§11. Fourier transform I

§11.1. Schwartz functions

__Defn.__ For $f \in L^1(\mathbb{R}^n)$, its __Fourier transform__ is

$$\hat{f}(\xi) := \int_{\mathbb{R}^n} e^{-ix\cdot\xi} f(x)\,dx$$

where $x \cdot \xi := \sum_{j=1}^n x_j \xi_j$.

We see directly that $\hat{f} \in L^\infty(\mathbb{R}^n)$:

$$\|\hat{f}\|_{L^\infty} \le \|f\|_{L^1}.$$

But it is __not__ true that the Fourier transform is an isomorphism $L^1 \to L^\infty$.

We instead will study it on the much smaller space of __Schwartz functions__:

<u>Defn</u> A function $\varphi \in C^\infty(\mathbb{R}^n)$ is called <u>Schwartz</u> if all derivatives of φ decay faster than any power of x, i.e. $\forall \alpha, \beta$,

$$x^\alpha \partial_x^\beta \varphi \in L^\infty(\mathbb{R}^n)$$

(Here $x^\alpha := x_1^{\alpha_1} \dots x_n^{\alpha_n}$.)

- Denote by $S(\mathbb{R}^n)$ the space of all Schwartz functions.
  It is a Fréchet space with seminorms $\sup |x^\alpha \partial_x^\beta \varphi|$ for all α, β
  Convergence in S: $\varphi_n \to 0$ if $\sup |x^\alpha \partial_x^\beta \varphi_n| \to 0$ $\forall \alpha, \beta$.
- $C_c^\infty(\mathbb{R}^n) \subset S(\mathbb{R}^n)$.
- $S(\mathbb{R}^n) \subset L^1(\mathbb{R}^n)$.
- x_j (multiplication operator) and ∂_{x_j} are continuous operators $S(\mathbb{R}^n) \circlearrowleft$

Fourier transform & derivatives:

Denote $D_j = D_{x_j} := -i\partial_{x_j}$.

Then for all $\varphi \in \mathcal{S}(\mathbb{R}^n)$, we have

$$\widehat{D_j\varphi}(\xi) = \xi_j \hat{\varphi}(\xi) \qquad (1)$$

$$\widehat{x_j\varphi}(\xi) = -D_{\xi_j}\hat{\varphi}(\xi). \qquad (2)$$

<u>Proof</u>: (1) $\widehat{D_j\varphi}(\xi) = \int_{\mathbb{R}^n} e^{-ix\cdot\xi}(-i\partial_{x_j}\varphi(x))\,dx$

$= \int_{\mathbb{R}^n} i\partial_{x_j}(e^{-ix\cdot\xi})\cdot\varphi(x)\,dx$ by integration by parts

$= \int_{\mathbb{R}^n} \xi_j e^{-ix\cdot\xi}\varphi(x)\,dx = \xi_j\hat{\varphi}(\xi)$.

(2) $-D_{\xi_j}\hat{\varphi}(\xi) = i\partial_{\xi_j}\int_{\mathbb{R}^n} e^{-ix\cdot\xi}\varphi(x)\,dx$

$= \int_{\mathbb{R}^n} x_j e^{-ix\cdot\xi}\varphi(x)\,dx = \widehat{x_j\varphi}(\xi)$

Both IBP (1) and differentiating the $\int$ (2) work because φ is rapidly decaying.

$\square$

Thm The Fourier transform is a continuous operator

$$\mathcal{F} : \mathcal{S}(\mathbb{R}^n) \to \mathcal{S}(\mathbb{R}^n).$$

Proof We know that $\forall \varphi \in \mathcal{S}(\mathbb{R}^n)$,

$$\sup|\hat{\varphi}| \le \|\varphi\|_{L^1} \le C\|(1+|x|)^{n+1}\varphi\|_{L^\infty}$$

$\le$ Some $\mathcal{S}$-seminorm of φ.

Now, iterating the statement on the previous page, we get $\forall \alpha, \beta$

$$\xi^\beta D_\xi^\alpha \hat{\varphi}(\xi) = \widehat{D^\beta (-x)^\alpha \varphi}(\xi)$$

So $\sup|\xi^\beta D_\xi^\alpha \hat{\varphi}| \le$ some $\mathcal{S}(\mathbb{R}^n)$-seminorm of φ

which gives the needed continuity. $\square$

- The Fourier transform is its own transpose:

$$(\hat{\varphi}, \psi) = (\varphi, \hat{\psi}) \quad \forall \varphi, \psi \in \mathcal{S}(\mathbb{R}^n)$$

Indeed, both are equal to

$$\int_{\mathbb{R}^{2n}} e^{-ix\cdot\xi} \varphi(x)\psi(\xi)\,dx\,d\xi$$

- Fourier transform of convolution:

if $\varphi, \psi \in \mathcal{S}(\mathbb{R}^n)$ then

$\varphi * \psi \in \mathcal{S}(\mathbb{R}^n)$ and

$$\widehat{\varphi * \psi}(\xi) = \hat{\varphi}(\xi)\,\hat{\psi}(\xi).$$

Indeed, $\widehat{\varphi * \psi}(\xi) = \int_{\mathbb{R}^n} e^{-ix\cdot\xi}\,\varphi * \psi(x)\,dx$

$$= \int_{\mathbb{R}^{2n}} e^{-ix\cdot\xi}\,\varphi(y)\,\psi(x-y)\,dx\,dy$$

$\boxed{x \to x+y}$

$$= \int_{\mathbb{R}^{2n}} e^{-i(x+y)\cdot\xi}\,\varphi(y)\,\psi(x)\,dx\,dy = \hat{\varphi}(\xi)\hat{\psi}(\xi)$$

- Fourier transform of Gaussians:

if $G(x) = e^{-\frac{|x|^2}{2}}$

then $\hat{G}(\xi) = (2\pi)^{\frac{n}{2}} e^{-\frac{|\xi|^2}{2}}$

i.e. $\hat{G} = (2\pi)^{\frac{n}{2}} G$

(See Pset 7)

- **Fourier Inversion Formula:**

if $\varphi \in S(\mathbb{R}^n)$ then $\forall x \in \mathbb{R}^n$,

$$\varphi(x) = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{ix\cdot\xi} \hat{\varphi}(\xi)\, d\xi.$$

From this one can see that the Fourier transform $\mathcal{F} : S(\mathbb{R}^n) \circlearrowleft$ is invertible, with inverse

$$\mathcal{F}^{-1}\psi(x) = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{ix\cdot\xi} \psi(\xi)\, d\xi$$

Proof Take the Gaussian $G(\xi) = e^{-\frac{|\xi|^2}{2}}$.

We write

$$\int_{\mathbb{R}^n} e^{ix\cdot\xi} \hat{\varphi}(\xi)\, d\xi \overset{\text{by D.C.T.}}{=} \lim_{\varepsilon \to 0+} \int_{\mathbb{R}^n} G(\varepsilon\xi) e^{ix\cdot\xi} \hat{\varphi}(\xi)\, d\xi$$

$$= \lim_{\varepsilon \to 0+} \int_{\mathbb{R}^{2n}} G(\varepsilon\xi) e^{i(x-y)\cdot\xi} \varphi(y)\, dy\, d\xi$$

$$= \varepsilon^{-n} \lim_{\varepsilon \to 0+} \int_{\mathbb{R}^{2n}} G(\xi) e^{\frac{i}{\varepsilon}(x-y)\cdot\xi} \varphi(y)\, dy\, d\xi$$

$$= \varepsilon^{-n} \lim_{\varepsilon \to 0+} \int_{\mathbb{R}^n} \hat{G}\left(\frac{y-x}{\varepsilon}\right) \varphi(y)\, dy$$

$(y = x + \varepsilon z)$

$$= \lim_{\varepsilon \to 0+} \int_{\mathbb{R}^n} \hat{G}(z)\, \varphi(x + \varepsilon z)\, dz$$

$$= \int_{\mathbb{R}^n} \hat{G}(z)\, dz \cdot \varphi(x) \quad \text{by D.C.T.}$$

(since $\varphi(x+\varepsilon z) \xrightarrow{\varepsilon \to 0} \varphi(x)$ $\forall z$,
$\hat{G} \in S \subset L^1$, $\sup|\varphi| < \infty$)

$$= (2\pi)^n \cdot \varphi(x) \quad \text{since}$$

$$\hat{G}(z) = (2\pi)^{\frac{n}{2}} e^{-\frac{|z|^2}{2}},$$

$$\int_{\mathbb{R}^n} \hat{G}(z)\, dz = (2\pi)^{\frac{n}{2}} \int_{\mathbb{R}^n} e^{-\frac{|z|^2}{2}} dz = (2\pi)^n. \quad \square$$

<u>Corollary</u>: $\forall \varphi \in S(\mathbb{R}^n)$,

$$\|\hat{\varphi}\|_{L^2} = (2\pi)^n \|\varphi\|_{L^2}.$$

So the Fourier transform extends to a bounded isomorphism $L^2(\mathbb{R}^n) \circlearrowleft$

§11.2. Tempered distributions

Defn. We call a linear map

$u : \mathcal{S}(\mathbb{R}^n) \to \mathbb{C}$ a **tempered distribution**

if u is continuous, i.e.

$$\varphi_n \to 0 \text{ in } \mathcal{S} \Rightarrow (u, \varphi_n) \to 0$$

Equivalently, $\exists C, N : \forall \varphi \in \mathcal{S}(\mathbb{R}^n)$,

$$|(u,\varphi)| \leq C \sum_{|\alpha|, |\beta| \leq N} \sup |x^\alpha \partial_x^\beta \varphi|.$$

• Denote by $\mathcal{S}'(\mathbb{R}^n)$ the space of tempered distributions.

We say $u_n \to u$ in $\mathcal{S}'(\mathbb{R}^n)$ if $(u_n, \varphi) \to (u, \varphi)$ $\forall \varphi \in \mathcal{S}(\mathbb{R}^n)$.

• $L^p(\mathbb{R}^n) \subset \mathcal{S}'(\mathbb{R}^n)$ $\forall p \in [1, \infty]$:

$$(f, \varphi) := \int_{\mathbb{R}^n} f(x)\varphi(x)\,dx \qquad \forall f \in L^p(\mathbb{R}^n),\ \varphi \in \mathcal{S}(\mathbb{R}^n)$$

$$|(f,\varphi)| \leq \|f\|_{L^p} \|\varphi\|_{L^q} \text{ by Hölder } \quad \tfrac{1}{p} + \tfrac{1}{q} = 1$$

In particular, $\mathcal{S}(\mathbb{R}^n) \subset \mathcal{S}'(\mathbb{R}^n)$.

- $\mathcal{E}'(\mathbb{R}^n) \subset \mathcal{S}'(\mathbb{R}^n) \subset \mathcal{D}'(\mathbb{R}^n)$,

Since $C^\infty(\mathbb{R}^n) \supset \mathcal{S}(\mathbb{R}^n) \supset C_c^\infty(\mathbb{R}^n)$

- $C_c^\infty(\mathbb{R}^n)$ is <u>dense</u> in $\mathcal{S}'(\mathbb{R}^n)$ (see Pset 7).

- <u>Differentiation</u>: if $u \in \mathcal{S}'(\mathbb{R}^n)$ then define $\partial_{x_j} u \in \mathcal{S}'(\mathbb{R}^n)$ by

$$(\partial_{x_j} u, \varphi) = -(u, \partial_{x_j}\varphi) \quad \forall \varphi \in \mathcal{S}(\mathbb{R}^n).$$

(agrees with the usual derivative when $u \in \mathcal{S}$)

- <u>Multiplication</u>: see Pset 7

- <u>Fourier Transform</u>:

if $u \in \mathcal{S}'(\mathbb{R}^n)$ then define $\hat{u} \in \mathcal{S}'(\mathbb{R}^n)$ by

$$(\hat{u}, \varphi) := (u, \hat{\varphi}) \quad \forall \varphi \in \mathcal{S}(\mathbb{R}^n).$$

<u>Note</u>: this agrees with Fourier transform on $\mathcal{S}$ when $u \in \mathcal{S}$

Basic example:

$u = \delta_0 \in \mathcal{E}'(\mathbb{R}^n) \subset \mathcal{S}'(\mathbb{R}^n)$.

Then $\forall \varphi \in \mathcal{S}(\mathbb{R}^n)$,

$$(\hat{\delta}_0, \varphi) = (\delta_0, \hat{\varphi}) = \hat{\varphi}(0)$$

$$= \int_{\mathbb{R}^n} \varphi(x)\, dx = (1, \varphi)$$

So $\boxed{\hat{\delta}_0 = 1}$

Basic properties of Fourier Transform on $\mathcal{S}'$:

1 Fourier Inversion Formula:

if $u \in \mathcal{S}'(\mathbb{R}^n)$ and $v = \hat{u} \in \mathcal{S}'(\mathbb{R}^n)$ then

$$u(x) = (2\pi)^{-n} \hat{v}(-x).$$

Here for $w \in \mathcal{S}'(\mathbb{R}^n)$, define $w(-x) \in \mathcal{S}'(\mathbb{R}^n)$ by

$$(w(-x), \varphi(x)) := (w(x), \varphi(-x)) \quad \forall \varphi \in \mathcal{S}$$

Can be proved using density of $\mathcal{S}$ in $\mathcal{S}'$.

But we give a more direct

<u>Proof</u>: For any $\varphi \in S(\mathbb{R}^n)$ we have (putting $\psi(x) := \varphi(-x)$)

$(2\pi)^{-n}(\hat{v}(-x), \varphi(x)) = (2\pi)^{-n}(\hat{v}, \psi)$

$= (2\pi)^{-n}(v, \hat{\psi}) = (2\pi)^{-n}(\hat{u}, \hat{\psi})$

$= (2\pi)^{-n}(u, \hat{\hat{\psi}})$. But

$$(2\pi)^{-n}\hat{\hat{\psi}}(x) = (2\pi)^{-n}\int_{\mathbb{R}^n} e^{-ix\xi}\hat{\psi}(\xi)\,d\xi =$$

(by the Fourier Inversion Formula in S)

$= \psi(-x) = \varphi(x)$. So

$(2\pi)^{-n}(\hat{v}(-x), \varphi(x)) = (u, \varphi) \Rightarrow$

$\Rightarrow u(x) = (2\pi)^{-n}\hat{v}(-x)$. $\square$

<u>Application</u>: $\hat{\delta}_0 = 1 \Rightarrow \boxed{\hat{1} = (2\pi)^n \delta_0.}$

$\boxed{2}$ <u>Differentiation & Multiplication</u>: $\forall u \in S'(\mathbb{R}^n)$

$$\widehat{D_x^\alpha u}(\xi) = \xi^\alpha \hat{u}(\xi), \quad \widehat{x^\alpha u}(\xi) = (-1)^{|\alpha|} D_\xi^\alpha \hat{u}(\xi)$$

<u>Proof</u>: for any $\varphi \in S(\mathbb{R}^n)$, compute

$$(\widehat{D_x^\alpha u}, \varphi) = (D_x^\alpha u, \hat{\varphi})$$
$$= (-1)^{|\alpha|} (u, D_x^\alpha \hat{\varphi})$$
$$= (-1)^{|\alpha|} (u, \widehat{(-\xi)^\alpha \varphi})$$
$$= (u, \widehat{\xi^\alpha \varphi}) = (\hat{u}, \xi^\alpha \varphi)$$
$$= (\xi^\alpha \hat{u}, \varphi)$$

and

$$(\widehat{x^\alpha u}, \varphi) = (x^\alpha u, \hat{\varphi}) = (u, x^\alpha \hat{\varphi})$$
$$= (u, \widehat{D_\xi^\alpha \varphi}) = (\hat{u}, D_\xi^\alpha \varphi)$$
$$= (-1)^{|\alpha|} (D_\xi^\alpha \hat{u}, \varphi).$$

$\boxed{3}$ Fourier transform on $\mathcal{E}' \subset \mathcal{S}'$:

if $u \in \mathcal{E}'$ then $\hat{u} \in C^\infty(\mathbb{R}^n)$ and

$$\hat{u}(\xi) = (u(x), \underbrace{e^{-ix\cdot\xi}}_{\in C^\infty(\mathbb{R}^n)})$$

Proof (1) If we define

$V(\xi) := (u(x), e^{-ix\cdot\xi})$ then $V \in C^\infty(\mathbb{R}^n)$

(because $u \in \mathcal{E}'$, $(x,\xi) \mapsto e^{-ix\cdot\xi} \in C^\infty$)

Moreover, v is polynomially bounded in ξ: $\boxed{e_\xi(x) = e^{-ix\cdot\xi}}$

$$|v(\xi)| = |(u(x), e_\xi(x))|$$
$$\leq C \|e_\xi\|_{C^N(B(0,R))}$$
$$\leq \tilde{C}(1+|\xi|)^N.$$

for some C, N, R depending only on u

(2) We need to show that $\forall \varphi \in \mathcal{S}(\mathbb{R}^n)$,

$$(v,\varphi) = (\hat{u},\varphi) \overset{\text{def}}{=} (u,\hat{\varphi})$$

Compute $(u,\hat{\varphi}) = \left(u(x), \int_{\mathbb{R}^n} \varphi(\xi) e^{-ix\cdot\xi} d\xi\right) =$

(using Riemann sums similarly to ($\star$) in §6.2)

$$= \int_{\mathbb{R}^n} \varphi(\xi)\,(u(x), e^{-ix\cdot\xi})\,d\xi$$

$$= \int_{\mathbb{R}^n} \varphi(\xi)\, v(\xi)\, d\xi = (v,\varphi). \quad \square$$

<u>Note</u>: if $u \in \mathcal{E}'(\mathbb{R}^n)$ then $\hat{u}(\xi)$ admits a holomorphic extension to $\xi \in \mathbb{C}^n$: $\hat{u}(\xi) = (u, e_\xi)$ where $e_\xi(x) = e^{-ix\cdot\xi} \in C^\infty(\mathbb{R}^n_x)$ holomorphic in $\xi \in \mathbb{C}^n$.

In fact, one can use complex analysis to characterize the class of $\hat{u}$ for $u \in \mathcal{E}'(\mathbb{R}^n)$.

This is known as <u>Paley-Wiener Thm</u>

We won't cover it here but you can look at [Friedlander-Joshi, §§10.1-10.2] or [Hörmander, §7.3].

$\boxed{4}$ <u>Convolutions</u>: if $u \in S'(\mathbb{R}^n)$, $v \in \mathcal{E}'(\mathbb{R}^n)$ then $u * v \in S'(\mathbb{R}^n)$ and

$$\widehat{u*v}(\xi) = \hat{u}(\xi) \cdot \hat{v}(\xi)$$

<u>Proof</u>: if $u, v \in \mathcal{E}'(\mathbb{R}^n)$ then $(e_\xi(x) = e^{-ix\cdot\xi})$

$$\widehat{u*v}(\xi) = (u*v, e_\xi) = (u(x) \otimes v(y), e_\xi(x+y))$$

$$= (u(x) \otimes v(y), e_\xi(x) \otimes e_\xi(y))$$

$$= (u, e_\xi)(v, e_\xi) = \hat{u}(\xi)\hat{v}(\xi)$$

For general $u \in S'(\mathbb{R}^n)$ just need to make sense of $\hat{u}\hat{v}$ & make sure that $u*v \in S'(\mathbb{R}^n)$. See [Friedlander-Joshi, §8.4] for details.