18.118, SPRING 2022, PROBLEM SET 4

1. Consider the following vector field on $\mathbb{R}^2_{x,\xi}$:

$$V = \xi\partial_x + (\sin x)\partial_\xi.$$

(This vector field corresponds to motion of a pendulum, with x being the angle to the vertical axis and $x = 0$ corresponding to the pendulum pointing directly upwards.) Let e^{tV} be the flow of V and define the diffeomorphism φ on $\mathbb{R}^2$ as $\varphi := e^{1V}$ (i.e. the time-one map of the flow). Show that $(0, 0)$ is a hyperbolic fixed point of φ and find its global stable/unstable manifolds. (Hint: the energy $\frac{1}{2}\xi^2 + \cos x$ is conserved along the flow. Look at points with the same energy as the fixed point.)

2. Assume that X is a compact manifold, $\varphi : X \to X$ is a diffeomorphism, and x_0 be a hyperbolic fixed point of φ. Let $W^s(x_0)$ be a local stable manifold of φ at x_0. Fix a volume form on X; for $x \in X$ and $n \geq 0$ denote by $|\det d\varphi^n(x)|$ the absolute value of the determinant of $d\varphi^n(x) : T_xX \to T_{\varphi^n(x)}X$ with respect to this volume form. Show that even for large n, the values of $|\det d\varphi^n(x)|$ at different points $x \in W^s(x_0)$ are comparable to each other, namely there exists C such that

$$|\det d\varphi^n(x)| \leq C|\det d\varphi^n(y)| \quad \text{for all} \quad x, y \in W^s(x_0), \quad n \geq 0.$$

(Hint: use the Chain Rule and multiplicativity of the determinant, together with the fact that $d(\varphi^n(x), \varphi^n(y)) \to 0$ exponentially fast as $n \to \infty$. This exercise also works for arbitrary hyperbolic φ-invariant sets, as well as separately for the stable and the unstable Jacobians, i.e. determinants of $d\varphi^n(x)$ restricted to stable/unstable spaces; the latter uses Hölder dependence of E_u, E_s on the base point. However, the conclusion is false if we instead took $n \leq 0$.)

3. Assume that X is a compact manifold and $\varphi : X \to X$ is an Anosov diffeomorphism. This exercise shows that the number of periodic points of φ grows at most exponentially. For $n \geq 1$, define the set of periodic points of period n

$$Z_n := \{x \in X \mid \varphi^n(x) = x\}.$$

Denote by d a metric on X.

(a) Show that there exist $c > 0$, $\Lambda > 1$ such that for all $n \geq 1$

$$x, y \in Z_n,\ d(x, y) \leq c\Lambda^{-n} \quad \Longrightarrow \quad x = y.$$

(Hint: you can take Λ to be the Lipschitz constant of φ with respect to the metric d and $c := \varepsilon_0$ be chosen in the Stable/Unstable Manifold Theorem; recall from that theorem that if the orbits of x, y stay ε_0-close to each other for all times then $x = y$.)

(b) Use part (a) to show that as $n \to \infty$

$$|Z_n| = \mathcal{O}(\Lambda^{dn}) \quad \text{where} \quad d := \dim X.$$

(Hint: the radius $\frac{1}{3}c\Lambda^{-n}$ balls centered at the points of Z_n are nonintersecting. Now count the volume.)

4. (Optional) This exercise studies a particular kind of hyperbolic fixed point, namely an attractive point in dimension 1, showing that near such a point, there are coordinates in which the map is linear. To simplify the setup, we construct a C^1 coordinate system, but this argument can give C^∞ coordinates. Put $I := (-1, 1)$ and assume

$$\varphi : I \to \varphi(I) \ \subset \ \mathbb{R}$$

is a C^∞ diffeomorphism such that

$$\varphi(0) = 0, \quad \varphi'(0) = a \in (0, 1). \tag{1}$$

Show that there exists an open interval $J \subset I$ containing 0 and a C^1 diffeomorphism

$$\psi : J \to \psi(J) \ \subset \ \mathbb{R}$$

such that $\psi(0) = 0$ and for all $x \in J$

$$\psi(\varphi(x)) = a \cdot \psi(x).$$

Follow the steps below:

(a) From (1) we can write

$$\varphi(x) = ax(1 + \widetilde{\varphi}(x)), \quad x \in I,$$

where $\widetilde{\varphi} \in C^\infty(I; \mathbb{R})$ satisfies $\widetilde{\varphi}(0) = 0$. (You don't need to prove this.) We put $J := (-\delta, \delta)$ where $\delta > 0$ will be chosen to be small later. Let $\mathcal{X}$ be the Banach space of C^1 functions g on $[-\delta, \delta]$ such that $g(0) = 0$, with the norm

$$\|g\|_{\mathcal{X}} := \sup_{|x| \leq \delta} |g'(x)|.$$

Define the linear operator $\Phi : \mathcal{X} \to \mathcal{X}$ by (here we use that $|\varphi(x)| \leq |x|$ for x small enough)

$$\Phi g(x) := (1 + \widetilde{\varphi}(x))g(\varphi(x)), \quad |x| \leq \delta.$$

Show that it suffices to prove that for δ small enough, the equation

$$g = \Phi g + \widetilde{\varphi} \tag{2}$$

has a solution $g \in \mathcal{X}$. (Hint: take $\psi(x) := x(1 + g(x))$.)

(b) Show that the equation (2) has a solution $g \in \mathcal{X}$ if δ is small enough, by proving that $\|\Phi\|_{\mathcal{X} \to \mathcal{X}} < a + C\delta$.