On Q-binomial Polynomials and Quantum Integer-Valued Polynomials

Jason Liu

under the direction of

Robert Burklund
Department of Mathematics
Massachusetts Institute of Technology

Research Science Institute
July 30, 2018

Abstract

We study rings of compactified q-deformed integer-valued polynomials defined by Harman and Hopkins, along with another ring of multi-variable q-deformed integer-valued polynomials. We disprove a conjecture of theirs about how one such compactified ring may be generated andfind a basis for a generalization of these sets of polynomials. We also find a basis for the multi-variable analog of these polynomials.

Summary

We analyze certain polynomials that send modified integers to other types of modified integers. We find a characterization of such polynomials and disprove a conjecture about these polynomials made by previous authors. We also work with polynomials in multiple variables of roughly the same form and find a characterization for these polynomials as well.

1 Introduction

In the seventeenth century, Newton worked on polynomial interpolation in his *Principia Mathematica* [1, Book 3, Lemma 5, Case 1], where he studied the polynomial of lowest degree passing through a set of points. One such problem considered polynomials on lattice points, which led naturally to an attempt to characterize all polynomials that mapped integer inputs to integer outputs. These polynomials are known as integer-valued polynomials. Newton determined that any integer-valued polynomial could be uniquely expressed as the linear combination, with integer coefficients, of the set of binomial polynomials. These binomial polynomials are polynomials that output specific binomial coefficients when evaluated at integer values.

Pólya [2] and Ostrowski [3] formalized this theory in 1919, adding proofs where Newton neglected to include them. They focused on finding a "regular basis" for the set of integer-valued polynomials for general fields, including the one that Newton worked in. That is, they focused on finding a collection of integer-valued polynomials, one of each degree, such that any integer-valued polynomial could be uniquely represented as the linear combination with integer coefficients of polynomials in the basis. For more background on integer-valued polynomials, consult Cahen and Chabert's *Integer-Valued Polynomials* [4].

In 2016, Harman and Hopkins [5] defined q-deformations of these polynomials and discovered a number of properties about the q-binomial coefficients and some associated operations. A q-deformation is a modification of existing structures to include an auxiliary variable q. By convention, setting q as 1 should recover the normal definitions of the q-deformed objects. This paper answers some of the open questions at the end of Harman and Hopkins' paper (see Section 9 of [5]).

Harman and Hopkins [5] defined the q-binomial polynomials to produce specific q-binomial coefficients when evaluated at the q-integers and proved that these q-binomial polynomials satisfy a number of nice positivity properties. They proceed to define a q-deformation, $\mathcal{R}_q$, of the ring of integer-valued polynomials, denoted $\mathcal{R}$. These quantum integer-valued polynomials map the q-integers to polynomials in q and q^{-1}. Harman and Hopkins showed that this ring is spanned by the set of q-binomial polynomials. They found connections between the q-binomial coefficients and finite fields, finite Grassmannians, and the Young tableau [5]. For a more detailed discussion of q-deformations, see Section 2. Harman and Hopkins (see Section 4 of [5]) also noted connections between the set of quantum integer-valued polynomials and quantum groups, specifically the group $U_q(\mathfrak{sl}_2)$. These quantum groups, in turn, have connections to quantum mechanics [6, 7].

We extend Harman and Hopkins' results, but work with polynomials that map positive q-integers to polynomials in q and negative q-integers to polynomials in q^{-1}. They suggested that this ring, denoted $\mathcal{R}_q^+ \cap \mathcal{R}_q^-$, might be spanned, with integer coefficients, by the q-binomial polynomials. We show that this is not the case and prove a basis for $\mathcal{R}_q^+ \cap \mathcal{R}_q^-$.

In Section 2, we recall Harman and Hopkins' definitions of q-analogs, and find some counterexamples to their conjecture about $\mathcal{R}_q^+ \cap \mathcal{R}_q^-$. In Section 3, we recall some generalizations of $\mathcal{R}_q^+ \cap \mathcal{R}_q^-$ from Harman and Hopkins' work, and construct a basis on these generalizations. In Section 4, we consider multi-variable quantum integer-valued polynomials by extending Harman and Hopkins' definition of $\mathcal{R}_q$. We prove in this section a natural and regular basis for multi-variable quantum integer-valued polynomials that extends an analogous folklore result for multi-variable integer-valued polynomials. We proceed to extend Harman and Hopkins' definition of $\mathcal{R}_q^+$ and $\mathcal{R}_q^-$ to multiple dimensions in Section 5, and prove a natural and regular basis for these rings. Section 6 describes some avenues of further research.

2 q-deformations

2.1 Definitions and properties

We begin by formally defining the ring $\mathcal{R}$ of integer-valued polynomials, which is defined as

$$\mathcal{R} := \left\{ P(x) \in \mathbb{Q}[x] \middle| P(n) \in \mathbb{Z},\ \forall n \in \mathbb{Z} \right\}.$$

The set

$$\left\{ \binom{x}{k},\ k \in \mathbb{N}_0 \right\}$$

forms a regular basis for $\mathcal{R}$ as a $\mathbb{Z}$-algebra [2, 3].

We now recall the the definitions of the q-deformations of various important objects. The q-deformation of the integer n, denoted $[n]_q$, is given by $[n]_q = \sum_{i=0}^{n-1} q^i = \frac{q^n - 1}{q - 1}$. The q-deformation of the factorial $n!$ is denoted $[n]_q!$ and is given by $\prod_{i=1}^{n} [i]_q$. The q-binomial coefficient is defined as

$$\begin{bmatrix} n \\ k \end{bmatrix}_q := \frac{[n]_q!}{[k]_q![n-k]_q!}.$$

Note that setting $q = 1$ recovers the standard expressions for $n, n!$, and $\binom{n}{k}$.

The q-deformation of the kth binomial polynomial, denoted $\begin{bmatrix} x \\ k \end{bmatrix}$, is given by

$$\begin{bmatrix} x \\ k \end{bmatrix} := \frac{1}{q^{\binom{k}{2}} [k]_q!} \prod_{i=0}^{k-1} (x - [i]_q).$$

This expression is chosen so that $\begin{bmatrix} x \\ k \end{bmatrix}$ satisfies the property

$$\begin{bmatrix} [n]_q \\ k \end{bmatrix} = \begin{bmatrix} n \\ k \end{bmatrix}_q.$$

In order to define the q-analog of $\mathcal{R}$, let $\mathbb{Q}(q)$ be the set of rational expressions in q over $\mathbb{Q}$. Harman and Hopkins use $\mathbb{Q}(q)$ to define q-deformations of integer-valued polynomials, also known as quantum integer valued polynomials, by generalizing the definition of $\mathcal{R}$ to the rings

$$\mathcal{R}_q := \left\{ P(x) \in \mathbb{Q}(q)[x] \middle| P([n]_q) \in \mathbb{Z}[q, q^{-1}],\ \forall n \in \mathbb{Z} \right\},$$
$$\mathcal{R}_q^+ := \left\{ P(x) \in \mathbb{Q}(q)[x] \middle| P([n]_q) \in \mathbb{Z}[q],\ \forall n \in \mathbb{N}_0 \right\},$$
$$\mathcal{R}_q^- := \left\{ P(x) \in \mathbb{Q}(q)[x] \middle| P([-n]_q) \in \mathbb{Z}[q^{-1}],\ \forall n \in \mathbb{N}_0 \right\}.$$

The rings $\mathcal{R}_q^-$ and $\mathcal{R}_q^+$ can be thought of as a kind of positive part and negative part of $\mathcal{R}_q$, respectively. Harman and Hopkins then prove that the polynomials $\begin{bmatrix} x \\ k \end{bmatrix}$ form a basis for R_q and R_q^+ as a $\mathbb{Z}[q, q^{-1}]$ and $\mathbb{Z}[q]$ algebra, respectively.

Harman and Hopkins [5] also work with two involutions on $\mathcal{R}_q$, a shift operator S that maps $\mathbb{Q}(q)[x]$ to $\mathbb{Q}(q)[x]$ with $S(x) := qx + 1$, and a bar involution$^-$: $\mathbb{Q}(q)[x] \to \mathbb{Q}(q)[x]$ with $\bar{q} = q^{-1}$ and $\bar{x} = -qx$. These operators satisfy nice properties and arise in several identities about $\begin{bmatrix} x \\ k \end{bmatrix}$. For a list of some of these properties and other results of Harman and Hopkins, see Appendix A. The polynomials $\overline{\begin{bmatrix} x \\ k \end{bmatrix}}$, obtained by applying the bar involution to $\begin{bmatrix} x \\ k \end{bmatrix}$, form bases for $\mathcal{R}_q$ and $\mathcal{R}_q^-$ as a $\mathbb{Z}[q, q^{-1}]$ and $\mathbb{Z}[q^{-1}]$ algebra, respectively.

2.2 Properties of $\mathcal{R}_q^+ \cap \mathcal{R}_q^-$

The ring $\mathcal{R}_q^+ \cap \mathcal{R}_q^-$ has some very interesting properties. One reason why it is worth studying is its compactification property. In $\mathcal{R}_q^+$, it is reasonable to evaluate at $q = 0$, as all interesting expressions involved are in $\mathbb{Z}[q]$, but not to set $q = \infty$. Similarly, in $\mathcal{R}_q^-$, it is reasonable to evaluate at $q = \infty$ as all expressions are in $\mathbb{Z}[q^{-1}]$, but not to set $q = 0$. In $\mathcal{R}_q^+ \cap \mathcal{R}_q^-$, it is in some sense reasonable to set either $q = 0$ or $q = \infty$, thereby producing a compactification property. It is this property that leads the basis proven later for this set to have only a finite number of elements of each degree in x.

We examine further properties of $\mathcal{R}_q^+ \cap \mathcal{R}_q^-$ by applying the identities proven by Harman and Hopkins [5]. They had suggested that $\mathcal{R}_q^+ \cap \mathcal{R}_q^- = \mathrm{Span}_{\mathbb{Z}} \left\{ \begin{bmatrix} x \\ k \end{bmatrix}, \overline{\begin{bmatrix} x \\ k \end{bmatrix}},\ k \in \mathbb{N}_0 \right\}$. Note that they specify a linear span over $\mathbb{Z}$ instead of $\mathbb{Z}[q]$ or $\mathbb{Z}[q^{-1}]$ for the reason that while $\begin{bmatrix} x \\ k \end{bmatrix}$ and $\overline{\begin{bmatrix} x \\ k \end{bmatrix}}$ lie in $\mathcal{R}_q^+ \cap \mathcal{R}_q^-$, $q^{-1} \begin{bmatrix} x \\ k \end{bmatrix}$ and $q \overline{\begin{bmatrix} x \\ k \end{bmatrix}}$ do not. However, we have found polynomials in $\mathcal{R}_q^+ \cap \mathcal{R}_q^-$

that do not lie in $\mathrm{Span}_{\mathbb{Z}}\left\{\begin{bmatrix}x\\k\end{bmatrix}, \overline{\begin{bmatrix}x\\k\end{bmatrix}},\ k \in \mathbb{N}_0\right\}$, as demonstrated in the Proposition 2.1 and the following example.

Proposition 2.1. *For nonnegative integers k and integers m, we have*

$$q^m \begin{bmatrix}x\\k\end{bmatrix} \in \begin{cases} \mathcal{R}_q^- & \text{iff } m \le \binom{k+1}{2} \\ \mathcal{R}_q^+ & \text{iff } 0 \le m. \end{cases} \quad \text{and } q^{-m}\overline{\begin{bmatrix}x\\k\end{bmatrix}} \in \begin{cases} \mathcal{R}_q^+ & \text{iff } m \le \binom{k+1}{2} \\ \mathcal{R}_q^- & \text{iff } m \ge 0. \end{cases}$$

For the proof of Proposition 2.1, see Appendix B.

Many of the polynomials in Proposition 2.1 are not actually in

$$\mathrm{Span}_{\mathbb{Z}}\left\{\begin{bmatrix}x\\k\end{bmatrix}, \overline{\begin{bmatrix}x\\k\end{bmatrix}},\ k \in \mathbb{N}_0\right\}.$$

We provide an example with $q\begin{bmatrix}x\\2\end{bmatrix}$. We note that $\overline{\begin{bmatrix}x\\2\end{bmatrix}} = q^5\begin{bmatrix}x\\2\end{bmatrix} + q^3\begin{bmatrix}x\\1\end{bmatrix}$ by Proposition 6.3 of [5]. Because $\overline{\begin{bmatrix}x\\2\end{bmatrix}}$, $\begin{bmatrix}x\\2\end{bmatrix}$, and $q\begin{bmatrix}x\\2\end{bmatrix}$ have coefficients of $\begin{bmatrix}x\\2\end{bmatrix}$ with different degrees in q, we know that $q\begin{bmatrix}x\\2\end{bmatrix}$ cannot be written as a linear combination of the first two with integer coefficients.

These polynomials in Proposition 2.1, however, are not the only polynomials in $\mathcal{R}_q^+ \cap \mathcal{R}_q^-$. We find a basis for $\mathcal{R}_q^+ \cap \mathcal{R}_q^-$, and also a basis for a generalization of this set.

3 A basis for a generalization of $\mathcal{R}_q^+ \cap \mathcal{R}_q^-$

Before defining a basis for $\mathcal{R}_q^+ \cap \mathcal{R}_q^-$, we first define some generalizations of these rings. Other than the basic rings $\mathcal{R}_q^{+,0} = \mathcal{R}_q^+$ and $\mathcal{R}_q^{-,0} = \mathcal{R}_q^-$, Harman and Hopkins [5] also defined the rings $\mathcal{R}_q^{+,m}$ and $\mathcal{R}_q^{-,m}$ with

$$\mathcal{R}_q^{+,m} := \{P(x) \in \mathbb{Q}(q)[x] : P([n]_q) \in \mathbb{Z}[q],\ \forall n \ge m\} \text{ and}$$

$$\mathcal{R}_q^{-,m} := \left\{P(x) \in \mathbb{Q}(q)[x] : P([n]_q) \in \mathbb{Z}[q^{-1}],\ \forall n \le m\right\}.$$

Recall Harman and Hopkins' [5] definition of the shift operator S, which sends x to $qx+1$. Harman and Hopkins' [5] Proposition 5.1 shows that S^m is an isomorphism from $\mathcal{R}_q^+$ to $\mathcal{R}_q^{+,-m}$ and an isomorphism from $\mathcal{R}_q^-$ to $\mathcal{R}_q^{-,-m}$.

Also recall from Harman and Hopkins' paper [5, Proposition 6.3] that

$$\overline{\begin{bmatrix} x \\ k \end{bmatrix}} = (-1)^k q^{\binom{k+1}{2}} S^{k-1} \begin{bmatrix} x \\ k \end{bmatrix}.$$

As such, $\left\{S^{-m} \left[\begin{smallmatrix} x \\ k \end{smallmatrix}\right],\ k \in \mathbb{N}_0\right\}$ is a basis for $\mathcal{R}_q^{+,m}$ as a $\mathbb{Z}[q]$-algebra, and $\left\{q^{\binom{k+1}{2}} S^{k-1-m} \left[\begin{smallmatrix} x \\ k \end{smallmatrix}\right],\ k \in \mathbb{N}_0\right\}$ is a basis for $\mathcal{R}_q^{-,m}$ as a $\mathbb{Z}[q^{-1}]$-algebra.

We now define the sets

$$\mathrm{Bar}(k) := \left\{ q^i \begin{bmatrix} x \\ k \end{bmatrix},\ 0 \le i \le \binom{k+1}{2} \right\},$$

$$\mathrm{X}(k) := \left\{ q^i \begin{bmatrix} x \\ k \end{bmatrix},\ 0 \le i < k \right\},\ \text{and}$$

$$\mathrm{Y}(k) := \left\{ \sum_{i=0}^{m} (-1)^i q^{\binom{i}{2}} \begin{bmatrix} k-m+i \\ k-m \end{bmatrix}_q \begin{bmatrix} x \\ k-m+i \end{bmatrix},\ 0 \le m \le k \right\}.$$

These sets will be used to form a basis for $\mathcal{R}_q^{+,s} \cap \mathcal{R}_q^{-,r}$.

Note that the sets $\mathrm{X}(k), S\,\mathrm{X}(k), S^2\,\mathrm{X}(k), \ldots, S^{m-1}\,\mathrm{X}(k), S^m\,\mathrm{Bar}(k)$ have different orders in q on their $\left[\begin{smallmatrix} x \\ k \end{smallmatrix}\right]$ terms, which are shown in Figure 1. See Figure 2 to understand the notation in Figure 1.

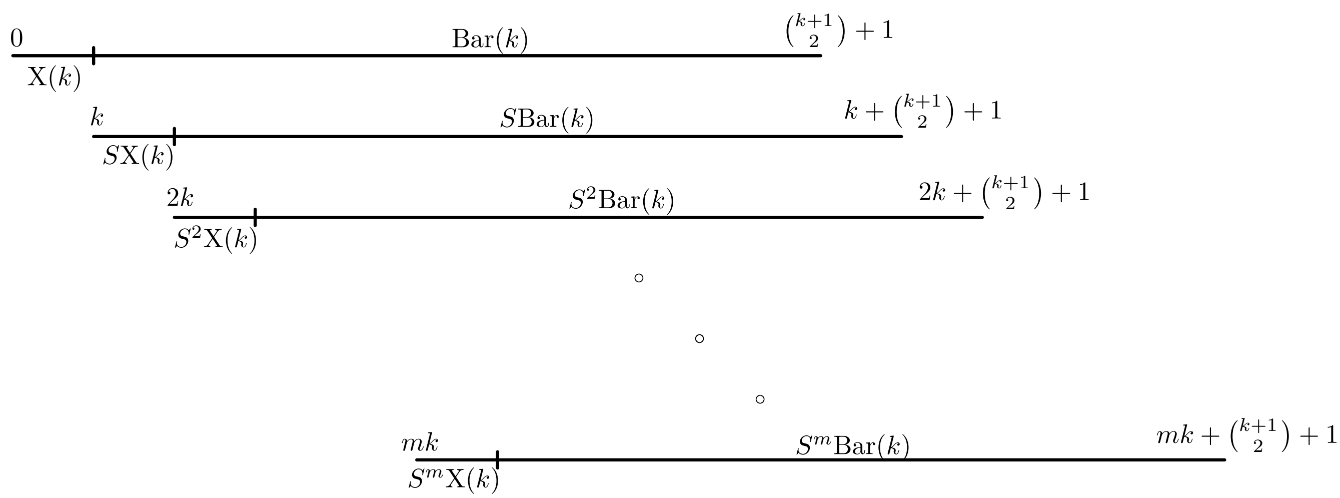

Figure 1: The orders of the polynomials in the sets $\mathrm{Bar}(k), S\,\mathrm{Bar}(k), \ldots, S^m\,\mathrm{Bar}(k)$ and $\mathrm{X}(k), S\,\mathrm{X}(k), \ldots, S^m\,\mathrm{X}(k)$.

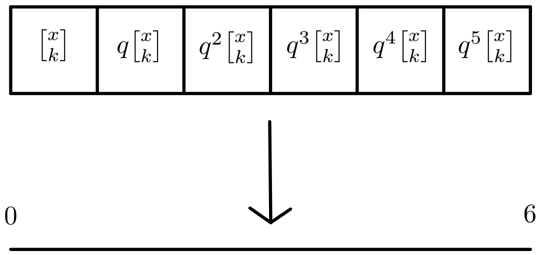

Figure 2: A depiction of the meaning of the lines in Figure 1

Note that $\mathrm{X}(k)$ is essentially the non-overlapping portion of $\mathrm{Bar}(k)$, that is, the part of $\mathrm{Bar}(k)$ that doesn't overlap with $S\,\mathrm{Bar}(k)$.

In order to show that sets of the form $S^m\,\mathrm{Bar}(k)$, $S^m\,\mathrm{X}(k)$, and $S^m\,\mathrm{Y}(k)$ can be used to form a basis, we must first show that they lie in the set $\mathcal{R}_q^{+,s} \cap \mathcal{R}_q^{-,r}$ for appropriate values of m and k.

Lemma 3.1. *For nonnegative integers k, we have the relations*

$$\begin{tikzcd}
\mathrm{X}(k) \arrow[r, hook] & \mathrm{Bar}(k) \arrow[r, hook] & \mathcal{R}_q^{+} \cap \mathcal{R}_q^{-,k-1} \\
& \mathrm{Y}(k) \arrow[r, hook] & \mathcal{R}_q^{+} \cap \mathcal{R}_q^{-,k}. \arrow[u, hook]
\end{tikzcd}$$

For the proof of Lemma 3.1, see Appendix C.

By repeatedly applying the shift operator to Lemma 3.1, we arrive at the following corollary.

Corollary 1. *For integer m and nonnegative k, we have the relations*

$$\begin{tikzcd}
S^m\,\mathrm{X}(k) \arrow[r, hook] & S^m\,\mathrm{Bar}(k) \arrow[r, hook] & \mathcal{R}_q^{+,-m} \cap \mathcal{R}_q^{-,k-1-m} \\
& S^m\,\mathrm{Y}(k) \arrow[r, hook] & \mathcal{R}_q^{+,-m} \cap \mathcal{R}_q^{-,k-m}. \arrow[u, hook]
\end{tikzcd}$$

To create a meaningful basis, we must have sets that are linearly independent. Before showing that the elements in the sets $S^m\,\mathrm{Bar}(k), S^m\,\mathrm{X}(k)$, and $S^m\,\mathrm{Y}(k)$ are linearly independent, we first show that these sets are disjoint under certain conditions on the respective values of k and m for the sets.

Lemma 3.2. *For any integers r, s, the sets of the form $S^{m-s}\,\mathrm{X}(k)$ or $S^{k-r-1}\,\mathrm{Bar}(k)$ with $\max(0, r-s+1) \leq k$ and $0 \leq m \leq k+s-r-2$ are pairwise disjoint. When $r \geq s$, the set $S^{-s}\,\mathrm{Y}(r-s)$ is also disjoint from the previous sets.*

For the proof of Lemma 3.2, see Appendix D.

We define the set

$$\mathrm{Z}(r,s) := \bigcup_{k=\max(0,r-s+1)}^{\infty} \left(S^{k-r-1}\,\mathrm{Bar}(k) \cup \bigcup_{m=0}^{k+s-r-2} S^{m-s}\,\mathrm{X}(k) \right) \cup \begin{cases} \emptyset & r < s \\ S^{-s}\,\mathrm{Y}(r-s) & r \geq s \end{cases}$$

and note that Lemma 3.2 essentially states that there is no overlap between the sets making up $\mathrm{Z}(r,s)$.

Lemma 3.3. *For any integers r, s, the elements of $\mathrm{Z}(r,s)$ are linearly independent over $\mathbb{Z}$.*

For the proof of Lemma 3.3, see Appendix E.

We now have the tools to prove a basis for $\mathcal{R}_q^{+,s} \cap \mathcal{R}_q^{-,r}$.

Theorem 3.4. *The set $\mathrm{Z}(r,s)$ forms a basis for $\mathcal{R}_q^{+,s} \cap \mathcal{R}_q^{-,r}$ as a $\mathbb{Z}$-algebra.*

Proof. By Lemma 3.3, we need only to show that $\mathrm{Z}(r,s)$ spans $\mathcal{R}_q^{+,s} \cap \mathcal{R}_q^{-,r}$ as a $\mathbb{Z}$-algebra.

Suppose, for the sake of contradiction, that we have some polynomial

$$P(x) = \sum_{i=0}^{k} \sum_{j=0}^{m_i} \alpha_{i,j} q^j S^{-s} \begin{bmatrix} x \\ i \end{bmatrix} = \sum_{i=0}^{k} f_i(q) S^{-s} \begin{bmatrix} x \\ i \end{bmatrix} \in \mathcal{R}_q^{+,s} \cap \mathcal{R}_q^{-,r}$$

that is not in the span of $\mathrm{Z}(r,s)$. Note that all elements in $\mathcal{R}_q^{+,s}$ are of same form as $P(x)$ with $m_i \geq 0$, so it is reasonable to assume that $P(x)$ is of this form. In this proof, we require that α_{k,m_k} is nonzero and pick $P(x)$ such that k is minimal and m_k is minimal among polynomials with minimal k. We split the problem into three cases based on the relative sizes of k and $r-s$.

Case 1: We assume $k > r-s$. We prove that $0 \leq m_k \leq \binom{k+1}{2} + (k+s-r-1)k$. Because $m_i \geq 0$ for all $0 \leq i \leq k$, we know that $m_k \geq 0$. We note that because $P(x)$ lies in $\mathcal{R}_q^{+,s} \cap \mathcal{R}_q^{-,r}$, the polynomial must also lie in

$$\mathcal{R}_q^{-,r} = \operatorname{Span}_{\mathbb{Z}[q^{-1}]} \left\{ S^{-r}\overline{\begin{bmatrix} x \\ k \end{bmatrix}},\ k \in \mathbb{N}_0 \right\}.$$

The degree in q of the coefficient on $S^{-s}\begin{bmatrix} x \\ k \end{bmatrix}$ for

$$S^{-r}\overline{\begin{bmatrix} x \\ k \end{bmatrix}} = (-1)^k q^{\binom{k+1}{2}} S^{k-1+s-r} \left(S^{-s}\begin{bmatrix} x \\ k \end{bmatrix} \right)$$

must be $\binom{k+1}{2} + k(k-1+s-r)$. Therefore, no term in $\mathcal{R}_q^{-,r}$ can have a higher degree of q for the coefficient of its $S^{-s}\begin{bmatrix} x \\ k \end{bmatrix}$ term. As such, we have $0 \le m_k \le \binom{k+1}{2} + k(k-1+s-r)$, as claimed.

We note that there are $\binom{k+1}{2} + 1 + k(k-1+s-r)$ polynomials in the set

$$S^{k-r-1}\,\mathrm{Bar}(k) \cup \bigcup_{m=0}^{k+s-r-2} S^{m-s}\,\mathrm{X}(k)$$

whose coefficients on their $S^{-s}\begin{bmatrix} x \\ k \end{bmatrix}$ terms all have different degrees in q. Furthermore, these degrees are integers between 0 and $\binom{k+1}{2} + k(k-1+s-r)$, inclusive, so each such integer must show up as a degree. As such, there must be a polynomial $P_{m_k,k}(x)$ in $\mathrm{Z}(r,s)$ with leading term $q^{m_k}S^{-s}\begin{bmatrix} x \\ k \end{bmatrix}$. The polynomial $P(x) - \alpha_{k,m_k}P_{m_k,k}(x)$ therefore has no $q^{m_k}S^{-s}\begin{bmatrix} x \\ k \end{bmatrix}$ term but still lies in $\mathcal{R}_q^{+,s} \cap \mathcal{R}_q^{-,r}$. By our conditions on $P(x)$ this new polynomial must lie in the linear span of the elements of $\mathrm{Z}(r,s)$. As such, as α_{k,m_k} is an integer, the polynomial $P(x)$ can also be written as the linear combination of elements in $\mathrm{Z}(r,s)$ with integer coefficients, contradicting our earlier assumption.

Case 2: We assume $k = r - s$. Due to space constraints, we will move this proof to Appendix F. In the proof, we find an explicit linear combination of polynomials in $S^{-s}\,\mathrm{Y}(k)$ that evaluates to $P(x)$.

Case 3: We assume $k < r-s$. Due to space constraints, we will move this proof to Appendix F. As a brief sketch of the proof, we find the degrees of each $f_i(q)$ and use these to show that $P([k+1]_q)$ is not an integer unless P is constant.

Because we cannot have any of $k < r-s$, $k = r-s$, or $k > r-s$, it must be the case that $P(x)$ does not exist, so the set $\mathrm{Z}(r,s)$ does indeed form a basis for $\mathcal{R}_q^{+,s} \cap \mathcal{R}_q^{-,r}$ as a $\mathbb{Z}$-algebra. $\square$

4 Multi-variable quantum integer-valued polynomials

A natural extension of many problems is to consider the multi-variable equivalent of the problem.

We start by defining the multi-variable binomial polynomials

$$\binom{\mathbf{x}}{\mathbf{k}} := \prod_{i=1}^{d} \binom{x_i}{k_i}$$

where $\mathbf{x} = (x_1, x_2, \dots, x_d)$ has real coordinates and $\mathbf{k} = (k_1, k_2, \dots, k_d)$ has nonnegative integer coordinates.

As with the multi-variable binomial polynomials, we can define multi-variable q-binomial polynomials as follows.

Definition 4.1. For $\mathbf{x} = (x_1, x_2, \dots, x_d) \in \mathbb{Z}[q, q^{-1}]^d$, $\mathbf{n} = (n_1, n_2, \dots, n_d) \in \mathbb{Z}^d$, and $\mathbf{k} = (k_1, k_2, \dots, k_d) \in \mathbb{N}_0^d$, we define

$$\begin{bmatrix} \mathbf{x} \\ \mathbf{k} \end{bmatrix} = \prod_{i=1}^{d} \begin{bmatrix} x_i \\ k_i \end{bmatrix} \text{ and } \begin{bmatrix} \mathbf{n} \\ \mathbf{k} \end{bmatrix}_q = \prod_{i=1}^{d} \begin{bmatrix} n_i \\ k_i \end{bmatrix}_q.$$

We also define

$$\mathbf{x}^{\mathbf{k}} = \prod_{i=1}^{d} x_i^{k_i}$$

for integers k_i and $[\mathbf{n}]_q = ([n_1]_q, [n_2]_q, \dots, [n_d]_q)$ for $\mathbf{n} \in \mathbb{Z}^d$.

For regular integer-valued polynomials, it is natural to define the ring $\mathcal{R}^d$ as

$$\mathcal{R}^d = \left\{ P(\mathbf{x}) \in \mathbb{Q}(q)[\mathbf{x}] : P(\mathbf{x}) \in \mathbb{Z},\ \forall \mathbf{x} \in \mathbb{Z}^d \right\}.$$

It is a classical forklore result that this ring has a basis given by $\left\{ \binom{\mathbf{x}}{\mathbf{k}},\ \mathbf{k} \in \mathbb{N}_0^d \right\}$.

A natural extension of considering the q-deformations of integer-valued polynomials is to examine the q-deformation of multi-variable integer-valued polynomials, $\mathcal{R}_q^d$.

Definition 4.2. We define

$$\mathcal{R}_q^d := \left\{ P(\mathbf{x}) \in \mathbb{Q}(q)[\mathbf{x}] : P([\mathbf{n}]_q) \in \mathbb{Z}[q, q^{-1}] \ \forall \mathbf{n} \in \mathbb{Z}^d \right\},$$

where we have $\mathbf{x} = (x_1, x_2, \ldots, x_d) \in \mathbb{Z}[q, q^{-1}]^d$.

We show that the basis for this ring is the set of multi-variable q-binomial coefficients.

Theorem 4.1. *The ring $\mathcal{R}_q^d$ has a basis given by*

$$\left\{ \begin{bmatrix} \mathbf{x} \\ \mathbf{k} \end{bmatrix}, k_i \geq 0 \right\}$$

as a $\mathbb{Z}[q, q^{-1}]$-module.

Proof. We start by showing that each polynomial of the form $\begin{bmatrix} \mathbf{x} \\ \mathbf{k} \end{bmatrix}$ is in $\mathcal{R}_q^d$.

When we plug in values $[n_i]_q$ for x_i, we find that $\begin{bmatrix} n_i \\ k_i \end{bmatrix}_q$ is in $\mathbb{Z}[q, q^{-1}]$, so the product of all these binomial coefficients is also in $\mathbb{Z}[q, q^{-1}]$. As such, $\begin{bmatrix} \mathbf{x} \\ \mathbf{k} \end{bmatrix}$ is in $\mathcal{R}_q^d$ for any $\mathbf{k} \in \mathbb{N}_0^d$.

Next, we show that the polynomials $\begin{bmatrix} \mathbf{x} \\ \mathbf{k} \end{bmatrix}$ are linearly independent. Note that the highest degree in x_i of any term in $\begin{bmatrix} x_i \\ k_i \end{bmatrix}$ is k_i, so the highest degree term of any polynomial $\begin{bmatrix} \mathbf{x} \\ \mathbf{k} \end{bmatrix}$, up to rational functions in q, is $\mathbf{x}^{\mathbf{k}}$. If the polynomials $\begin{bmatrix} \mathbf{x} \\ \mathbf{k} \end{bmatrix}$ are not independent, then there exist expressions $f_{\mathbf{k}}(q) \in \mathbb{Z}[q, q^{-1}]$ such that

$$\sum_{\mathbf{k} \in \mathbb{N}_0^d} f_{\mathbf{k}}(q) \begin{bmatrix} \mathbf{x} \\ \mathbf{k} \end{bmatrix} = 0.$$

If we consider the term with a nonzero coefficient $f_{\mathbf{k}}(q)$ and the largest value of $\sum_{i=1}^{d} k_i$, we have some element $\begin{bmatrix} \mathbf{x} \\ \mathbf{k} \end{bmatrix}$ whose $\mathbf{x}^{\mathbf{k}}$ term cancels, which implies that it must have a coefficient of 0, contradiction. Thus, the polynomials $\begin{bmatrix} \mathbf{x} \\ \mathbf{k} \end{bmatrix}$ must be linearly independent.

We now show that these polynomials span the ring $\mathcal{R}_q^d$. Suppose we have a polynomial $P(\mathbf{x}) \in \mathcal{R}_q^d$ with maximal degree k. We wish to show that $P(\mathbf{x})$ must be equal to some polynomial $P_{(k+1)^d}(\mathbf{x})$ that is in the set

$$\mathrm{Span}_{\mathbb{Z}[q,q^{-1}]}\left\{ \begin{bmatrix} \mathbf{x} \\ \mathbf{k} \end{bmatrix},\ \mathbf{k} \in \mathbb{N}_0^d \right\}.$$

We consider a sequence of distinct points $\mathbf{n}_1, \mathbf{n}_2, \ldots, \mathbf{n}_{(k+1)^d}$ in $\mathbb{Z}^d$ with positive integer coordinates between 0 and k such that the sum of the coordinates of $\mathbf{n}_i$ is less than or equal to the sum of the coordinates of $\mathbf{n}_j$ for any $i \leq j$. We also define $\mathbf{x}_i = [\mathbf{n}_i]_q$ for integers $1 \leq i \leq (k+1)^d$. We define the polynomials $P_1(\mathbf{x}), P_2(\mathbf{x}), \ldots, P_{(k+1)^d}(\mathbf{x})$ such that $P_1(\mathbf{x}) = P(\mathbf{x}_1)$ and

$$P_{j+1}(\mathbf{x}) = P_j(\mathbf{x}) + (P(\mathbf{x}_{j+1}) - P_j(\mathbf{x}_{j+1})) \begin{bmatrix} \mathbf{x} \\ \mathbf{n}_{j+1} \end{bmatrix}$$

for all $j < (k+1)^d$.

Because the sum of the coordinates of $\mathbf{n}_{j+1}$ is at least the sum of the coordinates of $\mathbf{n}_i$ for any $i \leq j$, there is an integer l with $1 \leq l \leq d$ such that the lth coordinate of $\mathbf{n}_{j+1}$ is larger than that of $\mathbf{n}_i$, so $\left[\begin{smallmatrix} \mathbf{x}_i \\ \mathbf{n}_{j+1} \end{smallmatrix}\right] = 0$ for all $i \leq j$. As such, $P_{j+1}(\mathbf{x}_i) = P_j(\mathbf{x}_i)$ for all $j \geq i$. At the same time, we know that $P_j(\mathbf{x}_j) = P_{j-1}(\mathbf{x}_j) + P(\mathbf{x}_j) - P_{j-1}(\mathbf{x}_j)$, which implies that $P_j(\mathbf{x}_l) = P(\mathbf{x}_l)$ for all $j \geq l$.

We show by induction on j that for $\mathbf{x} = [\mathbf{n}]_q$ with $\mathbf{n} \in \mathbb{Z}^d$, we have $P_j(\mathbf{x})$ is in $\mathbb{Z}[q, q^{-1}]$. To do so, we start with the observation that $P_1(\mathbf{x}) = P(\mathbf{x}_1)$ is always in $\mathbb{Z}[q, q^{-1}]$. If the polynomial $P_j(\mathbf{x})$ maps relevant values of $\mathbf{x}$ to $\mathbb{Z}[q, q^{-1}]$, then it must map $\mathbf{x}_{j+1}$ to $\mathbb{Z}[q, q^{-1}]$. Therefore, as the q-binomial polynomials map $[n]_q$ to $\mathbb{Z}[q, q^{-1}]$, the product

$$(P(\mathbf{x}_{j+1}) - P_j(\mathbf{x}_{j+1})) \begin{bmatrix} \mathbf{x} \\ \mathbf{x}_{j+1} \end{bmatrix}$$

maps relevant values of $\mathbf{x}$ to $\mathbb{Z}[q, q^{-1}]$. As such, the polynomial

$$P_{j+1}(\mathbf{x}) = P_j(\mathbf{x}) + (P(\mathbf{x}_{j+1}) - P_j(\mathbf{x}_{j+1})) \begin{bmatrix} \mathbf{x} \\ \mathbf{x}_{j+1} \end{bmatrix}$$

must also map relevant values of $\mathbf{x}$ to $\mathbb{Z}[q, q^{-1}]$, as claimed.

We now show that the polynomial $P_j(\mathbf{x})$ is an element of $\mathcal{R}_q^d$. Because we already know that $P_j(\mathbf{x})$ maps relevant values of $\mathbf{x}$ to $\mathbb{Z}[q,q^{-1}]$, we need only to show that $P_j(\mathbf{x})$ lies in $\mathbb{Q}(q)[\boldsymbol{x}]$. To do so, we proceed by induction on j, noting that $P_1(\mathbf{x}) = P(\mathbf{x}_1)$ is a constant expression in $\mathbb{Z}[q,q^{-1}]$, and therefore must lie in $\mathbb{Q}(q)[\mathbf{x}]$. We now suppose that the polynomial $P_j(\mathbf{x})$ lies in $\mathcal{R}_q^d$. Because $P_j(\mathbf{x}), P(\mathbf{x}) \in \mathcal{R}_q^d$, we know that $P(\mathbf{x}_{j+1}) - P_j(\mathbf{x}_{j+1})$ lies in $\mathbb{Z}[q,q^{-1}]$, while $\left[\begin{smallmatrix}\mathbf{x}\\ \mathbf{x}_{j+1}\end{smallmatrix}\right]$ lies in $\mathcal{R}_q^d$ because $\left[\begin{smallmatrix}x\\ k\end{smallmatrix}\right]$ is an element of $\mathcal{R}_q$. Thus, the product of these two terms is also in $\mathcal{R}_q^d$, so $P_{j+1}(\mathbf{x})$ must be in $\mathcal{R}_q^d$ as well, as desired.

We now know that $P_{(k+1)^d}(\mathbf{x})$ is in

$$\mathrm{Span}_{\mathbb{Z}[q,q^{-1}]}\left\{\begin{bmatrix}\mathbf{x}\\ \mathbf{x}_j\end{bmatrix},\ 1 \le j \le (k+1)^d\right\}.$$

As such, we have reduced our theorem to showing that $P_{(k+1)^d}(\mathbf{x}) = P(\mathbf{x})$.

To prove that $P_{(k+1)^d}(\mathbf{x}) = P(\mathbf{x})$ for all $\mathbf{x}$, we show that by induction on d that their difference is precisely the zero polynomial. We first note that if $d = 1$, then we have two polynomials in one variable that are equal at $k+1$ points, which means that their difference has $k+1$ zeros and therefore is the zero polynomial. Next, we show that if $P_{(k+1)^{d-1}}(\mathbf{x}) = P(\mathbf{x})$ for any polynomial $P(\mathbf{x})$ in $\mathcal{R}_q^{d-1}$, then $P_{(k+1)^d}(\mathbf{x}) = P(\mathbf{x})$ for any polynomial $P(\mathbf{x}) \in \mathcal{R}_q^d$. By setting $x_1, x_2, x_3, \ldots, x_{d-1}$ to values in $[0]_q, [1]_q, [2]_q, \ldots, [k]_q$, the resulting polynomial in x_d has degree at most k and evaluates to 0 at $k+1$ distinct values of x_d. As such, this polynomial is the zero polynomial no matter what values in $[0]_q, [1]_q, [2]_q, \ldots, [k]_q$ we plug in for $x_1, x_2, \ldots, x_{d-1}$. As such, when we consider the polynomial in $\mathcal{R}_q^{d-1}$ given by $P(x_1, x_2, \ldots, x_{d-1}, [n]_q)$ for any fixed $n \in \mathbb{Z}$, we know it is the zero polynomial by our inductive hypothesis. Thus, the difference $P_{(k+1)^d}(\mathbf{x}) - P(\mathbf{x})$ is precisely the zero polynomial, as claimed. $\square$

Other than generalizing $\mathcal{R}_q$ to d dimensions, we can also generalize $\mathcal{R}_q^+$ and $\mathcal{R}_q^-$, although these generalizations don't translate as nicely into non-q-deformed sets.

5 Generalizations of $\mathcal{R}_q^+$, $\mathcal{R}_q^-$ to d dimensions

We define the following d-dimensional generalizations of $\mathcal{R}_q^+$, $\mathcal{R}_q^-$, $\mathcal{R}_q^{+,s}$, and $\mathcal{R}_q^{-,r}$.

Definition 5.1. We define

$$\mathcal{R}_q^{d,+,\mathbf{s}} := \{P(\mathbf{x}) \in \mathbb{Q}(q)[\mathbf{x}] : P([\mathbf{n}]_q) \in \mathbb{Z}[q],\ n_i \geq s_i,\ 1 \leq i \leq d\},$$

where $[\mathbf{n}]_q = ([n_1]_q, [n_2]_q, \ldots, [n_d]_q)$ with $\mathbf{n} \in \mathbb{Z}^d$, $\mathbf{s} \in \mathbb{Z}^d$, and $\mathbf{x} \in \mathbb{Z}[q, q^{-1}]^d$.

Definition 5.2. We define

$$\mathcal{R}_q^{d,-,\mathbf{r}} := \left\{P(\mathbf{x}) \in \mathbb{Q}(q)[\mathbf{x}] : P([\mathbf{n}]_q) \in \mathbb{Z}[q^{-1}],\ n_i \geq s_i,\ 1 \leq i \leq d\right\},$$

where $[\mathbf{n}]_q = ([n_1]_q, [n_2]_q, \ldots, [n_d]_q)$ with $\mathbf{n} \in \mathbb{Z}^d$, $\mathbf{r} \in \mathbb{Z}^d$, and $\mathbf{x} \in \mathbb{Z}[q, q^{-1}]^d$.

We further define that

$$\mathcal{R}_q^{d,+} := \mathcal{R}_q^{d,+,\mathbf{0}}, \text{ and } \mathcal{R}_q^{d,-} := \mathcal{R}_q^{d,-,\mathbf{0}},$$

where we define $\mathbf{c} = (c, c, \ldots, c)$ for any constant c. We also define $\mathbf{n} + c = \mathbf{n} + \mathbf{c}$ for any constant c.

We extend the shift operator S to define

$$S^{\mathbf{m}} P(\mathbf{x}) = \left(\prod_{i=1}^{d} S_i^{m_i}\right) P(\mathbf{x})$$

where S_i acts only on x_i. Note that S^i is still an isomorphism on $\mathcal{R}_q^d$. The bar involution is still defined to act on all x_i.

We start by finding some bases for $\mathcal{R}_q^{d,+,\mathbf{s}}$ and $\mathcal{R}_q^{d,-,\mathbf{r}}$.

Lemma 5.1. *We have that* $\mathcal{R}_q^{d,+} \subseteq \mathcal{R}_q^d$.

For the proof of Lemma 5.1, see Appendix G.

Theorem 5.2. *We have that*

1. *The ring* $\mathcal{R}_q^{d,+,\mathbf{s}}$ *has a basis given by*

$$\left\{ S^{-\mathbf{s}} \begin{bmatrix} \mathbf{x} \\ \mathbf{k} \end{bmatrix}, \mathbf{k} \in \mathbb{N}_0^d \right\}$$

   as a $\mathbb{Z}[q]$*-module.*

2. *The ring* $\mathcal{R}_q^{d,-,\mathbf{r}}$ *has a basis given by*

$$\left\{ S^{-\mathbf{r}} \overline{\begin{bmatrix} \mathbf{x} \\ \mathbf{k} \end{bmatrix}}, \mathbf{k} \in \mathbb{N}_0^d \right\}$$

   as a $\mathbb{Z}[q^{-1}]$*-module.*

Proof. We note that $\mathcal{R}_q^{d,+,\mathbf{s}} = S^{-\mathbf{s}} \mathcal{R}_q^{d,+}$, so the first part of the theorem reduces to showing that $\left\{ \left[\begin{smallmatrix} \mathbf{x} \\ \mathbf{k} \end{smallmatrix}\right], k \in \mathbb{N}_0^d \right\}$ forms a basis for $\mathcal{R}_q^{d,+}$ as a $\mathbb{Z}[q]$-algebra. We note that by Lemma 5.1, the set $\mathcal{R}_q^{d,+} \subseteq \mathcal{R}_q^d$, hence, we can write that

$$P(\mathbf{x}) = \sum_{\mathbf{k}} f_{\mathbf{k}}(q) \begin{bmatrix} \mathbf{x} \\ \mathbf{k} \end{bmatrix}$$

for any $P(x) \in \mathcal{R}_q^{d,+}$, with some finite set of vectors $\mathbf{k} \in \mathbb{N}_0^d$ and functions $f_{\mathbf{k}}(q) \in \mathbb{Z}[q, q^{-1}]$. Suppose, for the sake of contradiction, that $P(\mathbf{x})$ is not in the linear span of $\left\{ \left[\begin{smallmatrix} \mathbf{x} \\ \mathbf{k} \end{smallmatrix}\right], k \in \mathbb{N}_0^d \right\}$ with coefficients in $\mathbb{Z}[q]$.

Because we have that $q^m \left[\begin{smallmatrix} \mathbf{x} \\ \mathbf{k} \end{smallmatrix}\right]$ must be in $\mathcal{R}_q^{d,+}$ for any nonnegative integer m, we can further assume that $f_{\mathbf{k}}(q) \in q^{-1}\mathbb{Z}[q^{-1}]$. By our assumption on $P(\mathbf{x})$, there must be some vector $\mathbf{k}$ where $f_{\mathbf{k}}(q)$ is still nonzero. We now plug in $\mathbf{x} = [\mathbf{k}]_q$ where $\mathbf{k}$ minimizes $\sum_{i=1}^{d} k_i$ over vectors where $f_{\mathbf{k}}(q) \in q^{-1}\mathbb{Z}[q^{-1}]$ is nonzero. We must therefore have

$$P(\mathbf{x}) = f_{\mathbf{k}}(q) \begin{bmatrix} \mathbf{x} \\ \mathbf{k} \end{bmatrix} = f_{\mathbf{k}}(q) \in q^{-1}\mathbb{Z}[q^{-1}].$$

At the same time, we must also have that $P(\mathbf{x}) \in \mathbb{Z}[q]$ since P lies in $\mathcal{R}_q^{d,+}$, so we must have that $f_{\mathbf{k}}(q) = 0$, contradiction.

As such, we have by strong induction that $\{ \left[\begin{smallmatrix} \mathbf{x} \\ \mathbf{k} \end{smallmatrix}\right], \mathbf{k} \in \mathbb{N}_0^d \}$ forms a basis for $\mathcal{R}_q^{d,+}$ as a $\mathbb{Z}[q]$-algebra. Therefore, the first part of this theorem is true.

The second part of this theorem follows simply by taking the bar involution everywhere.

□

6 Conclusion

We have examined the rings $\mathcal{R}_q^{+,s} \cap \mathcal{R}_q^{-,r}$, $\mathcal{R}_q^d$, $\mathcal{R}_q^{d,+\mathbf{s}}$, and $\mathcal{R}_q^{d,-,\mathbf{r}}$ and have found bases for all of these sets. The basis for $\mathcal{R}_q^d$, in particular, is a very natural generalization of the basis for standard integer-valued multi-variable polynomials. The bases for the rings of multi-variable polynomials are also very natural generalizations of the bases for the single variable analogues.

Further potential directions of research involve other generalizations made by Harman and Hopkins, including the Frobenius and quantum Frobenius maps (see their Section 7 [5]), a dilation operator defined but largely unexplored [5, Section 9.1], and the maximal ideals of $\mathcal{R}_q$ (see [5, Sections 8, 9.4]). Another potential direction of research is to attempt to generalize $\mathcal{R}_q^+ \cap \mathcal{R}_q^-$ to multiple variables in a meaningful way and to find its basis.

7 Acknowledgments

I would like to thank my mentor, Robert Burklund, of the Department of Mathematics at the Massachusetts Institute of Technology, for his help and guidance with my project, and especially for his helpful explanations of parts I did not understand. I would also like to thank my tutor, Dr. John Rickert, for helping me edit this project. Next, I would like to thank Dr. Tanya Khovanova for her advice for making my presentation and paper. I also offer my thanks to Dr. David Jerison and Dr. Ankur Moitra for their useful advice for my project, presentation, and paper. I would also like to thank Dr. Slava Gerovitch and Dr. Amy Sillman for helping to organize the MIT math portion of RSI and RSI itself, respectively. I also thank Rupert Li, Dimitar Chakarov, Austen Mazenko, Sean Elliott, and Dr. Linda Westrick for their advice in editing my paper. I would also like to thank CEE for giving me the opportunity to attend RSI, and MIT for its generosity in allowing us to stay on campus. Finally, I would like to thank my sponsors, the Leonetti/O'Connell Family Foundation, Dr. Michael Vogelbaum and Ms. Judy Rosman, and Dr. and Mrs. Edward D. Hon, who made my experience at RSI possible.

References

[1] I. Newton. *Philosophiae naturalis principia mathematica.* William Dawson & Sons Ltd., London, 1687.

[2] G. Pólya. Über ganzwertige Polynome in algebraischen Zahlkörpern. *J. Reine Angew. Math.*, 149:97–116, 1919.

[3] A. Ostrowski. Über ganzwertige Polynome in algebraischen Zahlkörpern. *J. Reine Angew. Math.*, 149:117–124, 1919.

[4] P.-J. Cahen and J.-L. Chabert. *Integer-Valued Polynomials.* Vol. 48. Mathematical Surveys and Monographs. American Mathematical Society, Providence, RI, 1997.

[5] N. Harman and S. Hopkins. Quantum integer-valued polynomials. *Journal of Algebraic Combinatorics*, 45(2):601–628, March 2017.

[6] A. Klimyk and K. Schmüdgen. *The Quantum Algebra Uq(sl2)and Its Representations*, pages 53–96. Springer Berlin Heidelberg, Berlin, Heidelberg, 1997.

[7] P. Woit. *Quantum theory, groups and representations.* Springer.

A Results from Harman and Hopkins

We provide here a list of several properties from Harman and Hopkins [5].

Proposition (See Equation 2.2 and Propositions 1.2, 5.1, and 6.3 of [5])**.** We have that

1. For nonnegative integers n, k,
$$\begin{bmatrix} n \\ k \end{bmatrix}_q = q^k \begin{bmatrix} n-1 \\ k \end{bmatrix}_q + \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_q.$$
2. $\mathcal{R}_q^+$ is freely generated as a $\mathbb{Z}[q]$-module by the q-binomial polynomials $\left[\begin{smallmatrix} x \\ k \end{smallmatrix}\right]$ for $k \in \mathbb{N}_0$.
3. For integers m, we have S^m is an isomorphism of $\mathbb{Z}[q]$-algebras mapping $\mathcal{R}_q^+$ to $\mathcal{R}_q^{+,-m}$ and an isomorphism of $\mathbb{Z}[q^{-1}]$-algebras mapping $\mathcal{R}_q^-$ to $\mathcal{R}_q^{-,-m}$.
4. For all nonnegative integers k we have
$$\begin{aligned} \overline{\begin{bmatrix} x \\ k \end{bmatrix}} &= (-1)^k q^{\binom{k+1}{2}} S^{k-1} \begin{bmatrix} x \\ k \end{bmatrix} \\ &= (-1)^k \sum_{i=0}^{k-1} q^{\binom{k+1}{2}+(k-1-i)(k-i)} \begin{bmatrix} k-1 \\ i \end{bmatrix}_q \begin{bmatrix} x \\ k-i \end{bmatrix}. \end{aligned}$$

B Proof of Proposition 2.1

We restate here the statement of Proposition 2.1 for the reader's convenience.

Proposition. For nonnegative integers k and integers m, we have

$$q^m \begin{bmatrix} x \\ k \end{bmatrix} \in \begin{cases} \mathcal{R}_q^- & \text{iff } m \leq \binom{k+1}{2} \\ \mathcal{R}_q^+ & \text{iff } 0 \leq m. \end{cases} \quad \text{and } q^{-m} \overline{\begin{bmatrix} x \\ k \end{bmatrix}} \in \begin{cases} \mathcal{R}_q^+ & \text{iff } m \leq \binom{k+1}{2} \\ \mathcal{R}_q^- & \text{iff } m \geq 0. \end{cases}$$

Proof. Note that for $m \geq 0$, we have $q^m \in \mathbb{Z}[q]$, so $q^m \left[\begin{smallmatrix} x \\ k \end{smallmatrix}\right] \in \mathcal{R}_q^+$ by Proposition 1.2 of [5]. For $m < 0$, we have $q^m \left[\begin{smallmatrix} k \\ k \end{smallmatrix}\right]_q = q^m \notin \mathbb{Z}[q]$, so $q^m \left[\begin{smallmatrix} x \\ k \end{smallmatrix}\right] \notin \mathcal{R}_q^+$ for $m < 0$.

We determine when $q^m\begin{bmatrix}x\\k\end{bmatrix} \in \mathcal{R}_q^-$. Note that

$$[-n]_q - [m]_q = \frac{q^{-n}-1}{q-1} - \frac{q^m-1}{q-1} = \frac{q^{-n}-q^m}{q-1} = q^{-n}\frac{1-q^{n+m}}{q-1} = -q^{-n}[n+m]_q$$

by the definition of $[n]_q$.

Therefore, we have that

$$\begin{bmatrix}[-n]_q\\k\end{bmatrix} = \frac{\prod_{i=0}^{k-1}([-n]_q - [i]_q)}{q^{\binom{k}{2}}[k]_q!}$$

can be rewritten as

$$(-1)^k q^{-nk-\binom{k}{2}}\frac{\prod_{i=0}^{k-1}[n+i]_q}{[k]_q!} = (-1)^k q^{-nk-\binom{k}{2}}\begin{bmatrix}n+k-1\\k\end{bmatrix}_q.$$

We note that $\begin{bmatrix}n+k-1\\k\end{bmatrix}_q \in \mathbb{Z}[q]$ is a polynomial in q, and therefore must have degree

$$\sum_{i=1}^{k}(n+i-2) - \sum_{i=1}^{k}(i-1) = k(n-1).$$

Hence, the highest degree term in $q^m\begin{bmatrix}[-n]_q\\k\end{bmatrix}$ has degree

$$m - \binom{k}{2} - nk + k(n-1) = m - \binom{k+1}{2}.$$

Because we already know that $q^m\begin{bmatrix}[-n]_q\\k\end{bmatrix}$ is in $\mathbb{Z}[q,q^{-1}]$, we know $q^m\begin{bmatrix}x\\k\end{bmatrix}$ lies in $\mathcal{R}_q^-$ if and only if the highest degree term of $q^m\begin{bmatrix}[-n]_q\\k\end{bmatrix}_q$ has nonpositive degree. As such, we see that $q^m\begin{bmatrix}x\\k\end{bmatrix} \in \mathcal{R}_q^-$ if and only if $m \leq \binom{k+1}{2}$.

The second part of the proposition follows by taking the bar involution everywhere. $\square$

C Proof of Lemma 3.1

We restate here the statement of Lemma 3.1 for the reader's convenience.

Lemma. For nonnegative integers k, we have the relations

$$\begin{tikzcd}
\mathrm{X}(k) \arrow[r, hook] & \mathrm{Bar}(k) \arrow[r, hook] & \mathcal{R}_q^+ \cap \mathcal{R}_q^{-,k-1} \\
& \mathrm{Y}(k) \arrow[r, hook] & \mathcal{R}_q^+ \cap \mathcal{R}_q^{-,k}. \arrow[u, hook]
\end{tikzcd}$$

Proof. For all $k \geq 1$, we have $\frac{k+1}{2} \geq 1$, so $\binom{k+1}{2} \geq k$, thus $\mathrm{X}(k) \subseteq \mathrm{Bar}(k)$. We also have that any polynomial $q^m \begin{bmatrix} x \\ k \end{bmatrix}$ is contained within $\mathcal{R}_q^+$. Furthermore, $S^{k-1} q^m \begin{bmatrix} x \\ k \end{bmatrix} = (-1)^k q^{m - \binom{k+1}{2}} \overline{\begin{bmatrix} x \\ k \end{bmatrix}} \in \mathcal{R}_q^-$ for $0 \leq m \leq \binom{k+1}{2}$. As such, we have $q^m \begin{bmatrix} x \\ k \end{bmatrix} \in \mathcal{R}_q^{-,k-1}$ whenever we have $0 \leq m \leq \binom{k+1}{2}$, so we must have $\mathrm{Bar}(k) \subseteq \mathcal{R}_q^+ \cap \mathcal{R}_q^{-,k-1}$.

We now show that $\mathrm{Y}(k) \subseteq \mathcal{R}_q^+ \cap \mathcal{R}_q^{-,k}$. Because

$$\mathcal{R}_q^+ = \mathrm{Span}_{\mathbb{Z}[q]} \left\{ \begin{bmatrix} x \\ k \end{bmatrix},\ k \geq 0 \right\},$$

we know that $\mathrm{Y}(k) \subseteq \mathcal{R}_q^+$. It remains to show that $\mathrm{Y}(k) \subseteq \mathcal{R}_q^{-,k}$.

Consider any polynomial of the form

$$P_m(x) = \sum_{i=0}^{m} (-1)^i q^{\binom{i}{2}} \begin{bmatrix} k-m+i \\ k-m \end{bmatrix}_q \begin{bmatrix} x \\ k-m+i \end{bmatrix}$$

for $0 \leq m \leq k-1$. We have

$$\begin{aligned}
P_m([n]_q) &= \sum_{i=0}^{m} (-1)^i q^{\binom{i}{2}} \begin{bmatrix} k-m+i \\ k-m \end{bmatrix}_q \begin{bmatrix} n \\ k-m+i \end{bmatrix}_q = \sum_{i=0}^{m} (-1)^i q^{\binom{i}{2}} \begin{bmatrix} n \\ k-m \end{bmatrix}_q \begin{bmatrix} n-k+m \\ i \end{bmatrix}_q \\
&= \begin{bmatrix} n \\ k-m \end{bmatrix}_q \sum_{i=0}^{m} (-1)^i q^{\binom{i}{2}} \begin{bmatrix} n-k+m \\ i \end{bmatrix}_q.
\end{aligned}$$

Using Harman and Hopkins' Equation (5.3), we see that

$$\begin{aligned}
P_m([n]_q) &= \begin{bmatrix} n \\ k-m \end{bmatrix}_q \sum_{i=0}^{m} \left((-1)^i q^{\binom{i+1}{2}} \begin{bmatrix} n-k+m-1 \\ i \end{bmatrix}_q + (-1)^i q^{\binom{i}{2}} \begin{bmatrix} n-k+m-1 \\ i-1 \end{bmatrix}_q \right) \\
&= (-1)^m q^{\binom{m+1}{2}} \begin{bmatrix} n \\ k-m \end{bmatrix}_q \begin{bmatrix} n-k+m-1 \\ m \end{bmatrix}_q.
\end{aligned}$$

We have

$$S^{m-1} \begin{bmatrix} x \\ m \end{bmatrix} \Bigg|_{x=[n]_q} = \begin{bmatrix} n+m-1 \\ m \end{bmatrix}_q,$$

so
$$\overline{\begin{bmatrix} x \\ m \end{bmatrix}}\Bigg|_{x=[n-k]_q} = (-1)^m q^{\binom{m+1}{2}} \begin{bmatrix} n-k+m-1 \\ m \end{bmatrix},$$
so hence
$$P_m([n]_q) = \begin{bmatrix} n \\ k-m \end{bmatrix}_q \overline{\begin{bmatrix} x \\ m \end{bmatrix}}\Bigg|_{x=[n-k]_q}.$$
Thus,
$$P_m([n]_q) \in \mathbb{Z}[q^{-1}]$$
for $n < k-m$.

For $k-m+1 \leq n \leq k$, we know $\begin{bmatrix} n-k+m-1 \\ m \end{bmatrix}_q = 0$, so $P_m([n]_q) = 0$. At $n = k-m$, we have
$$P_m([n]_q) = (-1)^m q^{\binom{m+1}{2}} \begin{bmatrix} [-1]_q \\ m \end{bmatrix}.$$
Because
$$[-1]_q - [k]_q = \frac{q^{-1}-1}{q-1} - \frac{q^k-1}{q-1} = -q^{-1}[k+1],$$
we have
$$\begin{bmatrix} [-1]_q \\ m \end{bmatrix} = \frac{(-q^{-1})^m}{q^{\binom{m}{2}}} = (-1)^m q^{-\binom{m+1}{2}}.$$
Therefore, $P_m([k-m]_q) = 1$, so $P_m([n]_q) \in \mathbb{Z}[q^{-1}]$ for all $n \leq k$.

This shows that $\mathrm{Y}(k) \subseteq \mathcal{R}_q^+ \cap \mathcal{R}_q^{-,k}$, as claimed. □

D Proof of Lemma 3.2

We restate here the statement of Lemma 3.2 for the reader's convenience.

Lemma. For any integers r, s, the sets of the form $S^{m-s}\,\mathrm{X}(k)$ or $S^{k-r-1}\,\mathrm{Bar}(k)$ with $\max(0, r-s+1) \leq k$ and $0 \leq m \leq k+s-r-2$ are pairwise disjoint. When $r \geq s$, the set $S^{-s}\,\mathrm{Y}(r-s)$ is also disjoint from the previous sets.

Proof. We note that the leading term of a polynomial $S^m q^i \begin{bmatrix} x \\ k \end{bmatrix}$ is $q^{(m+s)k+i} S^{-s} \begin{bmatrix} x \\ k \end{bmatrix}$, so the leading terms of elements in the sets $S^{m-s}\,\mathrm{X}(k)$ and $S^{k-r-1}\,\mathrm{Bar}(k)$ are all different. As such, these sets must be disjoint. Note that because $k > r - s$, none of these sets have elements with highest order terms of the form $(-1)^i q^{\binom{i}{2}} S^{-s} \begin{bmatrix} x \\ r-s \end{bmatrix}$. For $r \geq s$, the set $S^{-s}\,\mathrm{Y}(r-s)$ has elements whose highest order terms are of the form $(-1)^i q^{\binom{i}{2}} S^{-s} \begin{bmatrix} x \\ r-s \end{bmatrix}$, so $S^{-s}\,\mathrm{Y}(r-s)$ is disjoint from the other sets. Therefore, the desired sets are indeed distinct. $\square$

E Proof of Lemma 3.3

We restate here the statement of Lemma 3.3 for the reader's convenience.

Lemma. For any integers r, s, the elements of $\mathrm{Z}(r, s)$ are linearly independent over $\mathbb{Z}$.

Proof. As in the proof of Lemma 3.2, we note that most of the elements do not have the same leading term, so the only possible linear dependence must have highest order terms in $S^{-s}\,\mathrm{Y}(r-s)$. We know, however, that the elements in $S^{-s}\,\mathrm{Y}(r-s)$ have the lowest order in x of any elements in $\mathrm{Z}(r, s)$. As such, if the elements of $\mathrm{Z}(r, s)$ are not linearly independent over $\mathbb{Z}$, then the elements of $S^{-s}\,\mathrm{Y}(r-s)$ cannot be linearly independent over $\mathbb{Z}$. It remains to show that the elements of $S^{-s}\,\mathrm{Y}(r-s)$ are linearly independent over $\mathbb{Z}$.

We first note that the degree in q of $\begin{bmatrix} k-m+i \\ k-m \end{bmatrix}_q$ is $\binom{k-m+i}{2} - \binom{k-m}{2} - \binom{i}{2} = i(k-m)$, so the highest order term of a polynomial in $S^{-s}\,\mathrm{Y}(r-s)$ is

$$(-1)^m q^{\binom{m}{2} + m(k-m)} S^{-s} \begin{bmatrix} x \\ k \end{bmatrix} = (-1)^m q^{\binom{k}{2} - \binom{k-m}{2}} S^{-s} \begin{bmatrix} x \\ k \end{bmatrix}.$$

By making some substitutions and swapping the sums, we know that the sum of the terms in $S^{-s}\,\mathrm{Y}(r-s)$ is

$$\begin{aligned}
&\sum_{m=0}^{k}\sum_{i=0}^{m}(-1)^i q^{\binom{i}{2}} \begin{bmatrix} k-m+i \\ k-m \end{bmatrix}_q \begin{bmatrix} x \\ k-m+i \end{bmatrix} \\
&= \sum_{m=0}^{k}\sum_{i=0}^{m}(-1)^{m-i} q^{\binom{m-i}{2}} \begin{bmatrix} k-i \\ k-m \end{bmatrix}_q \begin{bmatrix} x \\ k-i \end{bmatrix} \\
&= \sum_{i=0}^{k}\sum_{m=i}^{k}(-1)^{m-i} q^{\binom{m-i}{2}} \begin{bmatrix} k-i \\ k-m \end{bmatrix}_q \begin{bmatrix} x \\ k-i \end{bmatrix} \\
&= \sum_{i=0}^{k}\sum_{m=0}^{k-i}(-1)^{m} q^{\binom{m}{2}} \begin{bmatrix} k-i \\ m \end{bmatrix}_q \begin{bmatrix} x \\ k-i \end{bmatrix}.
\end{aligned}$$

Now, by using using a result from Harman and Hopkins' [5, Equation 2.2], the preceding sum expands to

$$\sum_{i=0}^{k} \begin{bmatrix} x \\ k-i \end{bmatrix} \sum_{m=0}^{k-i}(-1)^m q^{\binom{m}{2}} \left(q^m \begin{bmatrix} k-i-1 \\ m \end{bmatrix}_q + \begin{bmatrix} k-i-1 \\ m-1 \end{bmatrix}_q \right). \tag{1}$$

In Equation 1, we note that

$$(-1)^m q^{\binom{m}{2}} q^m \begin{bmatrix} k-i-1 \\ m \end{bmatrix}_q + (-1)^{m+1} q^{\binom{m+1}{2}} \begin{bmatrix} k-i-1 \\ m+1-1 \end{bmatrix} = 0,$$

so our sum simplifies to

$$\sum_{i=0}^{k} \begin{bmatrix} x \\ k-i \end{bmatrix} \left(\begin{bmatrix} k-i-1 \\ -1 \end{bmatrix}_q + (-1)^{k-i} q^{\binom{k-i+1}{2}} \begin{bmatrix} k-i-1 \\ k-i \end{bmatrix}_q \right).$$

Because $\begin{bmatrix} k-i-1 \\ k-i \end{bmatrix}_q = 0$ is always true while $\begin{bmatrix} k-i-1 \\ -1 \end{bmatrix}_q = 1$ is only true for $k-i=0$, we see that our sum simplifies to $\begin{bmatrix} x \\ k-k \end{bmatrix}\begin{bmatrix} k-k-1 \\ -1 \end{bmatrix}_q = 1$.

As such, we can replace the

$$\sum_{i=0}^{k}(-1)^i q^{\binom{i}{2}} \begin{bmatrix} x \\ i \end{bmatrix}$$

element in $\mathrm{Y}(k)$ with 1, thereby producing a set of $k+1$ elements whose leading coefficients have distinct orders in q. Therefore, the elements in $\mathrm{Y}(k)$ are linearly independent, so those in $\mathrm{Z}(k)$ are linearly independent as well. $\square$

F Proof of Theorem 3.4, Cases 2 and 3

We restate some background about Theorem 3.4, Cases 2 and 3 for the reader's convenience. In Theorem 3.4, we show that $\mathrm{Z}(r,s)$ is a basis for $\mathcal{R}_q^{+,s} \cap \mathcal{R}_q^{-,r}$ as a $\mathbb{Z}$-algebra. In Cases 2 and 3, we consider a specific polynomial $P(x)$ in this set with degree $k \leq r-s$ that we assume is not in the span of $\mathrm{Z}(r,s)$. Our goal is to derive a contradiction and show that $P(x)$ is, in fact, in the span of $\mathrm{Z}(r,s)$ as a $\mathbb{Z}$-algebra.

Proof. **Case 2:** We assume that $k = r-s$. The polynomial $S^s P(x)$ lies in $\mathcal{R}_q^{+} \cap \mathcal{R}_q^{-,r-s}$, so $S^s P([n]_q) \in \mathbb{Z}$ for each integer n in $0 \leq n \leq k$. We have from the proof of Lemma 3.1 (see Appendix C for the proof and for the definitions of P_m) that for integer values of n,

$$P_m([n]_q) = \sum_{i=0}^{m} (-1)^i q^{\binom{i}{2}} \begin{bmatrix} k-m+i \\ k-m \end{bmatrix}_q \begin{bmatrix} n \\ k-m+i \end{bmatrix}_q = \begin{cases} 0 & 0 \leq n < k-m \\ 0 & k-m < n \leq k \\ 1 & n = k-m. \end{cases}$$

Therefore, for any sequence of $k+1$ integers $P([0]_q), P([1]_q), \ldots, P([k]_q)$, we see that

$$P'(x) = \sum_{n=0}^{k} S^s P(x)\Big|_{x=[n]_q} P_n(x) = \sum_{n=0}^{k} P([n+s]_q) P_n(x)$$

is a linear combination of the elements of $\mathrm{Y}(k)$ such that $P'([n]_q) = P([n]_q)$ for each integer n in $0 \leq n \leq k$. As such, our polynomial $P'(x) - S^s P(x)$ has degree k and $k+1$ zeros, so $P'(x) = S^s P(x)$. Thus, we see that

$$P(x) = \sum_{n=0}^{k} P([n+s]_q) S^{-s} P_n(x),$$

contradicting our assumption that $P(x)$ is not in $\mathrm{Span}_{\mathbb{Z}}\, \mathrm{Z}(r,s)$.

Case 3: We assume that $k < r-s$. We recall that $S^s P(x) \in \mathcal{R}_q^{+} \cap \mathcal{R}_q^{-,r-s}$, so $S^s P([n]_q) \in \mathbb{Z}$ for all $0 \leq n \leq r-s$. We show that $P(x)$ is a constant polynomial, and suppose for the sake of contradiction that it is not a constant polynomial.

Let n be the smallest positive integer such that the coefficient in $S^sP(x)$ on $\begin{bmatrix} x \\ n \end{bmatrix}$ is not the zero polynomial. we have $0 < n \le k < r - s$, so we plug in $x = [n]_q$, which gives that $S^sP([n]_q) = f_n(q)$ is an integer. As such, we have $m_n = 0$. We now use strong induction on i to show that $m_{n+i} = \binom{n+i}{2} - \binom{n}{2}$.

We have already shown the base case, $i = 0$, so it remains to show that if $m_{n+j} = \binom{n+j}{2} - \binom{n}{2}$ for all $0 \le j < i$, then we have $m_{n+i} = \binom{n+i}{2} - \binom{n}{2}$. We note that the degree of $\begin{bmatrix} n+i \\ n+j \end{bmatrix}_q$ is $\binom{n+i}{2} - \binom{n+j}{2} - \binom{i-j}{2} = (n+j)(i-j)$. As such, the degree of $\begin{bmatrix} n+i \\ n+j \end{bmatrix}_q f_{n+j}(q)$ is

$$(n+j)(i-j) + m_{n+j} = (n+j)(i-j) + \binom{n+j}{2} - \binom{n}{2} = \frac{n+j}{2}(n+2i-j-1) - \binom{n}{2}.$$

This is maximized when $n+j$ and $n+2i-j-1 \ge n+2(j+1)-j-1 = n+j+1$ are as close as possible, which is when $j = i-1$. This means that the polynomial

$$\sum_{m=n}^{n+i-1} f_m(q) \begin{bmatrix} n+i \\ m \end{bmatrix}_q$$

has degree $\binom{n+i}{2} - \binom{n}{2}$. Also note that the polynomial $\begin{bmatrix} n+i \\ m \end{bmatrix}_q f_m(q)$ is identically zero for $0 < m < n$. For $m = 0$, we note that $S^sP(0) = f_0(q)$. Hence, $f_0(q)$ is a constant polynomial, and $\begin{bmatrix} n+i \\ 0 \end{bmatrix}_q f_0(q)$ is also constant. As such,

$$\sum_{m=0}^{n-1} f_m(q) \begin{bmatrix} n+i \\ m \end{bmatrix}_q$$

is a constant polynomial, so the degree of

$$\sum_{m=0}^{n+i-1} f_m(q) \begin{bmatrix} n+i \\ m \end{bmatrix}_q$$

is $\binom{n+i}{2} - \binom{n}{2}$. Because $P([n+i]_q)$ evaluates to an integer, $f_{n+i}(q)$ must also have the same degree. Thus, by strong induction we know that $f_{n+i}(q)$ has degree $\binom{n+i}{2} - \binom{n}{2}$ for $0 \le i \le k - n$.

However, in this case, plugging in $x = [k+1]_q$ must give a polynomial in q of degree $\binom{k+1}{2} - \binom{n}{2} > 0$. Therefore, $S^sP([k+1]_q) \notin \mathbb{Z}$ even though we have $k+1 \le r-s$. As such,

$P(x)$ does not lie in $\mathcal{R}_q^{+,s} \cap \mathcal{R}_q^{-,r}$, contradiction. Therefore, the only possible polynomials $P(x)$ are the constant polynomials. However, because $S^{-s}(1) = 1$, we have from the proof of Lemma 3.3 (see Appendix E for the proof) that 1 is a linear combination of the elements in $S^{-s}\,\mathrm{Y}(k)$, so constant polynomials are also in $\mathrm{Span}_{\mathbb{Z}}\,\mathrm{Z}(r,s)$. Therefore, there is no polynomial $P(x) \in \mathcal{R}_q^{+,s} \cap \mathcal{R}_q^{-,r}$ with degree in x less than $r-s$ that is not in the $\mathrm{Span}_{\mathbb{Z}}\,\mathrm{Z}(r,s)$. $\square$

G Proof of Lemma 5.1

We repeat here the statement of Lemma 5.1 for the reader's convenience.

Lemma G.1. *We have that* $\mathcal{R}_q^{d,+} \subseteq \mathcal{R}_q^{d}$.

Proof. We shall prove that for any polynomial $P(\mathbf{x}) \in \mathcal{R}_q^{d,+}$, we have $P(\mathbf{x}) \in \mathcal{R}_q^{d}$ by inducting on the degree of $P(\mathbf{x})$. Let the highest degree term in $P(\mathbf{x})$ have degree k.

The base case, $k = 0$, is true because constants lying in $\mathbb{Z}[q]$ must also lie in $\mathbb{Z}[q, q^{-1}]$.

We now show that if $P(\mathbf{x}) \in \mathcal{R}_q^{d}$ for all polynomials $P(\mathbf{x}) \in \mathcal{R}_q^{d,+}$ with degree less than k, then $P(\mathbf{x}) \in \mathcal{R}_q^{d}$ for all polynomials $P(\mathbf{x}) \in \mathcal{R}_q^{d,+}$ with degree k as well.

We note that $P'(\mathbf{x}) = S^{\mathbf{1}}P(\mathbf{x}) - P(\mathbf{x})$ must have degree at most $k-1$, and must lie in $\mathcal{R}_q^{d,+}$. As such, $P'(\mathbf{x})$ is in $\mathcal{R}_q^{d}$ by the inductive hypothesis. We have from the definition of P' that $P([\mathbf{n}-1]_q) = P([\mathbf{n}]_q) - P'([\mathbf{n}-1]_q)$. As such, we have that for $\mathbf{n} \in \mathbb{Z}^d$,

$$P([\mathbf{n}]_q) = P([\mathbf{n}+n]_q) - \sum_{i=0}^{n-1} P'([\mathbf{n}+i]_q),$$

where we set $n = \max\limits_{1\le i\le d} |n_i|$.

We must have that $\mathbf{n}+n \in \mathbb{N}_0^d$, so $P([\mathbf{n}+n]_q) \in \mathbb{Z}[q]$. We also have that $P'([\mathbf{n}+i]_q) \in \mathbb{Z}[q, q^{-1}]$ for any integer i, so we therefore have that $P([\mathbf{n}]_q) \in \mathbb{Z}[q, q^{-1}]$ as well for any $\mathbf{n} \in \mathbb{Z}^d$. As such, we have that $P(\mathbf{x})$ is in $\mathcal{R}_q^{d}$, so by strong induction, we have that $\mathcal{R}_q^{d,+} \subseteq \mathcal{R}_q^{d}$.

$\square$