) = (2\pi)^{-n} \int_{\mathbb{R}^{2n}} \frac{|e^{iw\cdot\xi}-1|^2}{|w|^{n+2s}} |\hat{u}(\xi)|^2 d\xi dw.$$

Now integrate out w:

$$F(\xi) := (2\pi)^{-n} \int_{\mathbb{R}^n} \frac{|e^{iw\cdot\xi}-1|^2}{|w|^{n+2s}} dw.$$

- The integral converges: $\begin{cases} \text{at } |w|\to\infty \text{ as } s>0 \\ \text{at } w\to 0 \text{ as } s<1 \end{cases}$
- $F(\xi)$ depends only on $|\xi|$
  ($|w|$ is rotation invariant ...)
- For $t>0$,

$$F(t\xi) = (2\pi)^{-n} \int_{\mathbb{R}^n} \frac{|e^{itw\cdot\xi}-1|^2}{|w|^{n+2s}} dw =$$

$w \to w/t$

$$= (2\pi)^{-n} \int_{\mathbb{R}^n} \frac{|e^{iw\cdot\xi}-1|^2}{|w/t|^{n+2s}} \frac{dw}{t^n} = t^{2s} F(\xi)$$

So $F(\xi) = c|\xi|^{2s}$ for some $c>0$

and $$I_s(u) = \int_{\mathbb{R}^n} F(\xi)\cdot|\hat{u}(\xi)|^2 d\xi$$

$$= c\int_{\mathbb{R}^n} |\xi|^{2s} \cdot |\hat{u}(\xi)|^2 d\xi.$$

We see that $I_s(u) < \infty \iff \langle\xi\rangle^s \hat{u}(\xi) \in L^2$
$\iff u \in H^s(\mathbb{R}^n)$. $\square$

§12.2. Elliptic regularity II

Here we study a constant coefficient differential operator of order $m \geq 0$

$$P = \sum_{|\alpha| \leq m} c_\alpha D_x^\alpha$$

where $c_\alpha \in \mathbb{C}$ and $D_x = -i\partial_x$,

$$D_x^\alpha = (-i)^{|\alpha|} \partial_x^\alpha$$

Defn. The principal symbol of P is the function

$$p_0(\xi) := \sum_{|\alpha| = m} c_\alpha \xi^\alpha, \quad \xi^\alpha := \xi_1^{\alpha_1} \cdots \xi_n^{\alpha_n}$$

The full symbol of P is

$$p(\xi) := \sum_{|\alpha| \leq m} c_\alpha \xi^\alpha.$$

Note: p_0, p are polynomials of degree m and p_0 is homogeneous.

Examples:

① $P = -\Delta = -\sum_{j=1}^n \partial_{x_j}^2$

$$p_0(\xi) = p(\xi) = |\xi|^2.$$

2. $n=2$, $P = \partial_{x_1} + i\partial_{x_2}$
(Cauchy-Riemann operator)

$p_0(\xi) = p(\xi) = i(\xi_1 + i\xi_2)$

3. $P = \partial_{x_0}^2 - \Delta_{x'}$ (wave operator)

$p_0(\xi) = p(\xi) = -\xi_0^2 + |\xi'|^2$

4. $P = \partial_{x_0} - \Delta_{x'}$ (heat operator)

$p(\xi) = i\xi_0 + |\xi'|^2$, $p_0(\xi) = |\xi'|^2$.

Key relation between P and p:

$\forall u \in \mathcal{S}'(\mathbb{R}^n)$,

$$\widehat{Pu}(\xi) = p(\xi)\hat{u}(\xi)$$

as follows from $\widehat{D_{x_j} u}(\xi) = \xi_j \hat{u}(\xi)$.

Defn. We say P is elliptic, if the equation $p_0(\xi) = 0$ has no roots $\xi \in \mathbb{R}^n$, $\xi \neq 0$.

Examples above: 1, 2 are elliptic
3, 4 are not elliptic.

Thm. [Elliptic Regularity II]

Assume that P is elliptic.
Then $\forall\, U \subset \mathbb{R}^n$ open, $\forall u \in \mathcal{D}'(U)$
we have $\operatorname{sing\,supp} u \subset \operatorname{sing\,supp}(Pu)$.
The proof proceeds in several steps:

Step 1: construction of elliptic parametrix

We claim $\exists\, E \in \mathcal{D}'(\mathbb{R}^n)$ such that

- $PE - \delta_0 \in C^\infty(\mathbb{R}^n)$
  (i.e. $PE = \delta_0$ modulo C^∞)
- $\operatorname{sing\,supp} E \subset \{0\}$.

We construct E lying in $\mathcal{S}'(\mathbb{R}^n)$.
Since P is elliptic, $\exists\, c > 0$ s.t.
$|p_0(\xi)| \geq c|\xi|^m$ for all $\xi \neq 0$.
Since $p(\xi) = p_0(\xi) + O(|\xi|^{m-1})$
there exists $R > 0$ s.t.
$|p(\xi)| \geq \frac{c}{2}|\xi|^m$ for all ξ s.t. $|\xi| \geq R$

We fix $\chi \in C_c^\infty(\mathbb{R}^n)$,
$\chi = 1$ near $B(0,R)$
and define

$$q(\xi) := \frac{1-\chi(\xi)}{p(\xi)} \in C^\infty(\mathbb{R}^n).$$

Note that $q \in L^\infty$, so $q \in \mathcal{S}'(\mathbb{R}^n)$.
Define $E \in \mathcal{S}'(\mathbb{R}^n)$ such that

$$\hat{E} = q.$$

Then $\widehat{PE}(\xi) = p(\xi) q(\xi)$
$= 1 - \chi(\xi)$

Since $\chi \in C_c^\infty \subset \mathcal{S}(\mathbb{R}^n)$,
it is the Fourier transform
of a Schwartz function.
Also, $\hat{\delta}_0 = 1$. So

$$PE - \delta_0 \in \mathcal{S}(\mathbb{R}^n) \subset C^\infty(\mathbb{R}^n)$$

as needed.

Why sing supp $E \subset \{0\}$?

We use that q is a

Kohn–Nirenberg symbol

of degree $-m$, i.e. $\forall \alpha$

$$\partial_\xi^\alpha q(\xi) = O(\langle \xi \rangle^{-m-|\alpha|}). \qquad (*)$$

To see $(*)$ we may assume that $|\xi|$ is large enough so that $\xi \notin \operatorname{supp} \chi$, then

$$\partial_\xi^\alpha q(\xi) = \partial_\xi^\alpha \left(\frac{1}{p(\xi)} \right)$$

$=$ linear combination of expressions of the form $\dfrac{\partial^{\alpha_1} p \cdots \partial^{\alpha_k} p}{p^{k+1}}$

where $\alpha_1 + \cdots + \alpha_k = \alpha$.

Now, $\partial^{\alpha_\ell} p = O(\langle \xi \rangle^{m - |\alpha_\ell|})$

since p is a polynomial

and $\dfrac{1}{p^{k+1}} = O(\langle \xi \rangle^{-(k+1)m})$

So $\dfrac{\partial^{\alpha_1} p \cdots \partial^{\alpha_k} p}{p^{k+1}} = O(\langle \xi \rangle^{-(k+1)m + km - |\alpha_1| - \cdots - |\alpha_k|})$

$= O(\langle \xi \rangle^{-m-|\alpha|})$, giving $(*)$

How does $(*)$ imply that

Sing Supp $E \subset \{0\}$?

We have $\forall \alpha, \beta$

some powers of $\pm i$

$$\widehat{x^\alpha \partial_x^\beta E}(\xi) = (\ldots)\, \partial_\xi^\alpha \left(\xi^\beta \hat{E}(\xi)\right)$$

$$= (\ldots)\, \partial_\xi^\alpha \left(\xi^\beta q(\xi)\right) \in C^\infty(\mathbb{R}^n)$$

and it is $O\left(\langle \xi \rangle^{-m+|\beta|-|\alpha|}\right)$.

If $-m+|\beta|-|\alpha| \leq -n-1$ then

$$\widehat{x^\alpha \partial_x^\beta E}(\xi) \in L^1(\mathbb{R}^n)$$

which means that

$$|\alpha| \geq n+1-m+|\beta| \Rightarrow x^\alpha \partial_x^\beta E \in C^0(\mathbb{R}^n).$$

In particular, if $2\ell \geq n+1-m+|\beta|$

then $|x|^{2\ell} \partial_x^\beta E \in C^0$.

This implies that $\forall \beta$,

the restriction of $\partial_x^\beta E$ to $\mathbb{R}^n \setminus \{0\}$

is in C^0, so $E|_{\mathbb{R}^n \setminus \{0\}} \in C^\infty$.

<u>Step 2</u>: use elliptic parametrix + cutoffs similarly to §9.4.

Let $u \in \mathcal{D}'(U)$ and

$x_0 \in U$, $x_0 \notin \text{sing supp } Pu$.

We need to show that $x_0 \notin \text{sing supp } u$.

Take $\chi \in C_c^\infty(U)$, $\chi = 1$ near x_0.

Write $PE = \delta_0 + R$, $R \in C^\infty(\mathbb{R}^n)$.

Put $\boxed{v := \chi u} \in \mathcal{E}'(\mathbb{R}^n)$. Then

$$v = \delta_0 * v = (PE - R) * v$$
$$= (PE) * v - R * v$$
$$= E * Pv - R * v.$$

Since $R \in C^\infty(\mathbb{R}^n)$, we have $R * v \in C^\infty(\mathbb{R}^n)$.

So $\text{sing supp } v \subset \text{sing supp } (E * Pv)$

$\subset \text{sing supp } Pv$

Since $\text{sing supp } E \subset \{0\}$.

And $Pv = P\chi u$

$$= \chi P u + [P, \chi] u.$$

Now $x_0 \notin$ sing supp Pu

and $[P, \chi]$ is a diff. operator with coefficients supported away from x_0,

So $x_0 \notin \operatorname{supp} [P, \chi] u$.

So $x_0 \notin$ sing supp Pv

$$\Downarrow$$

$x_0 \notin$ sing supp v

$$\Downarrow$$

$x_0 \notin$ sing supp u as needed.

$\square$