**Interesting bases**

These notes concern some particularly interesting or useful bases of $\mathbb{R}^n$ and $\mathbb{C}^n$. The *standard basis* is

$$(e_1, e_2, \dots, e_n), \quad e_i = (0, \dots, 0, 1, 0, \dots, 0) \quad (1 \text{ in the } i\text{th place}).$$

One advantage of the standard basis is that it's easy to write down a vector in the standard basis:

$$(a_1, \dots, a_n) = \sum_{i=1}^{n} a_i e_i.$$

It's also clear exactly what each coefficient means: the coefficient of $e_i$ is the $i$th coordinate.

One reason that this basis might not be a good choice is if the values of individual coordinates are not so interesting. If each coordinate is the brightness of some pixel in an image, then no individual coordinate is interesting by itself. You might care about something like the sum of all the coordinates (which measures the brightness of the whole image), or something like

$$a_1 - a_2 + a_3 - \cdots - (-1)^n a_n,$$

which is a measure of how much the pixels are jumping around: how jagged the image is. So it's natural to look for other bases, in which the coefficients offer some of this other information.

A good example is the basis

$$v_1 = (1, 1), \ v_2 = (1, -1)$$

of $\mathbb{R}^2$. The expression of a vector in this basis is

$$(a_1, a_2) = \frac{a_1 + a_2}{2} v_1 + \frac{a_1 - a_2}{2} v_2.$$

The first coefficient is the average value of the coordinates of the vector, and the second measures how much they are jumping around.

Here is a version of this basis for $\mathbb{R}^4$:

$$v_1 = (1, 1, 1, 1), \quad v_2 = (1, -1, 1, -1), \quad v_3 = (1, 1, -1, -1), \quad v_4 = (1, -1, -1, 1).$$

The way to express any vector in this basis is

$$\begin{aligned}(a_1, a_2, a_3, a_4) = &\frac{a_1 + a_2 + a_3 + a_4}{4} v_1 + \frac{a_1 - a_2 + a_3 - a_4}{4} v_2 \\ &+ \frac{a_1 + a_2 - a_3 - a_4}{4} v_3 + \frac{a_1 - a_2 - a_3 + a_4}{4} v_4.\end{aligned}$$

The first coefficient is the average value, and the last three are different versions of "jaggedness."

In order to generalize this example to a basis of $\mathbb{R}^{2^n}$, it's easier to number the coordinates $0, 1, 2 \dots, 2^n - 1$, and to number the new basis vectors in the same way. If we do that, and write the indices in base 2, here is what the case of $\mathbb{R}^4$ looks like.

| | 00 | 01 | 10 | 11 |
|---|---|---|---|---|
| $v_{00}$ | 1 | 1 | 1 | 1 |
| $v_{01}$ | 1 | $-1$ | 1 | $-1$ |
| $v_{10}$ | 1 | 1 | $-1$ | $-1$ |
| $v_{11}$ | 1 | $-1$ | $-1$ | 1 |

**Basis in** $\mathbb{R}^4$

Here's a way to say this in $\mathbb{R}^{2^n}$.

**Definition 1.** *Fix an integer $n \geq 0$. Write the numbers from 0 to $2^n - 1$ in base two as strings of zeros and ones:*

$$\epsilon = \epsilon_{n-1}\epsilon_{n-2}\cdots\epsilon_1\epsilon_0 \longleftrightarrow \sum_{j=0}^{n-1} \epsilon_j 2^j \quad (\epsilon_j \in \{0,1\}).$$

*Label the coordinates of $\mathbb{R}^{2^n}$ by these numbers $\epsilon$. We define a new basis $v^n_\delta$ (indexed by these same binary numbers $\delta$) by*

$$(v^n_\delta)_\epsilon = ((-1)^{\sum_{j=0}^n \epsilon_j\delta_j}).$$

*That is, the zeroth coordinate of $v_\delta$ is always 1; the one-th coordinate ($\epsilon = 0\cdots01$) of $v_\delta$ is $(-1)^{\delta_0}$ (alternating $+1$ and $-1$); and so on.*

*Another way to describe these vectors is inductively: if we write*

$$\epsilon' = \epsilon_{n-2}\cdots\epsilon_1\epsilon_0$$

*for the last $n-1$ bits of $\epsilon$, then*

$$v^n_\epsilon = (v^{n-1}_{\epsilon'}, (-1)^{\epsilon_{n-1}} v^{n-1}_{\epsilon'}).$$

We could write the inductive definition without resorting to base 2 and the renumbering of everything from 0 to $2^n - 1$. In terms of the ordinary labeling of the coordinates, it looks like

$$v^n_i = \begin{cases} (v^{n-1}_i \quad , \quad v^{n-1}_i) & 1 \leq i \leq 2^{n-1} \\ (v^{n-1}_{i-2^{n-1}}, -v^{n-1}_{i-2^{n-1}}) & 2^{n-1} < i \leq 2^n. \end{cases}$$

You should look at how the inductive description works in the example of $2^n = 4$ above. In the first two rows of the table, the first two entries are the same as the last two entries; and in the last two rows, there is a sign change.

It turns out that the basis $(v^n_\delta \mid 0 \leq \delta < 2^n)$ is *orthogonal*, a notion we'll look at more in about six weeks. In any case it makes writing any vector in terms of the basis very easy:

$$v = \sum_{i=1}^{2^n} \frac{v \cdot v^n_i}{2^n} v^n_i.$$

Here the $\cdot$ means the inner product on $\mathbb{R}^N$:

$$v \cdot w = \sum_{j=1}^{N} v_j w_j \qquad (v, w \in \mathbb{R}^N).$$