18.700 Problem Set 5

Due in class Monday October 21; late work will not be accepted. Your work on graded problem sets should be written entirely on your own, although you may consult others before writing.

1. (6 points) Suppose we are given

$$\text{three } \textit{distinct} \text{ elements } x_1,\ x_2, \text{ and } x_3 \text{ in } F; \tag{1}$$

and

$$\text{three } \textit{arbitrary} \text{ elements } a,\ b, \text{ and } c \text{ in } F; \tag{2}$$

The problem is to find all polynomials

$$p(x) = u_0 + u_1 x + u_2 x^2 + u_3 x^3 \tag{3}$$

of degree less than or equal to three satisfying the conditions

$$p(x_1) = a, \qquad p(x_2) = b, \qquad p(x_3) = c. \tag{4}$$

a) The conditions (4) on p can be written as a system of three simultaneous linear equations in four unknowns. Write the augmented matrix of this system of equations.

b) Perform elementary row operations to bring this augmented matrix to reduced row-echelon form. (This is a bit disconcerting, because some of the entries of the matrix are not "numbers" like 7, but symbols for numbers, like x_2. You know from algebra how to add, subtract, and multiply such symbols. What requires care is dividing: before you divide by something like x_2, you need to explain why it is not zero, or else worry separately about the case when it *is* zero. But you should be able to manage.

c) Write all the polynomials of degree less than or equal to three satisfying the condition (4).

2. (8 points) This problem is about the sequence of integers defined by

$$a_0 = 0, \quad a_1 = 1, \quad a_{n+1} = a_n + 2a_{n-1} \qquad (n \geq 1).$$

(This is like the formula for Fibonacci numbers, but more complicated because I'm mean.) Explicitly,

$$\begin{aligned} &a_0 = 0, && a_1 = 1, && a_2 = 1 + 2 \cdot 0 = 1, \\ &a_3 = 1 + 2 \cdot 1 = 3 && a_4 = 3 + 2 \cdot 1 = 5, && a_5 = 5 + 2 \cdot 3 = 11, \dots \end{aligned}$$

This sequence is related to the matrix $A = \begin{pmatrix} 1 & 2 \\ 1 & 0 \end{pmatrix}$: the defining condition can be written $\begin{pmatrix} a_{n+1} \\ a_n \end{pmatrix} = A \begin{pmatrix} a_n \\ a_{n-1} \end{pmatrix}$.

a) Find all eigenvalues and eigenvectors of A.

b) Write $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ as a linear combination of eigenvectors of A.

c) Write a formula for $A^n \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ not using the matrix A. (A good answer is a vector of formulas depending on n: something like $\begin{pmatrix} 2n+1 \\ n^2 \end{pmatrix}$.)

d) Write a formula for a_n as a function of n.

3. (6 points) This problem is a generalization of (5.30) in the text, and can be proved in a similar way.

Suppose F is any field, and suppose $n = p + q$ with p and q positive integers. Suppose that A is a $p \times p$ matrix, B is a $p \times q$ matrix, and D is a $q \times q$ matrix. The $n \times n$ matrix

$$T = \begin{pmatrix} A & B \\ 0 & D \end{pmatrix}$$

is called *block upper triangular*; here 0 means the $q \times p$ matrix of zeros.

a) Show that T is invertible if and only if both A and D are invertible.

b) Show that λ is an eigenvalues of T if and only if *either* λ is an eigenvalue of A *or* λ is an eigenvalue of D (or both).

c) Give an example of a 4×4 real matrix T so that T has no (real) eigenvalues.