18.700 Problem Set 7

Due in class **Wednesday** November 13; late work will not be accepted. Your work on graded problem sets should be written entirely on your own, although you may consult others before writing.

1. (6 points) Suppose that V is a complex inner product space with orthogonal basis $(f_1, \dots, f_n)$, and $T \in \mathcal{L}(V)$.

a) Prove that any vector $v \in V$ can be written

$$v = \sum_{i=1}^{n} \frac{\langle v, f_i \rangle}{\langle f_i, f_i \rangle} f_i.$$

b) Find a formula involving inner products for the (i, j) entry a_{ij} of the matrix A of T in the basis $(f_1, \dots, f_n)$.

c) Find a formula involving inner products for the (i, j) entry b_{ij} of the matrix B of T^* in the basis $(f_1, \dots, f_n)$.

d) How do you pass from the matrix A to the matrix B?

2. (5 points) Suppose $V = \mathbb{R}^2$. The identity operator I is a positive selfadjoint operator, so according to Proposition 7.36 I has a unique positive square root (namely I). Find *all* selfadjoint square roots of I.

3. (6 points) Suppose V is a two-dimensional vector space over a field with q elements. How many square roots does the identity operator have? (Notice that "selfadjoint" doesn't appear in this problem.)

4. (2 points) Prove that there does not exist a selfadjoint operator $T \in \mathcal{L}(\mathbb{R}^3)$ such that $T(1, 2, 3) = 0$, $T(2, 5, 7) = (2, 5, 7)$.

5. (6 points)

a) Find a real number a so that there exists a selfadjoint operator $T \in \mathcal{L}(\mathbb{R}^3)$ with the properties $T(1, 2, 3) = (0, 0, 0)$, $T(2, 5, a) = (2, 5, a)$.

b) Find an eigenvector of T that is not in $\operatorname{Span}((1, 2, 3), (2, 5, a))$.

Hint: part (b) sounds like you're required to write down an operator T. You can do that, but you can also find the eigenvector without finding T.