PROLOGUE

In 18.155 we develop tools used in the modern study of
Partial Differential Equations (PDEs)
such as distributions,
Sobolev spaces,
elliptic regularity ...

Question: consider the following PDEs on $\mathbb{R}^2$:

ⓐ $(\partial_x^2 + \partial_y^2)u = 0$

ⓑ $(\partial_x + i\partial_y)u = 0$

ⓒ $(\partial_x^2 - \partial_y^2)u = 0$

Is it true that every solution $u(x,y)$ is smooth (i.e. infinitely differentiable)?

Answer: true for

(a) (harmonic functions)

(b) (analytic functions of $z = x+iy$)

but false for (c):

e.g. if $f \in C^2$ (but not necessarily C^∞)

then $u(x,y) = f(x+y)$ is a solution.

What is the difference between (a), (b), (c)?

Replace ∂_x with $\xi \in \mathbb{R}$

∂_y with $\eta \in \mathbb{R}$

Get $p(\xi,\eta)$ a polynomial

(a) $p(\xi,\eta) = \xi^2 + \eta^2$

(b) $p(\xi,\eta) = \xi + i\eta$

(c) $p(\xi,\eta) = \xi^2 - \eta^2$

For (a), (b), the equation $p(\xi,\eta) = 0$ has no real solutions except $(0,0)$ (elliptic regularity, studied later)

We will eventually get to elliptic regularity & some applications e.g.

- discreteness of the spectrum of the Laplacian on a compact Riemannian manifold
- Hodge theory: harmonic forms $\cong$ de Rham cohomology

But first we need to develop basic tools e.g. distributions

<u>Example:</u> if $f \in C_c^\infty(\mathbb{R}^2)$ $\uparrow$ smooth & compactly supported

then a solution to the equation

$$(\partial_x^2 + \partial_y^2) u = f$$

is given by

$$u(x,y) = \frac{1}{4\pi} \int_{\mathbb{R}^2} \log(|x-\tilde{x}|^2 + |y-\tilde{y}|^2) f(\tilde{x},\tilde{y}) \, d\tilde{x} \, d\tilde{y}.$$

Here are a couple of things to think about:

1. We can define u for any f measurable, bounded, compactly supported. But then u might not be in C^2. How do we understand the PDE then?

In the sense of <u>weak solutions</u>: for all $\varphi \in C_c^\infty(\mathbb{R}^2)$,

$$\int_{\mathbb{R}^2} u \cdot \left((\partial_x^2 + \partial_y^2)\varphi \right) = \int_{\mathbb{R}^2} f \cdot \varphi$$

Check this is true when $u \in C^2$, $f = (\partial_x^2 + \partial_y^2)u$, using integration by parts

2 We can try to show that $\Delta u = f$ by differentiating: $(f \in C_c^\infty(\mathbb{R}^2))$

$$u(x,y) = \int_{\mathbb{R}^2} E(x-\tilde{x}, y-\tilde{y}) f(\tilde{x},\tilde{y})\, d\tilde{x}\, d\tilde{y}$$

where $E(x) = \frac{1}{4\pi} \log(x^2+y^2)$

So $(\partial_x^2 + \partial_y^2) u(x,y) =$

$$= \int_{\mathbb{R}^2} ((\partial_x^2 + \partial_y^2) E)(x-\tilde{x}, y-\tilde{y}) f(\tilde{x},\tilde{y})\, d\tilde{x}\, d\tilde{y}$$

But we compute $(\partial_x^2 + \partial_y^2) E = 0$.

Why no contradiction?

Cannot differentiate under $\int$ and in fact, in distributions

$$(\partial_x^2 + \partial_y^2) E = \delta(x,y)$$

"Dirac mass" at 0

§1.1. The spaces L^p

We review various vector spaces of functions $f: \mathbb{R}^n \to \mathbb{C}$.
We start with the L^p spaces

- Lebesgue Integral (a very brief review)

Lebesgue measure:

$A \subset \mathbb{R}^n$ measurable set $\mapsto \lambda(A) \in [0, \infty]$
its measure
("volume")

For practical purposes,
"all sets are measurable"

Lebesgue Integral:

- for any $f: \mathbb{R}^n \to [0, \infty]$ which is measurable, i.e. $\{x \in \mathbb{R}^n : f(x) \leq a\}$ is a measurable set for all $a \in \mathbb{R}$

can define $\int_{\mathbb{R}^n} f(x)dx \in [0, \infty]$

- If $f: \mathbb{R}^n \to \mathbb{C}$ measurable and $\int_{\mathbb{R}^n} |f|\,dx < \infty$ then can define $\int_{\mathbb{R}^n} f(x)\,dx \in \mathbb{C}$.
- On $\mathbb{R}$ this coincides with the Riemann integral (if f is Riemann integrable)
- Fubini's/Tonnelli's Thm: if we write points in $\mathbb{R}^n$ as $(x,y) \in \mathbb{R}^k \times \mathbb{R}^{n-k}$

then
$$\int_{\mathbb{R}^n} f(x,y)\,dx\,dy = \int_{\mathbb{R}^k} \left(\int_{\mathbb{R}^{n-k}} f(x,y)\,dy \right) dx$$
if f is measurable and either $\int_{\mathbb{R}^n} |f| < \infty$ or $f \geq 0$

- The space L^1

Define the seminorm
$$\|f\|_{L^1(\mathbb{R}^n)} := \int_{\mathbb{R}^n} |f(x)|\,dx$$
on the space of all measurable functions $f: \mathbb{R}^n \to \mathbb{C}$

It is not a norm:

$$\|f\|_{L^1} = 0 \iff f(x) = 0$$

for almost every x
(a.e.)

i.e. $\{x \in \mathbb{R}^n : f(x) \neq 0\}$
has measure 0

Define

$$L^1(\mathbb{R}^n) = \{f : \mathbb{R}^n \to \mathbb{C} \text{ measur., } \|f\|_{L^1} < \infty\} / \sim$$

where $f \sim g$ if $f(x) = g(x)$
for a.e. x

Together with $\|\cdot\|_{L^1}$

$L^1(\mathbb{R}^n)$ forms a Banach space,

i.e.
- $L^1(\mathbb{R}^n)$ is a vector space (over $\mathbb{C}$);
- $\|\cdot\|_{L^1}$ is a norm on $L^1(\mathbb{R}^n)$;
- $L^1(\mathbb{R}^n)$ is complete

(every Cauchy sequence converges)

Note: For any (positive measure) set $U \subset \mathbb{R}^n$ we can define

$$L^1(U) = \{ f \in L^1(\mathbb{R}^n) : f(x) = 0 \text{ for all } x \notin U \}$$

$\downarrow$

L^1 functions $f : U \to \mathbb{C}$

The spaces L^p

For $1 \leq p < \infty$ define

$$\|f\|_{L^p} := \left(\int |f(x)|^p \, dx \right)^{1/p}$$

Similarly to L^1, get a Banach space $L^p(\mathbb{R}^n)$

For $p = \infty$:

$$\|f\|_{L^\infty} := \inf \{ a \geq 0 : |f(x)| \leq a \text{ for a.e. } x \}$$

The space $\underline{\underline{L^2(\mathbb{R}^n)}}$ is very special: it is a Hilbert space

That is,

$$\|f\|_{L^2} = \sqrt{\langle f,f\rangle_{L^2}}$$

where the Hermitian inner product $\langle \circ, \circ \rangle_{L^2}$ is defined by

$$\langle f,g\rangle := \int_{\mathbb{R}^n} f(x)\overline{g(x)}\,dx$$

<u>Riesz Representation Thm</u>:

<u>1</u>. Let $g \in L^2(\mathbb{R}^n)$. Then $T_g : f \mapsto \langle f, g\rangle$ defines a <u>bounded linear functional</u> on $L^2(\mathbb{R}^n)$ and $\|T_g\|_{L^2 \to \mathbb{C}} \overset{def}{=} \sup_{\substack{f \in L^2 \\ \|f\|_{L^2}=1}} |T_g(f)|$ is equal to $\|g\|_{L^2}$.

<u>2</u>. Every bounded linear functional on $L^2(\mathbb{R}^n)$ has the form T_g for some $g \in L^2(\mathbb{R}^n)$

This establishes an isomorphism

$$L^2(\mathbb{R}^n) \simeq (L^2(\mathbb{R}^n)')$$

the space of bdd linear functionals $L^2(\mathbb{R}^n) \to \mathbb{C}$ with operator norm

(L^2 is self-dual)

[More generally we have

$$(L^p(\mathbb{R}^n))' = L^q(\mathbb{R}^n)$$

for $1 \leq p < \infty$

where $\frac{1}{p} + \frac{1}{q} = 1$]

§ 1.2. The Space C_c^∞

Let $U \subset \mathbb{R}^n$ be a (nonempty) open set

$C^0(U)$ = the set of continuous functions $f: U \to \mathbb{C}$

Note: $\|f\|_{C^0} := \sup_{x \in U} |f(x)|$ is not a norm because we may have $\|f\|_{C^0} = \infty$ (since U is not compact)

One way to fix this is to consider functions with compact support.

Defn. For $f: U \to \mathbb{C}$ the support $\operatorname{supp} f$ is the closure of the set $\{x \in U : f(x) \neq 0\}$ inside U.

Why take closure?

This way, if f is differentiable,

$$\operatorname{supp}(\partial_{x_j} f) \subset \operatorname{supp} f$$

Indeed, if $x \in U$ does not lie in $\operatorname{supp} f$ then $\exists\, \varepsilon > 0$:

$f = 0$ on the open ball $B(x, \varepsilon)$

$\Downarrow$

$\partial_{x_j} f = 0$ on the same ball

$\Downarrow$

$x \notin \operatorname{supp}(\partial_{x_j} f)$.

Define the space

$$C_c^0(U) := \{ f \in C^0(U) \mid \operatorname{supp} f \text{ is compact} \}$$

For $f \in C_c^0(U)$, $\|f\|_{C^0} < \infty$ & this defines a normed vector space (but not a Banach space)

<u>Note</u>: $C_c^0(U) \subset L^p(U)$ for any p

Differentiating

Define

$$C^1(\overline{U}) := \{ f \in C^0(\overline{U}) : \partial_{x_1} f, \dots, \partial_{x_n} f \text{ exist and lie in } C^0(\overline{U}) \}$$

$$C^{k+1}(\overline{U}) := \{ f \in C^k(\overline{U}) : \partial_{x_1} f, \dots, \partial_{x_n} f \in C^k(\overline{U}) \}$$

Multiindex notation:

$\alpha = (\alpha_1, \dots, \alpha_n)$ multiindex

where $\alpha_1, \dots, \alpha_n \geq 0$ integers

$$|\alpha| := \alpha_1 + \cdots + \alpha_n$$

$$\partial_x^\alpha f := \partial_{x_1}^{\alpha_1} \cdots \partial_{x_n}^{\alpha_n} f$$

Now put

$$\|f\|_{C^k(\overline{U})} := \max_{|\alpha| \leq k} \sup_{\overline{U}} |\partial_x^\alpha f|$$

Get spaces $C_c^k(\overline{U})$ similarly to $C_c^0(\overline{U})$

Define the spaces

$$C^\infty(U) := \bigcap_k C^k(U)$$

Smooth functions on U

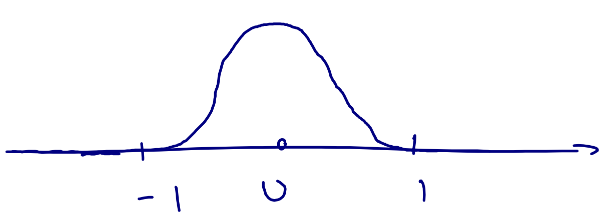
$$C_c^\infty(U) := \{ f \in C^\infty(U) : \operatorname{supp} f \text{ compact} \}$$

Compactly supported smooth functions on U.

<u>Fact</u>: $C_c^\infty(U)$ has nontrivial functions

If $U = B(0,r)$ is the ball of radius $r > 1$ can use the <u>bump function</u>

$$f(x) = \begin{cases} e^{-\frac{1}{1-|x|^2}}, & |x| < 1 \\ 0, & |x| \geq 1 \end{cases}$$



- $\operatorname{supp} f = \overline{B(0,1)}$
- $f > 0$ on $B(0,1)$

Theorem: $C_c^\infty(U)$ is dense in $L^p(U)$ for $1 \le p < \infty$,
that is for all $f \in L^p(U)$ there exists a sequence $f_n \in C_c^\infty(U)$ such that $\|f_n - f\|_{L^p} \xrightarrow[n\to\infty]{} 0$.

The proof proceeds in 2 steps:

Step 1: $C_c^0(U)$ is dense in $L^p(U)$.

We only sketch the proof:

1. $L_c^p(U) = \{ f \in L^p(U) \,|\, \operatorname{supp} f \text{ compact} \}$ is dense in $L^p(U)$:
take a sequence of compact sets
$K_1 \subset K_2 \subset \dots \subset K_j \subset \dots \subset U$
s.t. $U = \bigcup_{j \ge 1} K_j$,
then for each $f \in L^p(U)$,
$f \cdot 1_{K_j} \longrightarrow f$ in L^p

(note: will use that A dense in B, B dense in C implies A dense in C)

2. The set of linear combinations of 1_Q where $Q \subset U$ is a rectangle is dense in $L^p_c(U)$

3. Each 1_Q can be approximated in any L^p, $p < \infty$ by continuous functions in $C^0_c(U)$:

<u>Step 2</u>: $C^\infty_c(U)$ is dense in $C^0_c(U)$:

for each $f \in C^0_c(U)$ there exists a sequence $f_n \in C^\infty_c(U)$ such that

$$f_n \to f \text{ uniformly}$$

(and thus, since WLOG U is bounded,

$$f_n \to f \text{ in any } L^p)$$

To prove this we introduce <u>convolution</u>:

<u>Defn.</u> Let $f, g \in C^0_c(\mathbb{R}^n)$. Define $f * g \in L^\infty(\mathbb{R}^n)$ by

$$f * g(x) = \int_{\mathbb{R}^n} f(y) g(x-y)\, dy.$$

Basic properties of convolution:

- $f*g = g*f$
- $\operatorname{supp}(f*g) \subset \operatorname{supp} f + \operatorname{supp} g =$

$$= \{x+y \mid x \in \operatorname{supp} f, y \in \operatorname{supp} g\}$$

- $f,g \in C_c^0(\mathbb{R}^n) \Rightarrow f*g \in C_c^0(\mathbb{R}^n)$

Indeed, $|f*g(x) - f*g(\tilde{x})|$

$$\leq \int_{\mathbb{R}^n} |f(y)| \cdot |g(x-y) - g(\tilde{x}-y)| dy$$

$\to 0$ as $\tilde{x} \to x$ since

$$\sup_y |g(x-y) - g(\tilde{x}-y)| \xrightarrow[\tilde{x}\to x]{} 0$$

as g is uniformly continuous

- If $f \in C_c^1(\mathbb{R}^n)$, $g \in C_c^0(\mathbb{R}^n)$ then $f*g \in C_c^1(\mathbb{R}^n)$ and

$$\partial_{x_j}(f*g) = (\partial_{x_j} f)*g.$$

To prove this, differentiate under the integral sign!

$$\frac{f*g(x+te_j)-f*g(x)}{t}$$

$$=\frac{1}{t}\int_{\mathbb{R}^n}\left(f(x+te_j-y)-f(x-y)\right)g(y)\,dy$$

$$\downarrow t\to 0$$

$$\int_{\mathbb{R}^n}\partial_{x_j}f(x-y)\,g(y)\,dy$$

Since $\frac{f(x+te_j-y)-f(x-y)}{t}\to\partial_{x_j}f(x-y)$

uniformly in y by the Mean Value Thm + uniform continuity of $\partial_{x_j}f$

- Iterating the last point, we get:

$f\in C_c^k(\mathbb{R}^n)$, $g\in C_c^0(\mathbb{R}^n)$ $\Rightarrow$

$\Rightarrow$ $f*g\in C_c^k(\mathbb{R}^n)$,

$\partial^\alpha(f*g)=(\partial^\alpha f)*g$ when $|\alpha|\le k$

Back to the proof of Step 2

($C_c^\infty(U)$ dense in $C_c^0(U)$)

Fix $f \in C_c^0(U)$.

Fix also a "bump function"

$$\chi \in C_c^\infty(B(0,1)), \quad \int_{\mathbb{R}^n} \chi(x)\,dx = 1.$$

For $\varepsilon > 0$, define

$$\chi_\varepsilon(x) = \varepsilon^{-n} \chi\left(\frac{x}{\varepsilon}\right)$$

Note: $\int \chi_\varepsilon = 1$, $\operatorname{supp} \chi_\varepsilon \subset B(0,\varepsilon)$

Now, consider the mollifying sequence

$$f_\varepsilon := f * \chi_\varepsilon.$$

Note that $f_\varepsilon \in C_c^\infty(\mathbb{R}^n)$ (as $\chi_\varepsilon \in C_c^\infty$)

and $\operatorname{supp} f_\varepsilon \subset \operatorname{supp} f + B(0,\varepsilon) \subset U$

for ε small; i.e. $f_\varepsilon \in C_c^\infty(U)$

Claim: $f_\varepsilon \xrightarrow[\varepsilon \to 0]{} f$ uniformly

(which shows that C_c^∞ is dense in C_c^0)

Proof of claim: write

$$\left| f(x) - f_\varepsilon(x) \right| \leq \int_{\mathbb{R}^n} \left| \left(f(x) - f(x-y)\right) \chi_\varepsilon(y) \right| dy$$

$$= \int_{B(0,\varepsilon)} \left| \left(f(x) - f(x-y)\right) \chi_\varepsilon(y) \right| dy$$

$$\leq \|\chi\|_{L^1} \cdot \sup_{|y| \leq \varepsilon} |f(x) - f(x-y)| \longrightarrow 0$$

uniformly in x since f is uniformly continuous.

---

§1.3. More on C_c^∞

If $K \subset U \subset \mathbb{R}^n$, K compact, U open, then there exists $\chi \in C_c^\infty(U)$ with $\chi = 1$ on K

(take an intermediate set V, $K \subset V \subset U$, and put $\chi = 1_V * \chi_\varepsilon$ for small enough ε)

Using these cutoff functions, one can prove

<u>Thm</u> [Partition of unity]
Let $K \subset U_1 \cup \dots \cup U_m$ where K is compact, $U_1, \dots, U_m \subset \mathbb{R}^n$ open.
Then there exist

$$\chi_1 \in C_c^\infty(U_1), \dots, \chi_m \in C_c^\infty(U_m)$$

such that $\chi_j \geq 0$,

$$0 \leq \chi_1 + \dots + \chi_m \leq 1, \text{ and}$$

$$\chi_1 + \dots + \chi_m = 1 \text{ on } K.$$

Finally, here is a lemma crucial for distribution theory:

<u>Lemma</u> Let $U \subset \mathbb{R}^n$ be open, $f \in L^1_{loc}(U)$. Assume that $\int_U f(x)\varphi(x) = 0$ for all $\varphi \in C_c^\infty(U)$.
Then $f = 0$ a.e.

Here $L^1_{loc}(U)$ consists of locally L^1 functions $f: U \to \mathbb{C}$, i.e. $f \cdot 1_K \in L^1$ for all compact $K \subset U$.

<u>Proof</u> If $f \in C^0(U)$ then this is easy: if $f \not\equiv 0$ then take $\varphi =$ a bump function supported where $\mathrm{Re} f > 0$ (or $\mathrm{Re} f < 0$)

For general $f \in L^1_{loc}(U)$, can reduce to the case $f \in L^1_c(U)$: enough to show $\chi f = 0$ a.e. for all $\chi \in C_c^\infty(U)$

Now, take χ_ε as before and define
$$f_\varepsilon(x) = f * \chi_\varepsilon(x) = \int_{\mathbb{R}^n} f(y)\chi_\varepsilon(x-y)\,dy$$
For ε small enough, $f_\varepsilon = 0$ everywhere.
Indeed, if $x \in \operatorname{supp} f + B(0,\varepsilon)$ then $y \mapsto \chi_\varepsilon(x-y)$ is in $C_c^\infty(U)$ once ε is small enough that $\operatorname{supp} f + \overline{B(0,2\varepsilon)} \subset U$ and otherwise $f_\varepsilon(x) = 0$ immediately

But now $f_\varepsilon(x) \xrightarrow[\varepsilon \to 0]{} f(x)$ for a.e. x:

$$|f(x) - f_\varepsilon(x)| = \left| \int_{\mathbb{R}^n} (f(x) - f(y)) \chi_\varepsilon(x-y)\, dy \right|$$

$$\leq \|\chi\|_{C^0} \cdot \varepsilon^{-n} \int_{B(x,\varepsilon)} |f(x) - f(y)|\, dy \longrightarrow 0$$

by the Lebesgue Differentiation Thm.

Since $f_\varepsilon \equiv 0$ we get $f = 0$ a.e