One-sided inverses

These notes are a small extension of the material on pages 53–55 of the text.

Definition 1. *Suppose V and W are vector spaces over a field F, and $T \in \mathcal{L}(V,W)$. A left inverse for T is a linear map $S \in \mathcal{L}(W,V)$ with the property that $ST = I_V$ (the identity map on V). That is, we require*

$$ST(v) = v \qquad (\text{all } v \in V).$$

A right inverse for T is a linear map $S' \in \mathcal{L}(W,V)$ with the property that $TS' = I_W$ (the identity map on W). That is, we require

$$TS'(w) = w \qquad (\text{all } w \in W).$$

What are these things good for? I've said that one of the most basic problems in linear algebra is solving an equation like

$$Tx = c \qquad (QUESTION)$$

(with $c \in W$ specified); you are to find the unknown $x \in V$. If S is a left inverse of T, then we can apply S to this equation and get

$$x = I_V(x) = STx = Sc. \qquad (LEFT)$$

What this calculation proves is

Proposition 2. *Suppose S is a left inverse of T. Then the only possible solution of (QUESTION) is $x = Sc$.*

This does not say that Sc really *is* a solution; just that it's the only candidate for a solution. Sometimes that's useful information.

On the other hand, suppose S' is a right inverse of T. Then we can try $x = S'c$ and get

$$Tx = TS'c = I_W c = c. \qquad (RIGHT)$$

This calculation proves

Proposition 3. *Suppose S' is a right inverse of T. Then $x = S'c$ is a solution of (QUESTION).*

This time the ambiguity is uniqueness: we have found one solution, but there may be others. Sometimes that's all we need.

Example. *Suppose $V = W = \mathcal{P}(\mathbb{R})$ (polynomials), and $D = \frac{d}{dx}$. We would like to "undo" differentiation, so we integrate:*

$$(Jp)(x) = \int_0^x p(t)\,dt.$$

The fundamental theorem of calculus says that the derivative of this integral is p; that is, $DJ = I_{\mathcal{P}}$. So J is a right inverse of D; it provides a solution (not the only one!) of the differential equation $\frac{dq}{dx} = p$. If we try things in the other direction, there is a problem:

$$JD(p) = \int_0^x p'(t)\,dt = p(x) - p(0).$$

That is, JD sends p to $p - p(0)$, which is not the same as p. So J is not a left inverse to D; since D has a nonzero null space, we'll see that no left inverse can exist.

Theorem 4. *Suppose V and W are finite-dimensional, and that $T \in \mathcal{L}(V,W)$.*
1) *The operator T has a left inverse if and only if* $\mathrm{Null}(T) = 0$*.*
2) *If S is a left inverse of T, then* $\mathrm{Null}(S)$ *is a complement to* $\mathrm{Range}(T)$ *in the sense of Proposition 2.13 in the text:*

$$W = \mathrm{Range}(T) \oplus \mathrm{Null}(S).$$

3) *Assuming that* $\mathrm{Null}(T) = 0$*, there is a one-to-correspondence between left inverses of T and subspaces of W complementary to* $\mathrm{Range}(T)$*.*
4) *The operator T has a right inverse if and only if* $\mathrm{Range}(T) = W$*.*
5) *If S' is a right inverse of T, then* $\mathrm{Range}(S')$ *is a complement to* $\mathrm{Null}(T)$ *in the sense of Proposition 2.13 in the text:*

$$V = \mathrm{Null}(T) \oplus \mathrm{Range}(S').$$

6) *Assuming that* $\mathrm{Range}(T) = W$*, there is a one-to-correspondence between right inverses of T and subspaces of V complementary to* $\mathrm{Null}(T)$*.*
7) *If T has both a left and a right inverse, then the left and right inverses are unique and equal to each other. That, is there is a unique linear map $S \in \mathcal{L}(W,V)$ characterized by either of the two properties $ST = I_V$ or $TS = I_W$. If it has one of these properties, then it automatically has the other.*

The theorem is also true exactly as stated for possibly infinite-dimensional V and W, but the proof requires a little more cleverness.

Proof. For (1), suppose first that a left inverse exists. According to Proposition 2, the equation $Tx = 0$ has at most one solution, namely $x = S0 = 0$. That says precisely that $\mathrm{Null}(T) = 0$. Conversely, suppose $\mathrm{Null}(T) = 0$. Choose a basis $(v_1, \ldots, v_n)$ of V. By the proof of the rank plus nullity theorem, $(Tv_1, \ldots, Tv_n)$ is a basis of $\mathrm{Range}(T)$; so in particular it is a linearly independent set in W. We may therefore extend it to a basis

$$(Tv_1, \ldots, Tv_n, w_1, \ldots w_p)$$

of W.

To define a linear map S from W to V, we need to pick the images of these $n+p$ basis vectors; we are allowed to pick any vectors in V. If S is going to be a left inverse of T, we are *forced* to choose

$$S(Tv_i) = v_i;$$

the choices of Sw_j can be arbitrary. Since we have then arranged for the equation $STv = v$ to be true for all elements of a basis of V, it must be true for all of V. Therefore S is a left inverse of T.

For (2), suppose $ST = I_V$; we need to prove the direct sum decomposition shown. So suppose $w \in W$. Define $v = Sw$ and $r = Tv = TSw \in W$. Then $r \in \mathrm{Range}(T)$, and

$$n = w - r = w - TSw$$

satisfies

$$Sn = Sw - STSw = Sw - I_V Sw = Sw - Sw = 0;$$

so $n \in \text{Null}(S)$. We have therefore written $w = r + n$ as the sum of an element of $\text{Range}(T)$ and of $\text{Null}(S)$. To prove that the sum is direct, we must show that $\text{Null}(S) \cap \text{Range}(T) = 0$. So suppose Tv (in $\text{Range}(T)$) is also in $\text{Null}(S)$. Then

$$v = STv = 0$$

(since $Tv \in \text{Null}(S)$) so also $Tv = 0$, as we wished to show.

For (3), we have seen that any left inverse gives a direct sum decomposition of W. Conversely, suppose that $W = \text{Range}(T) \oplus N$ is a direct sum decomposition. Define a linear map S from W to V by

$$S(Tv + n) = v \qquad (v \in V, n \in N).$$

This formula makes sense because there is only one v with image Tv (by $\text{Null}(T) = 0$); it defines S on all of W by the direct sum hypothesis. This construction makes a left inverse S with $\text{Null}(S) = N$, and in fact it is the *only* way to make a left inverse with this null space.

Parts (4)–(6) are proved in exactly the same way.

For (7), if the left and right inverses exist, then $\text{Null}(T) = 0$ and $\text{Range}(T) = W$. So the only possible complement to $\text{Range}(T)$ is 0, so the left inverse S is unique by (3); and the only possible complement to $\text{Null}(T)$ is V, so the right inverse is unique by (6). To see that they are equal, apply S' on the right to the equation $ST = I_V$; we get

$$S' = I_V S' = STS' = SI_W = S,$$

so the left and right inverses are equal.