# 18.118, SPRING 2022, PROBLEM SET 3

Review/useful information:

- Lie group $\mathrm{SL}(2,\mathbb{R})$: consists of $2 \times 2$ real matrices with determinant 1.
- $\mathrm{PSL}(2,\mathbb{R})$: the quotient of $\mathrm{SL}(2,\mathbb{R})$ by the central subgroup $\{I, -I\}$ where $I$ denotes the identity matrix.
- Lie algebra $\mathfrak{sl}(2,\mathbb{R})$: consists of $2 \times 2$ real matrices with trace 0. Note that $\mathfrak{sl}(2,\mathbb{R})$ is the tangent space to $\mathrm{SL}(2,\mathbb{R})$ (and thus to $\mathrm{PSL}(2,\mathbb{R})$) at the identity. Also, if $\mathbf{a} \in \mathfrak{sl}(2,\mathbb{R})$, then the matrix exponential $\exp(\mathbf{a})$ lies in $\mathrm{SL}(2,\mathbb{R})$ and can also be viewed as an element of $\mathrm{PSL}(2,\mathbb{R})$.
- For $A \in \mathrm{PSL}(2,\mathbb{R})$, denote by $L_A : \mathrm{PSL}(2,\mathbb{R}) \to \mathrm{PSL}(2,\mathbb{R})$ the left multiplication by $A$: $L_A(B) = AB$.
- For $\mathbf{a} \in \mathfrak{sl}(2,\mathbb{R})$, define the vector field $Z_{\mathbf{a}}$ on $\mathrm{PSL}(2,\mathbb{R})$ by

$$Z_{\mathbf{a}}(A) = dL_A(I)\mathbf{a} \ \in \ T_A \,\mathrm{PSL}(2,\mathbb{R}).$$

  Note that $Z_{\mathbf{a}}$ is *left-invariant*, that is

$$dL_A(B) Z_{\mathbf{a}}(B) = Z_{\mathbf{a}}(L_A(B)).$$

- The flow $e^{tZ_{\mathbf{a}}} : \mathrm{PSL}(2,\mathbb{R}) \to \mathrm{PSL}(2,\mathbb{R})$ is the right multiplication by matrix exponential:

$$e^{tZ_{\mathbf{a}}}(A) = A \exp(t\mathbf{a}) \quad \text{for all} \quad A \in \mathrm{PSL}(2,\mathbb{R}), \quad \mathbf{a} \in \mathfrak{sl}(2,\mathbb{R}). \tag{1}$$

- Pushforward of left-invariant vector fields by the flows of left-invariant vector fields: if $\mathbf{a}, \mathbf{b} \in \mathfrak{sl}(2,\mathbb{R})$ and $A \in \mathrm{PSL}(2,\mathbb{R})$ then

$$de^{tZ_{\mathbf{a}}}(A) Z_{\mathbf{b}}(A) = Z_{\mathbf{c}}(e^{tZ_{\mathbf{a}}}(A)) \quad \text{where} \quad \mathbf{c} = \exp(-t\mathbf{a})\mathbf{b}\exp(t\mathbf{a}) \in \mathfrak{sl}(2,\mathbb{R}). \tag{2}$$

- We have the commutation identity

$$[Z_{\mathbf{a}}, Z_{\mathbf{b}}] = Z_{[\mathbf{a},\mathbf{b}]} \quad \text{for all} \quad \mathbf{a}, \mathbf{b} \in \mathfrak{sl}(2,\mathbb{R}) \tag{3}$$

  where the left-hand side is the Lie bracket of the vector fields $Z_{\mathbf{a}}, Z_{\mathbf{b}}$ on $\mathrm{PSL}(2,\mathbb{R})$ and the right-hand side features the commutator $[\mathbf{a}, \mathbf{b}] = \mathbf{ab} - \mathbf{ba}$ of the matrices $\mathbf{a}, \mathbf{b}$.
- Poincaré upper half-plane model for the hyperbolic plane:

$$\mathbb{H}^2 = \{z \in \mathbb{C} \mid \operatorname{Im} z > 0\}, \quad g = \frac{|dz|^2}{(\operatorname{Im} z)^2}.$$

- Action of $\mathrm{SL}(2,\mathbb{R})$ on $\mathbb{H}^2$:
$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2,\mathbb{R}) \quad \Longrightarrow \quad \gamma_A(z) = \frac{az+b}{cz+d}.$$
Acts on the left (i.e. $\gamma_{AB} = \gamma_A \circ \gamma_B$) by isometries and descends to $\mathrm{PSL}(2,\mathbb{R})$. Also defines the natural (left) action on the sphere bundle $S\mathbb{H}^2$:
$$(z,v) \in S\mathbb{H}^2 \quad \Longrightarrow \quad \widetilde{\gamma}_A(z,v) = (\gamma_A(z), \gamma_A'(z)v)$$
where $\gamma_A'(z) \in \mathbb{C}$ is the derivative of $\gamma_A(z)$ as a complex analytic function of $z$, which can be computed to be
$$\gamma_A'(z) = \frac{1}{(cz+d)^2}.$$

In this problemset we express the vector fields $V, W, V_\perp, U_+, U_-$ that we introduced on $S\mathbb{H}^2$ in terms of the Lie algebra of the Lie group
$$G := \mathrm{PSL}(2,\mathbb{R}).$$
(If $M = \Gamma\backslash\mathbb{H}^2$ is a hyperbolic surface, then one can show that $SM \simeq \Gamma\backslash G$ is the set of left $\Gamma$-cosets on $G$. Since the vector fields defined below are left-invariant, they descend to $SM$. This gives a Lie algebraic way of thinking about the geodesic and horocyclic flows on hyperbolic surfaces.)

**1.** The point $(z,v) = (i,i)$ lies in $S\mathbb{H}^2$. Show that the map
$$\Phi : G \to S\mathbb{H}^2, \quad \Phi(A) = \widetilde{\gamma}_A(i,i)$$
is a diffeomorphism. (Hint: injectivity can be checked directly. Bijectivity of the differential can be reduced to the case $A = I$ because we have a group action. For surjectivity, we need to show that the group $G$ acts transitively on $S\mathbb{H}^2$: to do this we can first show that any $(z,v) \in S\mathbb{H}^2$ can be mapped to $(i,w)$ for some $w \in \mathbb{C}$, $|w| = 1$, and then map $(i,w)$ to $(i,i)$ by another element of $S\mathbb{H}^2$.)

**2.** Let $V, W$ be the vector fields on $S\mathbb{H}^2$ such that $e^{tV}$ is the geodesic flow and $e^{sW}(z,v) = (z, R_s v)$ where $R_s : T_z\mathbb{H}^2 \to T_z\mathbb{H}^2$ is the counterclockwise rotation by angle $s$; if we think of $v$ as an element of $\mathbb{C} = T_z\mathbb{H}^2$ then $R_s v = e^{is}v$.

**(a)** Explain why the maps $e^{tV}$ and $e^{sW}$ commute with the map $\widetilde{\gamma}_A$ for any $A \in G$. Show that the pullbacks $\Phi^*V$, $\Phi^*W$ are left-invariant vector fields on $G$ and conclude that
$$\Phi^*V = Z_{\mathbf{v}}, \quad \Phi^*W = Z_{\mathbf{w}} \quad \text{for some} \quad \mathbf{v}, \mathbf{w} \in \mathfrak{sl}(2,\mathbb{R}).$$

**(b)** Recall the vector fields $V_\perp := [V,W]$, $U_+ := V_\perp + W$, $U_- := V_\perp - W$ on $S\mathbb{H}^2$. Show that
$$\Phi^*V = Z_{\mathbf{v}}, \quad \Phi^*W = Z_{\mathbf{w}}, \quad \Phi^*V_\perp = Z_{\mathbf{v}_\perp}, \quad \Phi^*U_\pm = Z_{\mathbf{u}_\pm} \tag{4}$$

for the following matrices in $\mathfrak{sl}(2,\mathbb{R})$:

$$\mathbf{v}=\begin{pmatrix}\frac12 & 0\\ 0 & -\frac12\end{pmatrix},\quad \mathbf{w}=\begin{pmatrix}0 & \frac12\\ -\frac12 & 0\end{pmatrix},\quad \mathbf{v}_\perp=\begin{pmatrix}0 & \frac12\\ \frac12 & 0\end{pmatrix},\quad \mathbf{u}_+=\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix},\quad \mathbf{u}_-=\begin{pmatrix}0 & 0\\ 1 & 0\end{pmatrix}.$$

(Hint: for $V$ and $W$, use part (a) and compute $V$ and $W$ at the point $(i,i)$ to find $\mathbf{v}=(\Phi^*V)(I)$, $\mathbf{w}=(\Phi^*W)(I)$. You can use the fact that $\delta(t)=e^t i$ is a geodesic on $\mathbb{H}^2$. For $V_\perp, U_\pm$ use their definitions and (3).)

**3.** (Optional) Assume that $a,b,a',b',t\in\mathbb{R}$ satisfy

$$ab=e^{-t/2}-1,\quad a'=e^{-t/2}a,\quad b'=e^{t/2}b.$$

Check that

$$\exp(b\mathbf{u}_-)\exp(a\mathbf{u}_+)=\exp(t\mathbf{v})\exp(a'\mathbf{u}_+)\exp(b'\mathbf{u}_-). \tag{5}$$

Using (4) and (1), show the following commutation identity for the geodesic and horocycle flows on $S\mathbb{H}^2$:

$$e^{aU_+}\circ e^{bU_-}=e^{b'U_-}\circ e^{a'U_+}\circ e^{tV}. \tag{6}$$

**4.** (Optional) Define the differential 3-form $\omega$ on $S\mathbb{H}^2$ uniquely by the condition

$$\omega(V,U_+,U_-)=1\quad\text{on the entire } S\mathbb{H}^2.$$

Show that it is invariant under the flows $e^{tV}$ and $e^{sU_+}$, namely $(e^{tV})^*\omega=(e^{sU_+})^*\omega=\omega$. (Hint: you can use the definition of pullback of a differential form together with Exercise 2 and the pushforward formula (2).) (One can similarly show invariance under $e^{sU_-}$. The volume form $\omega$ gives the Liouville measure and this exercise shows that the Liouville measure is invariant under the geodesic flow and the horocycle flows.)