Geometric Quaternionic Quantum Mechanics

Gerald Xu

Research Science Institute

Under the Direction of Ethan Sussman
MIT

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Abstract

In quantum mechanics, the symmetries of a physical system are closely related to the conservation laws within that system. As a result, a mathematical understanding of a system's symmetries allows us to accurately model and describe that system, and therefore we want to find methods of mathematically representing these symmetries. Past works by Eugene Wigner and Daniel Freed have proved methods of representing symmetries as complex matrices and complex projective spaces, respectively. This project closely follows the work of Daniel Freed, but with respect to quaternionic projective spaces. It is known that symmetries in physical systems preserve empirical quantities. In both Freed's and our project, we are interested in the symmetries that preserve transition probabilities, which are the probabilities that a state of the system would evolve to other states.

Our work focuses on quaternionic projective spaces, and proving the quaternionic analog to Freed's work. We give the quaternionic analog to the Fubini-Study metric on complex projective space, as well as describe its associated distance function. We prove the relation between this distance function and transition probability, demonstrating that symmetries in quantum mechanics can be represented by isometries in quaternionic projective space.

Summary

In quantum mechanics we study physical systems, which are simply a portion of the physical universe we choose to analyse. In our studies, we frequently find that certain properties within a system are preserved even under some transformations, which change the state of a physical system. Naturally, these transformations that act on a physical system, but preserves certain properties, are closely related to conservation laws, which describe a measurable property that remains unchanged despite a change in the system's states. Therefore, to help us solve problems within, and create models for physical systems, we want to mathematically describe these property-preserving transformations that act on the system.

This paper presents a method of representing these property-preserving transformations, known as symmetries, using the concept of quaternionic projective spaces in mathematics. Roughly speaking, a quaternionic projective space is an object with points on its surface that represent lines. In this paper, we show that functions between points on a quaternionic projective space that preserve distance can be used to represent the property-preserving symmetries in a physical system.

1 Introduction

In quantum mechanics, the study of symmetries is useful to gain a better understandings of physical systems. Symmetries are transformations that act on a physical system while preserving the time-evolution of the system. The features invariant under these transformations are closely related to the quantities we observe as being conserved in a physical system. Problem solving within a physical system frequently relies on conservation laws, and therefore is dependent on the study of symmetries [1]. To assist us in generating models that accurately represent a physical system, we look for ways to describe symmetries mathematically.

In 1931, Eugene Wigner [2] published his major contribution: a method to mathematically represent symmetries. Wigner proved a relation between the symmetries in quantum mechanics and complex matrices in mathematics, known as Wigner's theorem. Following Wigner's work, in 2012, Daniel Freed published an indirect proof of Wigner's theorem, in which he provided a mathematical representation of symmetries through complex projective spaces [3].

In their proofs, both Wigner and Freed provided a method of describing symmetries of physical systems using complex numbers. Their work can be generalised to the quaternions, a number system that extends the complex numbers, and Valentine Bargamnn proved the quaternionic analog to Wigner's theorem in 1964 [4]. To our best knowledge, an analog to Freed's corollary, Corollary 1, has not been proven for the quaternions. This project aims to demonstrate that symmetries in a physical system can be represented using quaternionic projective space. The proof in this paper follows a similar argument in Freed's paper. The Fubini-Study metric [5, 6] defines distance on complex projective space, and in this project we give the quaternionic analog to the Fubini-Study metric, through defining a map from quaternionic projective space to a sphere, and inducing the Fubini-Study metric from the sphere metric. Using the definition of distance on quaternionic projective space that follows from that metric, we prove a relation between distance and transition probability, a quantity preserved by symmetries. This relation allows us to show that symmetries preserving transition probability are equivalent to functions on quaternionic projective space that preserve distance, or the isometries on quaternionic projective space.

2 Past Works

In this section, we recall the separate contributions of Eugene Wigner and Daniel Freed in establishing methods of describing quantum symmetries.

Definition 1 (Rays)**.** The rays of a complex Hilbert space $\mathcal{H}$ are the equivalence classes of vectors $v \in \mathcal{H}$ with $v \neq 0$ and the relation $\sim$ given by $v \sim w$ when $v = \lambda w$ for nonzero $\lambda \in \mathbb{C}$.

Definition 2 (Symmetry transformation)**.** A ray transformation $T : R \to R'$ is a symmetry transformation if

$$T\underline{\Psi} \cdot T\underline{\Phi} = \underline{\Psi} \cdot \underline{\Phi}, \quad \forall \underline{\Psi}, \underline{\Phi} \in \mathcal{H}$$

where $\Psi \cdot \underline{\Phi}$ is the norm of their inner product in $\mathcal{H}$, and preserves transition probabilities.

The following theorem due to Wigner [2], establishes the correspondence between symmetries and unitary and antiunitary transformations.

Theorem 1 (E. Wigner, 1931)**.** *Let $\mathcal{H}$ and $\mathcal{K}$ be Hilbert spaces and $\underline{\Psi}$ denote a ray in Hilbert space. If $T : \underline{\Psi} \subset \mathcal{H} \mapsto T(\underline{\Psi}) \subset \mathcal{K}$ is a symmetry transformation, there exists a unitary or antiunitary transformation $V : \mathcal{H} \to \mathcal{K}$ which is compatible with T if* $\dim \mathcal{H} \geq 2$.

Definition 3 (Compatible)**.** We say that a transformation V of Hilbert space is compatible with a transformation T of ray space if $\forall \underline{\Psi}$, $V\underline{\Psi} \in T\underline{\Psi}$.

Definition 4 ($\mathbb{P}\mathcal{H}$)**.** A complex projective space of dimension n, denoted by $\mathbb{CP}^n$, is the set of vector lines in the complex vector space $\mathbb{C}^{n+1}$. $\mathbb{P}\mathcal{H}$ is defined as a finite or infinite dimensional complex projective space.

Definition 5 (Fubini-Study metric)**.** Let a point $\mathcal{Z}$ on $\mathbb{P}\mathcal{H}$ be described by homogeneous coordinates $[1 : z_1 \cdots : z_n]$ for $z_i \in \mathbb{C}$ and $z_0 \neq 0$. The Fubini-Study metric [5, 6], describing the length of an infinitesimal line segment in some z_i and z_j direction, is given by

$$ds^2 = g_{i\bar{j}} dz^i d\bar{z}^j$$

for integers $1 \leq i, j \leq n$, and $g_{i\bar{j}}$ defined as follows for $i = j$ and $i \neq j$

$$g_{i\bar{i}} = \frac{1 + |\mathcal{Z}|^2 - |z_i|^2}{(1 + |\mathcal{Z}|^2)^2}, \qquad g_{i\bar{j}} = \frac{-\bar{z}_i z_j}{(1 + |\mathcal{Z}|^2)^2}.$$

Definition 6 (Fubini-Study distance function)**.** We define the Fubini-Study distance function $d \colon \mathbb{P}\mathcal{H} \times \mathbb{P}\mathcal{H} \to \mathbb{R}$ as the distance function associated with the Fubini-Study metric. The Fubini-Study distance function gives the minimum length of all curves connecting two points in $\mathbb{P}\mathcal{H}$, where curve lengths are determined by the Fubini-Study metric.

We also recall the following corollary due to Freed [3], relating the group of symmetries, $\mathrm{Aut}_{\mathrm{qtm}}(\mathbb{P}\mathcal{H})$, with the group of distance-preserving bijections in finite or infinite dimensional complex projective space.

Corollary 1 (D. Freed, 2012)**.** $\mathrm{Aut}_{\mathrm{qtm}}(\mathbb{P}\mathcal{H})$ *is the group of isometries of* $\mathbb{P}\mathcal{H}$ *with the Fubini-Study distance function.*

3 Preliminaries

Definition 7 (Quaternions)**.** The set of quaternions is a real division algebra that contains the complex numbers as a subring. A quaternion q can be expressed in the form $a+bi+cj+dk$, where $a, b, c, d \in \mathbb{R}$ and i, j, k are quaternion units with multiplication between them defined in the chart below (Figure 1). The conjugate of q is denoted by $\bar{q}$ and given by $a-bi-cj-dk$.

×	1	i	j	k
1	1	i	j	k
i	i	-1	k	$-j$
j	j	$-k$	-1	i
k	k	j	$-i$	-1

Figure 1: Quaternionic multiplication [7]

We denote the group of quaternions by $\mathbb{H}$, and the quaternionic vector space of $\dim n$ by $\mathbb{H}^n$. The quaternionic projective space of $\dim n$, denoted by $\mathbb{HP}^n$, is defined as the set of vector lines in $\mathbb{H}^{n+1}$ following the description of $\mathbb{CP}^n$ in definition 4.

Definition 8 (Homogeneous Coordinates)**.** The homogeneous coordinates are a coordinate system on $\mathbb{HP}^n$, such that $[q_0 : q_1 : \cdots : q_n] \in \mathbb{HP}^n$ represents all vectors $\lambda(q_0, q_1, \cdots, q_n) \in \mathbb{H}^{n+1}$ for nonzero $\lambda \in \mathbb{H}$.

Definition 9 (Quaternionic Inner Product)**.** The quaternionic inner product $\langle \cdot, \cdot \rangle$ is given by

$$\langle q_1, q_2 \rangle = q_1 \bar{q}_2, \quad q_1, q_2 \in \mathbb{H}.$$

We see that the quaternionic inner product satisfies the relation $\langle q_i, q_i \rangle = |q_i|^2$, where $|q_i|$ denotes the Euclidean norm of q_i.

Definition 10 ($SU(2)$)**.** The special unitary group of degree 2, $SU(2)$, is the group of 2×2 complex unitary matrices with determinant 1.

A well known property of the unit quaternions and $SU(2)$ is given in the following proposition.

Proposition 2. *The set of unit quaternions* $\lambda \in \mathbb{H}$ *satisfying* $|\lambda| = 1$ *is isomorphic to the special unitary group* $SU(2)$.

4 Fubini-Study Metric for Quaternions

In the case of studying complex projective space $\mathbb{CP}^n$, we find that the Fubini-Study metric is the optimal metric, as the resulting shape of $\mathbb{CP}^n$ is symmetric. More specifically, the Fubini-Study metric is, up to proportionality or scaling, the unique Riemannian metric on $\mathbb{CP}^n$ that is invariant under an action in the unitary group $U(n+1)$.

Definition 11 ($U(n)$)**.** The group of $n \times n$ unitary matrices, denoted by $U(n)$, is the group of $n \times n$ complex matrices satisfying the property

$$u\bar{u}^T = \bar{u}^T u = I, \quad \forall u \in U(n),$$

where I is the $n \times n$ identity matrix, and $\bar{u}^T$ is the conjugate transpose of u.

In particular, this invariance of the Fubini-Study metric under $U(n+1)$ allows the preservation of the scalar product, which results in the symmetric shape of $\mathbb{CP}^n$. In this section, we define and give the analog to the Fubini-Study metric for $\mathbb{HP}^n$.

Definition 12 (Riemannian manifold)**.** A Riemannian manifold is a real, smooth manifold equipped with a metric g, and is denoted by (M, g). The Riemannian manifold is also equipped with an inner product, and a tangent space at each point $p \in (M, g)$.

For some Riemannian manifold (M_1, g_1) with an isometric group action G, we let $\pi : (M_1, g_1) \to (M_2, g_2)$ be the projection map, where (M_2, g_2) is the quotient space M/G equipped with the g_2. The metric g_2 is the naturally induced metric via the mapping of tangent vectors from M_1 to M_2. We require this mapping to be orthogonal to the fiber.

Definition 13 (Fiber)**.** From the definition of $\pi : (M_1, g_1) \to (M_2, g_2)$, the fiber of a point

$u \in M_2$ is defined to be it's preimage in M_1, namely the set of points in M_1 described by $\pi^{-1}(u)$ in M_1.

We formally give the definition of g_2, the induced metric, as follows.

Definition 14 (Induced Metric)**.** The induced metric g_2 is given by

$$g_2(x, y) = g_1(\tilde{x}, \tilde{y}),$$

where x, y are tangent vectors in in $T_Y M_1$, the tangent space at point $Y \in M_1$, and $\tilde{x}, \tilde{y} \in T_X M_2$, the tangent space at point $X \in M_2$. We say $\pi(Y) = X$, and we require $\tilde{x} \perp T_Y M_1(\pi^{-1}(x))$, $\tilde{y} \perp T_Y M_1(\pi^{-1}(x))$.

Definition 15 (Riemannian Submersion)**.** Given a group G that acts on isometrically on a Riemannian manifold (M, g) the projection $pi : M \to N$ to the quotient space $N = M/G$ equipped with the quotient metric is a Riemannian submersion.

The following lemma relating the metrics on Riemannian manifolds is taken from a statement in the notes of Fabrice Baudoin [8].

Lemma 3 (F.Baudoin, 2014)**.** *Recall the map $\pi : (M_1, g_1) \to (M_2, g_2)$ under an isometry group G. For a point $Y \in M_1$, if $T_Y M_1(\pi^{-1}(x))$ is the vertical space at X, denoted V_X, and H_X is the horizontal space at X, the orthogonal complement to V_X, then we have the orthogonal decomposition, $T_Y M_1 = H_X \oplus V_X$, and the corresponding splitting of the metric $g_1 = g_H + g_V$.*

The result from lemma 3 shows that our definition of the induced metric in definition 14, under the orthogonal mapping is well-defined. Therefore, we get the following proposition due to lemma 3.

Proposition 4. *The value of $g_2(x, y)$ for tangent vectors $x, y \in T_X$, is invariant under different choices for Y satisfying $\pi(Y) = X$.*

Definition 16 (Base Point)**.** For a choice of Y satisfying $\pi(Y) = X$, where π sends tangent vectors in T_X to tangent vectors in T_Y, we say Y is the base point of X.

We see that when we set (M_1, g_1) to be (S^{4n+3}, g_S), where g_S denotes the standard metric on the sphere S^{4n+3}, (M_2, g_2) to be $(\mathbb{HP}^n, g_{fs})$, where g_{fs} denotes the Fubini-Study metric on $\mathbb{HP}^n$, and the isometric group G on (M_1, g_1) to be $SU(2)$, g_{fs} is the induced metric from the metric on S^{4n+3}.

Here, we give the Fubini-Study metric for $\mathbb{HP}^n$ induced from pulling back the metric g_S on the sphere S^{4n+3}.

Definition 17 (Pull-Back Metric)**.** If we have a map $\rho : \mathbb{H}^n \to S^{4n+3}$, and a metric g_s on S^{4n+3}, the pull-back metric is the metric on $\mathbb{H}^n$ satisfying $(f^\star g_s)(u, v) = g_s(f(u), f(v))$, for $u, v \in \mathbb{HP}^n$.

The map $\rho : \mathbb{H}^n \to S^{4n+3}$ sends a point $[z_1, \cdots, z_n]$ on $\mathbb{H}^n$ to a corresponding point $(1, w_1, \cdots, w_n)$ on the sphere. Each coordinate z_i is sent to the coordinate w_1 of the form

$$w_i = \frac{z_i}{\sqrt{1 + |Z|^2}},$$

where $|Z| = z_1\bar{z_1} + z_2\bar{z_2} + \cdots + z_n\bar{z_n}$. The pull-back metric from this map ρ gives the Fubini-Study metric on $\mathbb{HP}^n$ written in terms of homogeneous coordinates.

Due to the non-commutative nature of quaternionic multiplication, we compute the Fubini-Study metric for quaternions with respect to the real numbers to obtain a more precise definition of the metric.

In order to simplify the computation of the Fubini-Study metric with respect to real numbers, we condense the computation using matrix and quaternionic shorthand. We have 2 interpretations of a quaternion in the context of real numbers. The first interpretation is to write a quaternion as a 4 by 1 column vector, with $a, b, c, d \in \mathbb{R}$ representing the 4 components of the quaternions. Alternatively, we can considers a quaternion as a linear transformation on a quaternionic vector in H^{n+1} via left multiplication. A quaternion of the form $a + bi + cj + dk$ can be expressed by the following 4×4 real matrix:

$$\begin{bmatrix} a & -b & -c & -d \\ b & a & -d & c \\ c & d & a & -b \\ d & -c & b & a \end{bmatrix}.$$

As follows from this real representation of the quaternion, in the computation of the Fubini-Study metric, we associate all quaternions terms z_i with its corresponding 4×4 matrix. We also associate all dz_i and $d\bar{z}_j$ terms as quaternionic vectors, expressed by da, db, dc, dd for real variables a, b, c, d. The products $z_i dz_i$ and $z_i \bar{z}_j$ follow the standard rules for matrix multiplication. The product $dz_i d\bar{z}_i$ is formally defined as their quaternionic inner product, $\langle dz_i, d\bar{z}_i \rangle$.

Recalling our map ρ, the Fubini-Study metric written in homogeneous coordinates is the pull-back metric of S^{4n+3}. The sphere metric g_s is given by

$$g_s = \sum_{i=0}^{n} dw_i d\bar{w}_i.$$

From the relation of w_i and z_i due to ρ, we see that

$$dw_i = \sum_{j=1}^{n} \frac{dw_i}{dz_j} dz_j, \qquad d\bar{w}_i = \sum_{j=1}^{n} \frac{d\bar{w}_i}{dz_j} dz_j.$$

We obtain the pull-back metric for H^n, or the Fubini-Study metric written in homogeneous coordinates via a substitution of equations. The Fubini-Study metric for $\mathbb{HP}^n$, expanded out, is given as follows

$$g_{fs} = \sum_{i=1}^{n} \frac{dz_i d\bar{z}_i}{1+|Z|^2} - \frac{1}{2} \sum_{i=1}^{n} z_i \left(\sum_{j=1}^{n} \frac{\bar{z}_j dz_j}{(1+|Z|^2)^2} \right) d\bar{z}_i - \frac{1}{2} \sum_{i=1}^{n} dz_i \left(\sum_{j=1}^{n} \frac{d\bar{z}_j z_j}{(1+|Z|^2)^2} \right) \bar{z}_i + \frac{1}{4} \sum_{i=1}^{n} z_i \left(\sum_{j,k=1}^{n} \frac{\bar{z}_j d\bar{z}_j d\bar{z}_k z_k}{(1+|Z|^2)^3} \right) \bar{z}_i.$$

Definition 18 (Distance on Riemannian Manifolds)**.** The distance between two points A, B on a Riemannian manifold (M, g) is given by

$$d_g(A,B) = \min \int_0^1 \sqrt{g(\gamma'(t), \gamma'(t))}dt,$$

where $\gamma(t) : [0,1] \to M$ is a parametrization of a smooth path connecting points A, B such that $\gamma(0) = A$ and $\gamma(1) = B$.

Following definition 18, the distance function between points u, v on $\mathbb{HP}^n$ is given from its the Fubini-Study metric through the following relation

$$d(u,v) = \min \int_0^1 \sqrt{g_{fs}(\gamma'(t), \gamma'(t))}dt,$$

where $\gamma(t)$ describes a path connecting u, v such that $\gamma(0) = u$ and $\gamma(1) = v$.

5 Distance and Transition Probability

Let $d\colon \mathbb{HP}^n \times \mathbb{HP}^n \to \mathbb{R}^{\geq 0}$ be the distance function associated to the Fubini-Study Metric for quaternions. Since the metric on S^{4n+3} is a complete metric, the Fubini-Study metric on $\mathbb{HP}^n$ is also complete. Therefore, we define $d(u, v)$ to be the minimum length of all smooth paths connecting points $u, v \in \mathbb{HP}^n$ [9]. We define $\tilde{d} : \mathbb{HP}^n \times \mathbb{HP}^n \to \mathbb{R}^{\geq 0}$ as the rescaled distance function for $\mathbb{HP}^n$ such that the Fubini-Study metric on $\mathbb{HP}^1$ is equal to the metric on S^4. We explicitly give $\tilde{d}$ later through Lemma 6.

Definition 19 (Transition Probability)**.** Let $p : \mathbb{HP}^n \times \mathbb{HP}^n \to [0, 1]$ be a function that acts on states $\mathcal{L}_1, \mathcal{L}_2 \in \mathbb{HP}^n$. The transition probability is given by $p(\mathcal{L}_1, \mathcal{L}_2)$: if $\psi_i \in \mathcal{L}_i$ is a unit norm vector in $\mathcal{L}_i$, then

$$p(\mathcal{L}_1, \mathcal{L}_2) = |\langle \psi_1, \psi_2 \rangle|^2.$$

A result from Freed's paper [3] states the relation between transition probability and the Fubini-Study distance function in the case of complex projective space. We state this relation for the quaternions in the following theorem.

Theorem 5. *The rescaled distance function $\tilde{d}$ and transition probability p are related by*

$$\cos^2\left(\frac{\tilde{d}}{2}\right) = p.$$

For the proof of this theorem, we first give some definitions and statements related to the proof of theorem 5. we define a map $f\colon \mathbb{HP}^n \backslash \mathbb{HP}^{n-1} \to S^{4n+3}$, such that f sends point $u = [z_0 : z_1 : \cdots : z_n]$ belonging to a subset of $\mathbb{HP}^n$ to a point on S^{4n+3} denoted by $\lambda u'$, where $\lambda \in SU(2)$ and $u' = (w_0, w_1, \cdots, w_n)$, and $w_i = \frac{z_i}{1+|Z|^{2^2}}$.

The following result, relating the metrics of $\mathbb{HP}^n$ and S^4 is taken from a book [10] by Francis Burstall, Franz Pedit, Dirk Ferus, Katrin Leschke, and Ulrich Pinkall.

Lemma 6 (F. Burstall *et al.*, 2000)**.** *The metric on $\mathbb{HP}^1$ is and the metric on S^4 are related by*

$$g_{\mathbb{HP}^1} = \frac{1}{2} g_{S^4}.$$

The following proposition is a well-known fact about the relation between the Euclidean inner product and the quaternionic inner product.

Proposition 7. *The Euclidean inner product of two quaternions q_1,q_2, is equal to the real components of their quaternionic inner product, denoted by $Re\langle q_1, q_2\rangle$.*

Proof. Let $q_1 = a + bi + cj + dk$ and $q_2 = a' + b'i + c'j + d'k$, for real values of a, b, c, d and a', b', c', d'. We have from definition 9, $Re\langle q_1, q_2\rangle = Re(q_1\bar{q_2})$. Expanding $Re(q_1\bar{q_2})$, we see that the real components are given by $aa' + bb' + cc' + dd'$, which is precisely equal to $E(q_1, q_2)$. ∎

Lemma 8. *For optimal choices of $\lambda, \mu \in SU(2)$, we can maximise $Re\langle \lambda u', \mu v'\rangle$ such that it is equal to $|\langle u', v'\rangle|$.*

Proof. Let μ be the identity in $SU(2)$. Then we choose a λ that maximises $Re\langle \lambda u', v'\rangle$. We have $Re\langle \lambda u', v'\rangle = Re(\lambda\langle u', v'\rangle)$, and $\langle u', v'\rangle$ can be written in the form of $r\alpha$ for $r \in \mathbb{R}^{\geq 0}$

and $\alpha \in SU(2)$. When we set λ as $\bar{\alpha}$, we eliminate the imaginary components of the inner product, and $Re\langle \lambda u', v' \rangle = Re(r)$. It is simple to verify that r is equal to $|\langle u', v' \rangle|$. □

Proof of Theorem 5. Let us recall the map $f\colon \mathbb{HP}^n \backslash \mathbb{HP}^{n-1} \to S^{4n+3}$, which sends points $u, v \in \mathbb{HP}^n \backslash \mathbb{HP}^{n-1}$ to $\lambda u', \mu v' \in S^{4n+3}$. Under the map f, λ, μ are unit elements of $\mathbb{H}$ that we identify with elements of $SU(2)$, and from here we use the notation $\lambda, \mu \in SU(2)$. We know S^{4n+3} induces the Fubini-Study metric on $\mathbb{HP}^n$, such that $g_{fs}(u, v) = g_s \lambda u', \mu v'$.

Then we say that the images of points u, v on $\mathbb{HP}^n$ are the sets of points on S^{4n+3} of the form $\lambda u'$ and $\mu v'$ respectively, for $\lambda, \mu \in SU(2)$. We see that f sends a point u to a particular element of its image, determined by a particular choice of λ.

We rescale the Fubini-Study metric with respect to the result of Lemma 6, so that the Fubini-Study metric on $\mathbb{HP}^1$ is identifiable with the standard metric on S^4. Under this rescaling of g_{fs}, we get the distance function on $\mathbb{HP}^n$ rescaled by a factor of 2, and given by

$$\tilde{d}(u, v) = 2 \min d_S(\lambda u', \mu v'),$$

where d_S denotes the distance function on S^{4n+3}.

It is known that the value of $d_S(\lambda u', \mu v')$ is the arc length between points $\lambda u', \mu v'$ on the sphere, and is given by $\arccos(E(\lambda u', \mu v'))$, where $E(\lambda u', \mu v')$ is the Euclidean inner product of quaternionic vectors $\lambda u', \mu v'$. It follows from proposition 7, that $\tilde{d}(u, v)$ is equal to the minimum value of $2 \arccos(Re\langle \lambda u', \mu v' \rangle)$ for different choices of $\lambda, \mu \in SU(2)$. Hence, we choose optimal values for $\lambda, \mu \in SU(2)$ that maximize $Re\langle \lambda u', \mu v' \rangle$. It is clear that $Re\langle \lambda u', \mu v' \rangle \leq |\langle u', v' \rangle|$, and due to Lemma 8, we can choose values of λ, μ such that $Re\langle \lambda u', \mu v' \rangle = |\langle u', v' \rangle|$.

Therefore, we get $\tilde{d}(u, v) = 2 \arccos(|\langle u', v' \rangle|)$. We see that this relation is equivalent to the relation in theorem 5. □

6 Conclusion and Future Directions

This work gives the Fubini-Study metric, using matrix and quaternionic shorthand, for $\mathbb{HP}^n$ with respect to real numbers. We also show that the distance function on $\mathbb{HP}^n$ associated with the Fubini-Study metric is aligned with the notion of distance in Riemannian geometry. Finally, this work relates the distance function on $\mathbb{HP}^n$ and transition probability, demonstrating that functions in $\mathbb{HP}^n$ that preserve distance also preserve transition probabilities. Therefore, isometries in $\mathbb{HP}^n$ can be used to represent symmetries in physical systems. In the future, one can study the relation between these isometries in $\mathbb{HP}^n$ and the the compact symplectic matrix group $\mathbb{SP}(n)$, to indirectly prove Bargmann's work [4] on the quaternionic analog to Wigner's Theorem.

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