

### 18.700 Problem Set 4

Due in class WEDNESDAY October 16; late work will not be accepted. Your work on graded problem sets should be written entirely on your own, although you may consult others before writing.

For the first problems, you may use the theorem I stated in class Monday October 7: there is a one-to-one correspondence

$$U \leftrightarrow \text{Row}(A)$$

between  $r$ -dimensional subspaces of  $F^n$  and  $r \times n$  reduced row-echelon matrices having exactly one pivot in each row. (That is, no rows of  $A$  are zero.)

1. (4 points) Suppose  $F = \mathbb{F}_5$  is the field with five elements. How many one-dimensional subspaces of  $F^4$  are there?

2. (4 points) Suppose  $F = \mathbb{F}_q$  is the field with  $q$  elements. How many two-dimensional subspaces of  $F^4$  are there? (The answer will be a formula depending on  $q$ , something like  $e^{2q} - \sin(q) + 7$ .)

3. (6 points) Still assume  $F = \mathbb{F}_q$ .

a) Suppose  $r \leq n$ . Explain why the number of  $r$ -element linear independent lists  $(v_1, \dots, v_r)$  in  $F^n$  is equal to

$$(q^n - 1)(q^n - q)(q^n - q^2) \cdots (q^n - q^{r-1}).$$

b) How many invertible  $n \times n$  matrices with entries in  $F$  are there? (Hint: parts (a) and (b) have something to do with each other.)

c) Suppose  $A$  is a very large random square matrix with entries in the field  $\mathbb{F}_2$ . Is there a 30% chance that  $A$  is invertible?

4. (3 points) Axler, page 139, exercise 8; but  $F$  is allowed to be any field, and you need to see whether the answer depends on  $F$ .

5. (3 points) Axler, page 140, exercise 19, with the same warning.

6. (5 points) Axler, page 141, exercise 23.