Clique Structure of Orthomodular Posets

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Abstract

Dacey graphs include looped structures of maximal cliques when two maximal cliques cover a third. Dacey established a class of simple, finite graphs, called Dacey graphs, which corresponds to the set of finite orthomodular posets, posets where the join of orthogonal elements exists and satisfy constructive properties. We begin an enumeration of Dacey graphs by presenting methods of partitioning Dacey graphs into equivalence classes based on the structures of maximal clique intersections and characterizing Dacey graphs which are edge-covered by m maximal cliques. This leads to a complete classification of Dacey graphs for $m \leq 4$. We then explore the implications of applying Dacey as a local condition to show each Dacey graph is locally-Dacey and investigate some bounds on such a local condition.

Summary

Orthomodular posets map out the logical structure of experiments which potentially hold implications in quantum mechanics. Dacey formulated the class of graphs, Dacey graphs, which represent orthomodular posets. Using this, we present ways to simplify graphs without altering their underlying structure and classify graphs whose edges can be covered by m fully connected parts of the graph. We find a complete classification of Dacey graphs for m up to four. We also show how the Dacey condition, when applied to the part of a graph close to each vertex, can affect the graph as a whole.

1 Introduction

We investigate the structure of graphs which correspond to orthomodular posets and describe methods by which such graphs can be generated. In John von Neumann's [1] mathematical formulation of quantum mechanics, a projector on a Hilbert space $\mathscr{H}$ acts on a state and a collection of these objects represents the effect and outcome of quantum physical measurements. The set of projectors $\mathscr{P}(\mathscr{H})$ on a Hilbert space are in one-to-one correspondence with the set of all closed subspaces of $\mathscr{H}$ [2]. The set of projectors $\mathscr{P}(\mathscr{H})$ and the relation of set-theoretic inclusion form an orthomodular *partially ordered set* (poset). A poset $\mathcal{P} = (P, \leq)$ consists of a set of elements P and a relation denoted by $\leq$ where for two elements $x, y \in P$, we have at most one of the following: $x \leq y$, $y \leq x$, or x and y are incomparable. Two elements are orthogonal in a complemented poset if the complement of one element is bounded below by the other.

Orthomodular structures are prominent in understanding quantum logic. In an orthomodular poset, the supremum of orthogonal subsets always exists and for two related elements $a \leq b$, element a can always be paired up with an orthogonal element so that the supremum of the pair is b. Orthomodular posets generally serve as structures to describe experimental setups on physical systems. Dacey [3] showed there exists a class of simple graphs which corresponds to orthomodular posets, where vertices in each graph correspond to the atoms of an orthomodular poset while edges correspond to pairs of orthogonal atoms. The condition necessary for a graph to correspond to an orthomodular poset pertains to the maximal cliques of the graphs, which represent maximal Boolean algebras in the poset. In this way, we take a graph-theoretic approach to begin enumerating algebraic orthomodular posets.

We begin by introducing modifications on graphs which preserve the relevant conditions. We use this to establish a classification of graphs which can be edge-covered by m cliques and enumerate the graphs which are Dacey for m up to four. Along the way, there are reasonable simplifications and bounds on the number of cliques. These results motivate simple structural theorems as well.

2 Orthomodular Posets to Graphs

Notation

We consider only simple, undirected, and finite graphs. We denote the set of vertices of G by $V(G)$. At times, the vertex set $V(G)$ may be used to denote the graph induced in G by the vertices in $V(G)$ as specified in context. Given two vertices $a, b \in V(G)$ we say $(a, b) \in E(G)$ if they are adjacent in G and $(a, b) \notin E(G)$ otherwise. Given a subset of vertices $A \subseteq V(G)$, we denote the set of neighbors of A by $N(A)$ where a vertex $v \in V(G)$ belongs to $N(A)$ if and only if we have $(v, a) \in E(G)$ for all $a \in A$. A *maximal clique* of G is a subset $M \subseteq G$ of vertices which induces a complete subgraph in G where there does *not* exist a vertex $v \in G$ such that $M \subseteq N(v)$. We denote the set of all maximal cliques and of all cliques in G as $\mathcal{M}(G)$ and $\mathcal{K}(G)$, respectively.

Previous Work

Dacey [3] shows the precise correlation between graphs and orthomodular posets, the result of Theorem 2.1.

Definition 2.1. Given a graph G, let $S \subseteq V(G)$. The set of vertices S is *closed* if and only if $N(N(S)) = S$.

The poset corresponding to G consists of elements which are the closed sets S. It is possible to define equivalence classes on $\mathcal{K}(G)$, the set of all cliques in G, such that each closed set of vertices S may be formed by the unions of equivalence classes on $\mathcal{K}(G)$. This allows us to define a poset on the set of closed S and the relation of set theoretic inclusion $\subseteq$, which leads to the key result of Dacey. We use the following definition of Dacey graphs:

Definition 2.2 (Sumner [4])**.** A graph G is a *Dacey* graph if and only if for each maximal clique $M \in \mathcal{M}(G)$ and any pair of distinct vertices $u, v \in V(G)$ we have

$$M \subseteq N(u) \cup N(v) \Rightarrow (u, v) \in E(G).$$

In this way, Dacey translates orthomodularity to a single condition on graphs. The correlation between graphs and orthomodular posets is as follows:

Theorem 2.1 (Dacey [3])**.** *Given graph G, let $\mathcal{P}$ be the pair*

$$(\{S \mid S \in V(G) \,\text{and}\, S \,\text{is closed}\}, \subseteq).$$

The pair $\mathcal{P}$ is an orthomodular poset if and only if G is Dacey.

The Dacey condition specifies every graph which corresponds to an orthomodular poset, but multiple Dacey graphs may correspond to the same orthomodular poset. To define a set of graphs where this mapping is bijective, we introduce the *clique-distant* condition. A natural choice for a set of graphs which form a bijection with finite orthomodular posets satisfies two conditions, as formulated by Definitions 2.2 and 2.3.

Definition 2.3. A graph G is *clique-distant* if and only if for any pair of distinct maximal cliques $M_1, M_2 \in \mathcal{M}(G)$, there exist four vertices $u_1, v_1, u_2, v_2 \in V(G)$ such that

$$u_1, v_1 \in M_1 \text{ and } u_1, v_1 \notin M_2,$$
$$u_2, v_2 \in M_2 \text{ and } u_2, v_2 \notin M_1.$$

There are no well-known, effective methods to enumerate all finite clique-distant Dacey graphs. We take steps in beginning this process to generate interesting orthomodular posets. We begin in sections 3 and 4 by establishing the structure behind graphs which are edge-coverable by m maximal cliques and confine the area of search for Dacey graphs to induced subgraphs of a specified finite graph based on m. This includes constructing the "inflation" process on graphs, under which the Dacey condition remains invariant. In section 5, we provide some bounds between m and the number of maximal cliques in total in a graph. Section 6 combines the methods of previous sections to provide a classification of all graphs for m at most four, while section 7 explores the Dacey condition as a local condition.

3 Graphs by Edge-Covering Maximal Cliques

We begin with a formal definition for graphs edge-coverable by m maximal cliques.

Definition 3.1. A graph G is *m-clique coverable* if there exists a set of maximal cliques $\mathcal{S}(G) \subseteq \mathcal{M}(G)$, where $|\mathcal{S}(G)| = m$, so that for each $K \in E(G)$ there exists some $M \in \mathcal{S}(G)$ such that $M \supseteq K$.

We say G satisfies the (m, n) condition when G is m-clique coverable and $|\mathcal{M}(G)| = n$.

For an m-clique coverable graph G, each vertex $v \in G$ belongs to some set of maximal cliques in $\mathcal{M}(G)$. To consider possible structures of an (m, n) graph G, we construct graphs based on the power set of some set, where again, each element corresponds to a maximal clique in the subset of m cliques in $\mathcal{M}(G)$ covering all edges in G.

Definition 3.2. Let $N \subset \mathbb{N}$ be a finite set. We say a graph G is an *power set intersection graph* if there exists a bijection $f : V(G) \to \mathbf{2}^N$ where $(u, v) \in E(G) \iff f(u) \cap f(v) \neq \emptyset$.

If G is an intersection graph with $|N| = m$, we say $G \cong PS_m$.

4 Properties of Inflation and Deflation

Let us denote the closed neighbor set of a vertex $v \in V(G)$ to be $N[v] = N(v) \sqcup v$. We define a binary relation $\sim \subseteq V(G) \times V(G)$ where $v \sim v'$ if and only if $N[v] = N[v']$. This is an equivalence relation on $V(G)$. The set $V(G)/\sim = \{\{v \in V(G) | v \sim v_0\} | v_0 \in V(G)\}$ of equivalence classes over $\sim$ is the collection of all maximal subsets of $V(G)$ containing vertices sharing the same closed neighbor set in G. Note $V(G)/\sim \subseteq \mathcal{K}(G)$.

Definition 4.1. Given graph G with some ordinal on $V(G) = \{v_1, v_2, \dots v_n\}$ and a sequence of positive integers $S = (s_1, s_2, \dots s_n)$, we say the *inflation* G^S of G with respect to S is the graph obtained from replacing each vertex $v_i \in V(G)$ with a clique $K_i \in \mathcal{K}(G^S)$ such that:

- $|K_i| = s_i$, where the size of the enumerated cliques correspond to terms in S
- $(u, v) \in E(G^S) \iff (v_i, v_j) \in E(G)$ for each pair $(u, v) \in V(G^S) \times V(G^S)$, with $u \in K_i$, $v \in K_j$, and $i \neq j$.

the above implies $K_i \in V(G^S)/\sim$.

Naturally, we say a graph G is *deflated* if there does not exist a pair $(u, v) \in V(G) \times V(G)$ and $u \sim v$. We see a deflated graph is not a non-trivial inflation of any other graph. Moreover, it minimally represents graphs with some particular structure of maximal clique intersection. Analogously, if some graph G is an inflation of a graph H, then we say H is a *deflation* of G. From this arises a natural partitioning of m-clique coverable graphs.

Theorem 4.1. *Every m-clique coverable graph G is isomorphic to some inflation of an induced subgraph of PS_m.*

Proof. For an m-clique coverable graph G, let us enumerate the maximal cliques in $\mathcal{M}(G) = \{M_i \mid i \in N\}$ and define a map $r : V(G) \to \mathbf{2}^N$ such that $r(v) = \{i \in N \mid M_i \ni v\}$. We see $(u, v) \in E(G) \iff r(u) \cap r(v) \neq \emptyset$, meaning the adjacency of vertices are interpreted through the intersections of their image sets.

Suppose we enumerate the *subsets* of maximal cliques contained in $\mathbf{2}^{\mathcal{M}(G)}$ so that $\mathbf{2}^{\mathcal{M}(G)} = \{\mathcal{A}_i \,|\, i \in \{1, 2, \cdots, 2^m - 1\}\}$. Say $r(v) = \mathcal{A}_i$ for exactly s_i vertices $v \in V(G)$. We define a set

$\overline{V}$ of vertices to remove from PS_m as follows:

$$\overline{V} = \{v \in PS_m |\, \exists\, u \in G \text{ where } f(v) = r(u) = \mathcal{A}_i \text{ and } s_i = 0\}.$$

We consider the induced subgraph $H = PS_m \backslash \overline{V}$. Then, the inflation H^S of H where $S = (s_i |\, s_i \neq 0)$ is isomorphic to G. Moreover, there exists a deflation G' of any m-clique coverable graph G to an induced subgraph $G' \subseteq PS_m$. $\square$

The mechanism behind this result is the correlation between the cliques of a graph and the cliques of some inflation of the graph. Deflated graphs serve as a minimal representation of some structure with regard to maximal cliques and the following result helps show this:

Proposition 4.2. *Two distinct vertices $u, v \in V(G)$ are contained in precisely the same set of maximal cliques in G if and only if $u \sim v$.*

Proof. See Appendix A. $\square$

It is necessary to consider how a deflation or inflation G^S of a graph G affects the properties described by Definitions 2.2 and 2.3.

Proposition 4.3. *For every graph G, there exists an inflation G^S of G such that G^S is clique-distant.*

Proof. Take an inflation G^S of G with respect to a sequence S where for each $s_i \in S$ we have $s_i = 2$. This means by Proposition 4.2, to every unique set of maximal cliques with a non-empty arbitrary intersection there corresponds at least two vertices contained in exactly those maximal cliques (and no others). So, given two distinct maximal cliques in G, where each must contain some vertex not contained by the other, there exist at least two vertices in G^S contained in one maximal clique but not the other, with respect to both cliques. This guarantees the graph G is clique-distant. $\square$

Moreover, there exists a graph G_0 of minimal size for any G which is clique-distant. We consider the deflation G_D of G which is deflated. Note that any two distinct maximal cliques differ by either one or two vertices. We then inflate G_D with respect to S such that we define the terms so that $s_i = 2$ if v_i belongs in some maximal clique M_1 where $M_1 \backslash v_i \subseteq M_2$, for some maximal clique $M_2 \neq M_1$, and $s_i = 1$ otherwise. It follows that given any Dacey graph, there exists some minimal inflation so that the resulting graph is clique-distant. With this proposition, we consider only the Dacey condition in constructing deflated graphs, because every graph may always be inflated to meet the clique-distant condition.

Proposition 4.4. *For every graph G and deflation G_D, the graph G_D is Dacey if and only if G is Dacey.*

Proof. We know G is Dacey if and only if there does not exist a pair of vertices $x, y \in V(G)$ such that $(x, y) \notin E(G)$ and $M \subseteq N(x) \cup N(y)$ for some maximal clique $M \in \mathcal{M}(G)$. Consider some deflation $D_0(G)$ obtained by contracting some edge $(u, v) \in E(G)$ to a vertex v_0, where $u \sim v$. By Proposition 4.2, there exists a natural bijection between the set of maximal cliques in $D_0(G)$ containing v_0 and the corresponding set of maximal cliques in G for either u or v. For each pair of distinct vertices $x, y \in D_0(G)$ both different from v_0 so that $x, y \in V(G)$, we see either there exists no maximal clique $M \in \mathcal{M}(G)$ such that $M \subseteq N(x) \cup N(y)$ or there does. If such a maximal clique M in $D_0(G)$ exists for a pair of vertices $x, y \in D_0(G)$ different from v_0 then $M \subseteq N(x) \cup N(y)$ in $D_0(G)$ only if $M \subseteq N(x) \cup N(y)$ in G since u and v share the same neighbors. If $x = u$ and y were such that $M \subseteq N(x) \cup N(y)$ in G then $M \subseteq N(x) \cup N(y)$ in $D_0(G)$ as well. So whether the Dacey condition is satisfied by G is always preserved by single-vertex deflations $D_0(G)$ of G. All deflations are some sequence of single-vertex deflations, so this holds for all deflations G_D. Similarly, the other direction follows since for all inflations G^S of G, there exists a deflation of G^S which results in G. So, the Dacey condition is preserved over inflations as well. $\square$

5 Bounds on Number of Maximal Cliques

By Theorem 4.1, every (m, n) graph is some inflation of some induced subgraph of PS_m, a defined, finite graph. This suggests there exist limitations to n, the number of maximal cliques in total contained in the graph. We explore these bounds on general graphs and on Dacey graphs.

Lemma 5.1. *Given $G' \subseteq G$ is an induced subgraph of G, the inequality $|\mathcal{M}(G')| \leq |\mathcal{M}(G)|$ holds.*

Proof. Let us define a map $f : \mathcal{M}(G') \to \mathcal{M}(G)$. By $G' \subseteq G$, we see each maximal clique $M \in \mathcal{M}(G')$ is a clique in G, hence $M \in \mathcal{K}(G)$. This implies M is contained in a maximal clique in G and there exists some $f(M) \in \mathcal{M}(G)$ for which $M \subseteq f(M)$. We show f is injective.

Assume on the contrary, there exist two distinct $M_1, M_2 \in \mathcal{M}(G')$ where $f(M_1) = f(M_2) = M$. This implies $M_1 \cup M_2 \subseteq M$. Since G' is an induced subgraph, all adjacencies are preserved between present vertices in G', and this implies $(u, v) \in E(G')$ for every $v \in M_1$ and $u \in M_2$, where $u \neq v$. This contradicts the maximality of the cliques M_1 and M_2 in G' and shows f is injective so that $|\mathcal{M}(G')| \leq |\mathcal{M}(G)|$. $\square$

Proposition 5.2. *For any (m, n) graph G, given $\lambda(m)$ is the number of distinct maximal cliques in graph PS_m, then we have $n \leq \lambda(m)$.*

Proof. We know G is some inflation of an induced subgraph $G_0 \subseteq PS_m$. By Lemma 5.1, the inequality $|\mathcal{M}(G_0)| \leq |\mathcal{M}(PS_m)|$ holds. We revisit the mapping $f : V(PS_m) \to \mathbf{2}^N$ where $(u, v) \in E(PS_m) \iff f(u) \cap f(v) \neq \emptyset$ explained in Definition 3.2. We can define

$$\mathcal{M}(PS_m) = \{\mathcal{S} \subseteq V(PS_m) | \, \forall \, u, v \in \mathcal{S}, f(u) \cap f(v) \neq \emptyset$$
$$\text{and } \nexists \; w \notin \mathcal{S} \text{ where } f(w) \cap f(v) \neq \emptyset \, \forall \, v \in \mathcal{S}\}.$$

This is the definition of maximally linked systems in $\mathcal{P}(\{1, 2, \cdots, m\})$ as described by Brouwer in [5]. Since each maximal clique of PS_m corresponds to a maximally linked system of $\mathcal{P}(\{1, 2, \cdots, m\})$, of which there are $\lambda(m)$ as listed in the OEIS sequence A001206, then there are at most $\lambda(m)$ maximal cliques in G. $\square$

Theorem 5.3. *For a Dacey, (m, n) graph G, we have the following:*

$$\begin{aligned} m = 2 \quad &\textit{implies} \quad n = 2, \\ m = 3 \quad &\textit{implies} \quad n = 3, \\ m = 4 \quad &\textit{implies} \quad n \leq 8. \end{aligned}$$

Proof. For a 2-clique coverable graph G, by Theorem 4.1, G is some inflation of an induced subgraph of PS_2. Any inflation of PS_2, which is not edge-coverable by only one clique, must be the clique-sum of two complete graphs, which contains exactly two distinct cliques overall.

For a 3-clique coverable graph G, we know G is some inflation of an induced subgraph of PS_3. By Proposition 4.4, given G is Dacey, the corresponding induced subgraph of PS_3 must be Dacey as well. Since PS_3 itself is not Dacey and neither is the graph $PS_3 \backslash v$ for the vertex v which satisfies $f(v) = \{1, 2, 3\}$ in PS_3, then the induced subgraph of PS_3 of which G is an inflation must differ from PS_3 by some vertex different from v. All such induced subgraphs of PS_3 have at most 3 maximal cliques.

For a 4-clique coverable graph, the result follows from Theorem 6.3. Under the deletion or addition of vertices adjacent to every other vertex in G, the number $\mathcal{M}(G)$ remains invariant. This quantity is invariant under inflations and deflations of G as well, so the most maximal cliques a 4-clique coverable Dacey graph can have is 8, the most maximal cliques any graph has under Theorem 6.3. $\square$

6 Classification of Dacey m-clique Coverable Graphs for $m \leq 4$

In addition to bounds on the number of maximal cliques, it is important to know how m-clique coverable Dacey graphs look. Aside from inflation and deflation, modifying a graph G by removing or appending a vertex v adjacent to all other vertices of G preserves the Dacey condition. We call such a vertex v a *cone* in G.

Proposition 6.1. *Given a graph G and $v \in V(G)$, where for every other vertex $u \in G \backslash v$, we have $(u, v) \in E(G)$, graph G is Dacey if and only if $G \backslash v$ is Dacey.*

Proof. See Appendix B. □

We should consider only graphs unique both under the inclusion or exclusion of a cone and under inflation or deflation. This forms partitioned classes of Dacey graphs. In all deflated graphs, there exists at most one cone. With this, we build the classification, starting with a new construction.

Definition 6.1. A connected graph G is *n-horned* if the following is true:

- G contains a clique K so that there exists a set $H = \{v_1, v_2, \dots v_n\}$ of n mutually-non-adjacent vertices, where $H \cap K = \emptyset$ and $H \cup K = V(G)$
- the map $N_H : K \to \mathbf{2}^H \backslash H$ is injective.

By the second condition, we see each vertex $v \in K$ corresponds to a subset $N_H(v) \in \mathbf{2}^H$ so that for some $v_i \in H$, the adjacency $(v, v_i) \in E(G)$ exists if and only if $v_i \in N_H(v)$. Note that in G, we cannot have an element $v_i \in H$ such that for all $v \in K$ it follows $v_i \in N_H(v)$. Otherwise, there is some $v_i \notin K$ where $N(v_i) \supseteq K$. The vertices $v_i \in H$ are referred to as *horns* and clique K is maximal in G.

Proposition 6.2. *An n-horned graph H is deflated.*

Proof. For all pairs $(u, v) \in E(G)$ either both $u, v \in K$ or $v = v_i \in H$ and $u \in K$. If $u, v \in K$, then we cannot have $N(u)\backslash v = N(v)\backslash u$ as $N_H(v)$ is injective. If $u \in K$ and $v = v_i \in H$, then $N(u)\backslash v = N(v)\backslash u$ only if $K \subseteq N(v)$, which cannot be the case as K is maximal. □

This helps lead to a classification theorem of 4-clique coverable graphs up to inflation, deflation, and inclusion or exclusion of cones.

Theorem 6.3. *A connected m-clique coverable graph G where $m \leq 4$ is Dacey if and only if G is the inflation of one of the following (up to addition or deletion of a cone):*

- *an n-horned graph with $n \leq 3$ where there does **not** exist some pair $v_i, v_j \in H$ such that $N_H(v) \cap \{v_i, v_j\}$ is non-empty for all $v \in M$*
- *one of the nine deflated Dacey graphs depicted in Figure 1.*

Proof. Given the result of Theorem 6.3, for $m \leq 4$ the only graphs to consider are induced subgraphs of PS_4. Proposition 4.4 tells we only need to check induced subgraphs of PS_4 for Dacey graphs to determine whether the inflations of those graphs are Dacey. This search for Dacey graphs spans all graphs for which $m \leq 4$. The task can be carried out computationally by inspection through the program in Appendix C. The program iterates through all connected, induced subgraphs of PS_4 and filters a list of those which are Dacey. After removing duplicate graphs up to isomorphism, the program deflates the graphs in the list. The program then displays the resulting graphs, which is classified above. □

Theorem 6.3 provides a simple enumeration of all 4-clique coverable graphs which are Dacey, which in turn allows us to consider all orthomodular posets covered by up to four Boolean algebras.

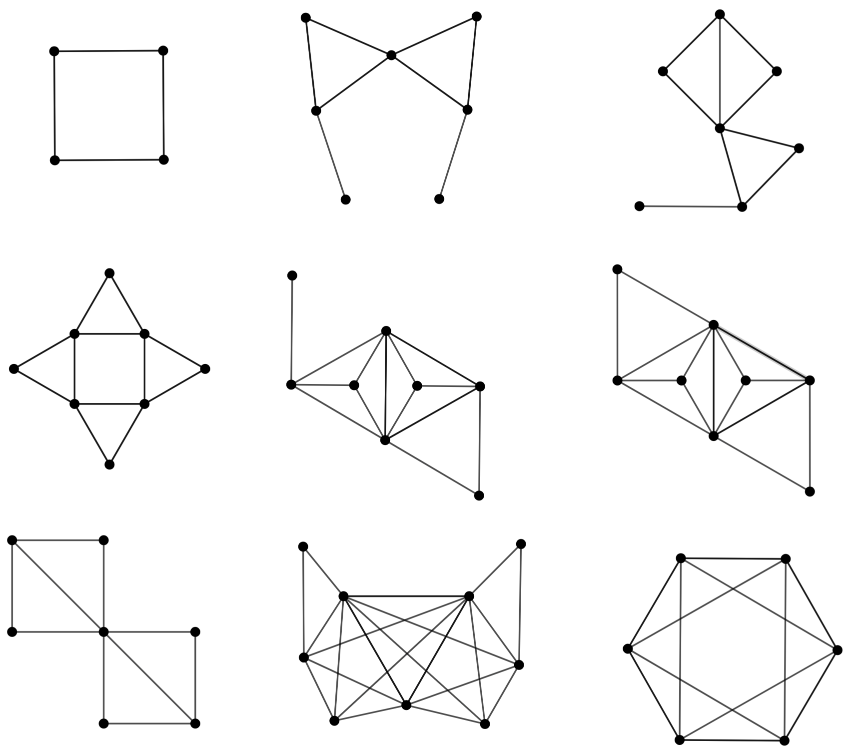

Figure 1: The small set of nine graphs which completes the classification of 4-clique coverable Dacey graphs. Note the last graph on the bottom row contains 8 distinct maximal cliques, the most possible among Dacey graphs for $m = 4$.

7 Structure Theorem

We look now at localizing the Dacey condition and investigate implications on Dacey graphs in general.

Definition 7.1. A graph G is *locally-Dacey* if for all $v \in V(G)$ the induced subgraph of G on the set $N(v)$ is Dacey.

Theorem 7.1. *Every Dacey graph G is a locally-Dacey graph.*

Proof. Assume, on the contrary, there exists $v \in V(G)$ such that $N(v)$, the graph induced by $N(v)$ in G, is not Dacey. This implies there exists some pair of distinct vertices $x, y \in V(N(v))$ such that $N(x) \cup N(y) \supseteq M$ in the graph $N(v)$ for some maximal clique $M \in N(v)$ and $(x, y) \notin E(N(v))$.

Considering the entire graph, the clique induced by $M \sqcup v = M' \subseteq G$ forms a maximal clique in G since there exist no vertex in G connected to all of the vertices in M'. But, then it must follow $N(x) \cup N(y) \supseteq M'$ in the graph G with $(x, y) \notin E(G)$ and $M' \in \mathcal{M}(G)$. So G is not Dacey, a contradiction. $\square$

The locally-Dacey condition is weaker than the Dacey condition. All Dacey graphs are locally-Dacey, but there exist locally-Dacey graphs which are not Dacey. In order to further investigate how local induced subgraphs apply to the Dacey condition on a graph as a whole, we introduce the concept of neighbors more than a graph theoretical distance of one away from some vertex.

Definition 7.2. Let v be some vertex in graph G. We define $N_d(v)$ in G to be the set of all vertices in G which are at most a graph-theoretic distance of d away from vertex v.

While $N(v)$ does not contain v, the set $N_1(v)$ does contain v. This leads to questions on the bounds of the Dacey condition's reach for each vertex in a graph.

Theorem 7.2. *If $G \backslash v$ is Dacey and v is such that the graph induced on $N_3(v)$ is Dacey, then G is Dacey.*

Proof. We restate the Dacey condition by its contrapositive. A graph G is Dacey if and only if there does not exist an induced subgraph B in G which consists of a maximal clique and two vertices which do not form an edge who collectively neighbor that maximal clique.

Given $G \backslash v$ is Dacey, we know such an induced subgraph B does not exist in $G \backslash v$. If we assume on the contrary, G is not Dacey, there must exist such an induced subgraph B in G. This implies B contains v. The maximum diameter of an induced sub graph which contains a maximal clique and two vertices connected to the maximal clique is 3. Since the maximum distance between any two vertices in B is 3 and v is in B, then B is an induced subgraph of G fully contained in $N_3(v)$. But, this makes $N_3(v)$ non-Dacey, a contradiction. So, G must be Dacey. $\square$

It is not true however that given graphs G and $G\backslash v$ are Dacey that $N_i(v)$ is necessarily Dacey for positive integers $i \geq 2$.

8 Future Work

Further work is needed on a more concise classification of m-clique coverable graphs which correspond to orthomodular posets. A relevant question is whether the set of nine graphs in Theorem 6.3 could be unified. Another modification of a graph G to consider is appending a vertex v so that v is adjacent to all the vertices of some maximal clique in G. We could consider repeating this process on all problematic maximal cliques of some non-Dacey graph G until it is Dacey. How this will be useful in an enumeration is not yet clear. Attempts to incorporate the Dacey condition to a set-theoretic property of collections of subsets of finite N were made to provide a characterization of Dacey induced subgraphs of PS_m, but more work is needed.

9 Conclusion

Simple, undirected, finite graphs serve as a compact representation of orthomodular posets. The correlation of graphs to induced subgraphs of PS_m graphs and related methods serve as a characterization of orthomodular posets which can be covered by m element, order coverings. With inflation, this offers a method of finding orthomodular posets which are covered by a certain number of Boolean algebras. We introduce a certain simplification of orthomodular posets which easily identifies those with the same underlying structure of maximal cliques. Inflation preserves this structure in a graph. Deflated graphs distinguish the equivalence classes which include infinitely many clique-distant graphs. Moreover, we find a complete classification of all clique-distant Dacey graphs which are up to 4-clique coverable, including structures which have not been widely studied in previous literature.

We also examine the effect of adding or cutting an atom from an orthomodular poset in the structure theorems. This gives some insight on the local nature of the Dacey condition, which is at the heart of what makes a graph correspond to an orthomodular poset. These general results on Dacey graphs help to characterize the effect of the Dacey condition on individual vertices.

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A Proof of Proposition 4.2

Proposition 4.2 *Two distinct vertices $u, v \in G$ are contained in precisely the same set of maximal cliques in G if and only if $u \sim v$.*

Proof. Given an (m, n) graph G, we enumerate $\mathcal{M}(G) = \{M_i \,|\, i \in N\}$ where $|N| = n$. Given $v \in V(G)$, we say v corresponds to a non-empty subset $S(v) \subseteq N$ where $i \in S(v)$ if and only if $M_i \ni v$. If we have $u \sim v$, then $M \ni u \iff M \ni v$ for every $M \in \mathcal{M}(G)$, so it follows $S(u) = S(v)$. To show the converse, assume $S(u) = S(v)$. Any vertex w adjacent to u is contained in some maximal clique $M \supseteq (w, u)$. Since $N(u) \sqcup \{u\} = N(v) \sqcup \{v\}$, we also have $v \in M$, so w is adjacent to v as well. So, we have $(u, v) \in E(G)$ and u and v must share precisely the same closed neighbor set. $\square$

B Proof of Proposition 6.1

Proposition 6.1 *Given a graph G which contains a vertex v such that every other vertex in G neighbors v, graph G is Dacey if and only if $G \backslash v$ is Dacey.*

Proof. For every maximal clique $M \in \mathcal{M}(G)$, we see $M \backslash v$ induces a maximal clique in $G \backslash v$, because no vertex in G neighbors all the vertices of $M \backslash v$. Similarly, for a maximal clique $M' \in \mathcal{M}(G \backslash v)$, the graph induced in G by the vertices $V(M') \sqcup v = M$ is a clique in G which is maximal by the maximality of M' in $G \backslash v$. Moreover, M is the only clique in G containing M', and v is the only vertex in G which is adjacent to every vertex in M'. This establishes a one-to-one correspondence between the sets $\mathcal{M}(G)$ and $\mathcal{M}(G \backslash v)$.

If graph $G \backslash v$ is Dacey, then for each pair of vertices $x, y \in V(G \backslash v)$ such that $N(x) \cup N(y)$ contains a maximal clique $M' \in \mathcal{M}(G \backslash v)$, we have the edge $(x, y) \in E(G \backslash v)$. If we now consider G containing vertex v, we see this is true for any pair of distinct vertices $u, v \in V(G)$ containing v. For each pair of vertices $u_1, u_2 \in V(G)$ each different from v, we note that if

$N(u_1) \cup N(u_2)$ contained a maximal clique $M' \in \mathcal{M}(G\backslash v)$, then with respect to G, the graph induced by $N(u_1) \cup N(u_2)$ contains the maximal clique M induced by the vertices $V(M') \sqcup v$. Moreover, in G the vertices in $N(u_1) \cup N(u_2)$ induce only the maximal cliques in G containing some maximal clique M' in $G\backslash v$. We see $(u_1, u_2) \in E(G)$ if and only if $(u_1, u_2) \in E(G\backslash v)$. Because two vertices different from v neighbor a maximal clique in G only if they neighbor a maximal clique in $G\backslash v$ and edges in $G\backslash v$ are present in G, graph G is Dacey only if $G\backslash v$ is Dacey.

By the same correspondence of maximal cliques between $G\backslash v$ and G, two vertices in $G\backslash v$ collectively neighbor a maximal clique in $G\backslash v$ only if the vertices collectively neighbor the corresponding maximal clique with v in G. Edges of $G\backslash v$ are always preserved in G as well, so given G is Dacey, we have $G\backslash v$ is Dacey. $\square$

C Program: Deflated Dacey Induced Subgraphs of PS_4

Runs on SageMath 9.1 with Python 3.7.

```
def Union(lst1, lst2):
    final_list = list(set(lst1) | set(lst2))
    return final_list
def Dacey(i):
    V = i.vertices()
    M = i.cliques_maximal()
    for u in V:
        for v in V:
            if i.has_edge(u,v) == False:
                for m in M:
                    if set(m).issubset(set(Union(i.neighbors(u), i.neighbors(v)))): return False
H=Graph({       1:[12, 13, 14, 123, 124, 134],
                2:[12, 23, 24, 123, 124, 234],
                3:[13, 23, 34, 123, 134, 234],
                4:[14, 24, 34, 234, 134, 124],
                12:[13, 14, 23, 24, 123, 124, 134, 234],
                13:[12, 14, 23, 34, 123, 124, 134, 234],
```

```
            14:[12, 13, 24, 34, 123, 124, 134, 234],
            23:[12, 13, 24, 34, 123, 124, 134, 234],
            24:[12, 14, 23, 34, 123, 124, 134, 234],
            34:[13, 14, 23, 24, 123, 124, 134, 234],
            123:[124, 134, 234],
            124:[123, 134, 234],
            134:[123, 124, 234],
            234:[123, 124, 134],
            1234:[1, 2, 3, 4, 12, 13, 14, 23, 24, 34, 123, 124, 134, 234]
        })
L = list(H.connected_subgraph_iterator())
D = []
for i in L:
    if not Dacey(i) == False:
        D.append(i)
for i in D:
    for j in D:
        if D.index(j)>D.index(i):
            if i.is_isomorphic(j) == True:
                D.remove(j)
for i in D:
    U = i.vertices()
    for u in U:
        for v in U:
            if u < v:
                if i.has_edge(u,v):
                    if set(i.neighbors(u, closed=True)) == set(i.neighbors(v, closed=True)):
                        i.delete_vertex(v); U = i.vertices()

for i in D:
    show(i)
```