# A Computational Approach to Intrinsic Linkedness in Complete Graphs

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## Abstract

Connections among topology, geometry, and graph theory arise when analyzing structural components of nonplanar graphs. Initial discovery into intrinsic knottedness and linkedness of graphs began with proofs that $K_6$ is intrinsically linked and $K_7$ is intrinsically knotted. Previously it had been shown that $K_9$ is intrinsically linked in linear embeddings and $K_{10}$ is intrinsically linked in all spatial embeddings with linking number $\geq 2$. We show through a computational approach that $K_7$ is not intrinsically linked in linear embeddings with linking number $\geq 2$, and that $K_8$ is not intrinsically linked in spatial embeddings with linking number $\geq 2$. It is still unknown whether $K_9$ is intrinsically linked in spatial embeddings or $K_8$ is intrinsically linked in linear embeddings with linking number $\geq 2$.

## Summary

When studying graphs, or points in space and the connections between them, certain structures can appear within the graph such as knots and links. Knots are closed curves, and links are collections of knots. Linking numbers are a way to associate certain properties of knots and links to graphs. Previous work was concerned with determining whether links of a certain linking number existed within all possible diagrams of complete graphs, or graphs within which each vertex is connected by an edge to all vertices by an edge. We show that complete graphs with 8 vertices do not possess links of linking number at least 2 in all drawings. Additionally, we show that a complete graphs with 7 vertices, in three dimensional drawings with straight lines, do not possess links of linking number at least 2 in all drawings. These results can be used in determining properties of molecules such as DNA and the effectiveness of veins in leaves.

# 1 Introduction

As networks and systems grow increasingly complex, so has the need to model and analyze those systems. This has led to the rapidly increasing relevance of mathematical fields such as graph theory. Graphs are used to model many complex systems, such as social networks, pathology and disease spread, and communication systems.

In complicated spatially embedded graphs, structures such as knots and links may appear, where knots are non-self intersecting curves in space and links are collections of knots. The relationships between these topological structures and graphs were first noted in Conway and Gordon's 1983 proof that $K_6$ is intrinsically linked and $K_7$ is intrinsically knotted [1]. Knots and links can appear in complex systems such as the knotting of DNA, other molecular polymers, and leaf venation in plants [2] [3]. Although in these situations, graphs are significant in that they reflect connections between elements, it is important to also acknowledge the presence of more complex structures that can be formed within the graph, such as links and knots. These structures can provide further insight into the modeled system than simply analyzing vertices and connections between them, allowing for the categorization of graphs and simplification of network analysis.

Previous analyses of linkedness and knottedness of graphs have been concerned with complete graphs, or graphs within which every vertex is connected by an edge to every other vertex, especially those which possess a given topological structure in every embedding [1]. Our research is concerned with identifying the prevalence of a specific type of link in both linear and general spatial embeddings of complete graphs, a 2-component link with linking number $\geq 2$, where linking number is a value quantifying crossings in a link. Although previous research has established the existence of these structures in specific complete graphs, it has not been shown whether these graphs are the smallest complete graphs that contain a 2-component link with linking number $\geq 2$ in every embedding.

# 2 Background

## 2.1 Graphs

A graph is an abstract structure, denoted $G = (V, E)$, identified by a vertex set $V$ and an edge set $E$, where $E \subseteq V \times V$. The elements of the edge set are the pairs of vertices that comprise all the edges of a graph. Two graphs are equivalent if they possess the same vertex set and edge set. Our research will primarily be concerned with complete graphs, denoted $K_n$, where $n$ is the total number of vertices within the graph. In a complete graph, all vertices are connected to all other vertices.

**Definition 2.1.** The *embedding* of a graph G onto $\mathbb{R}^n$ is a map $\phi : V \to \mathbb{R}^n$ such that $\phi(v_1) = \phi(v_2)$ if and only if $v_1 = v_2$, and a map $\psi : E \to \mathcal{E}(\mathbb{R}^n)$, where $\mathcal{E}(\mathbb{R}^n)$ is the set of non self-intersecting curves in $\mathbb{R}^n$ and for $\mathcal{E} \in E$, $\psi(\mathcal{E}) = \psi(v_i, v_j)$ is a curve with endpoints $\phi(v_i)$ and $\phi(v_j)$.

**Definition 2.2.** A *linear embedding* of a graph $G$ is an embedding where all edges are straight lines.

In an embedding of a graph, edges cannot cross, with the exception of adjacent edges meeting at their shared vertex. A graph that can be embedded in $\mathbb{R}^2$ is called planar, and graphs that cannot be embedded in $\mathbb{R}^2$ are nonplanar. Nonplanar graphs can be depicted in two dimensions if crossings are denoted in the planar drawing; a solid strand indicates a strand laying on top, while a corresponding broken strand represents the strand laying underneath. Every embedding we refer to from now on will be in $\mathbb{R}^3$, although they may be depicted in $\mathbb{R}^2$.

Two significant types of embeddings of graphs are linear embeddings and general spatial embeddings. In linear embeddings of graphs, all edges must be straight lines whereas in general spatial embeddings, edges can be curves. A notable difference between these two types of graph embeddings is that in the general embedding of a graph, two edges that share a vertex can cross each other; however, this can never be the case in a linear embedding because in a straight edge model, the only way for two edges adjacent on a vertex to intersect without being curved is if they are the same edge. A graph is said to be weighted if its edges are associated with quantities.

A path of a graph is a traversal in that graph between two vertices, characterized by an alternating, non-repeating sequence of the form $(v_1, e_1, v_2, e_2, v_3, \ldots, e_n, v_n)$, where each consecutive vertex and edge in the sequence are adjacent on the graph. A cycle is a path that begins and ends at the same vertex.

## 2.2 Links and Knots

A knot is a closed, non-self-intersecting curve in $\mathbb{R}^3$ that is homeomorphic to a circle, such as the trefoil in Figure 1. A link is a collection of two or more of these non-self intersecting curves, such as the Hopf link in Figure 2.

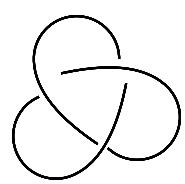

Figure 1: The trefoil, the most basic nontrivial knot.

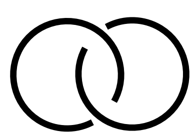

Figure 2: The Hopf link, the most basic nontrivial link.

Two knots or links are said to be isotopic, or equivalent, if they can be transformed into each other without the knot curve passing through itself. Procedures that deform a knot while still maintaining its identity are known as Reidemeister moves, depicted in Figure 3. Similarly, two different graph embeddings are considered identical if they share the same vertex and edge set and can be transformed into each other using traditional Reidemeister moves as well as two additional Reidemeister moves $R4$ and $R5$, which are depicted in Figure 4.

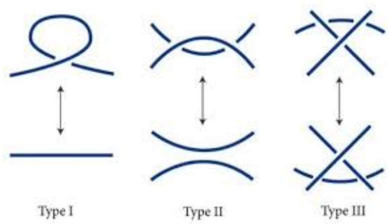


Figure 3: The three types of Reidemeister moves [4].

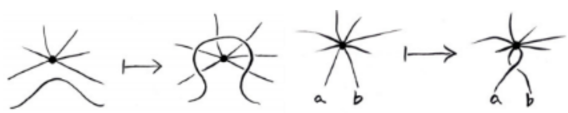


Figure 4: The two additional types of Reidemeister moves for graphs [5].

The crossing number for a knot or a link quantifies either the overlapping components of knots or the separate overlapping knots that comprise a link. The crossing number follows a sign convention illustrated in Figure 5. Consider the orientation of the top edge and the orientation of the bottom edge. If the top edge needs to be rotated clockwise to be parallel to and match the orientation of the bottom edge, then the signed crossing number is -1; if the top edge needs to be rotated counterclockwise, then the signed crossing number is +1. The linking number of a link is denoted $lk(L)$, where $lk(L) = \frac{1}{2}\sum_{c\in L} cr(c)$, and $cr(c)$ denotes the signed crossing number of some crossing $c$ between different components of the link.

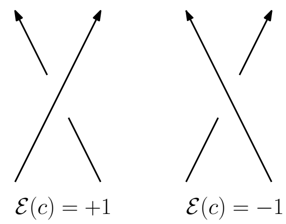


Figure 5: The rules for the signed crossing number of a link or knot: $\varepsilon(C)$ denotes the crossing number of a crossing $C$.

Structures such as links and knots can exist within the cycles of a nonplanar graph. In these cases, graph theory must be considered in conjunction with knot theory in order to identify these structures and classify the graphs that contain them. We proceed with definitions regarding graphs containing knots and links.

**Definition 2.3.** An *intrinsically knotted graph* is a graph that contains a knotted cycle in every embedding in $\mathbb{R}^3$.

**Definition 2.4.** An *intrinsically linked graph* is a graph that contains at least two linked cycles in every embedding in $\mathbb{R}^3$.

# 3 Previous Work

The first results regarding intrinsically knotted and intrinsically linked graphs were with mathematicians Conway and Gordon [1], while different proofs for their initial theorems are present in other literature [6]. Most studies of intrinsic knottedness and linkedness of graphs have only considered complete graphs. Two notable advancements in this field are the following theorems about $K_6$ and $K_7$:

**Theorem 3.1** (Conway, Gordon 1983 [1])**.** *Every spatial embedding of $K_6$ contains a non-trivial link.*

**Theorem 3.2** (Conway, Gordon 1983 [1]). *Every spatial embedding of $K_7$ contains a nontrivial knot.*

These theorems, as well as most investigations of knotted and linked graphs, rely on properties of graph equivalence, knot equivalence, and invariant functions to show the prevalence of a given structure in all embeddings of a graph.

In Kozai's [6] refinement of the proof of Theorem 3.1, the sum of all linking numbers across disjoint cycles in embeddings of graphs is shown to be an invariant function across all embeddings of the same graph. Taking the sum of linking numbers (mod 2) for embeddings of $K_6$, it was found that every embedding of $K_6$ possesses a sum of linking numbers across disjoint cycles congruent to 1 (mod 2). Thus, all cycles of $K_6$ were proven to contain a nontrivial link. A similar analysis was used to prove that every embedding of $K_7$ contains a nontrivial knot. By analyzing a knot invariant function of Hamiltonian cycles across embeddings of $K_7$, it was shown that $K_7$ is intrinsically knotted [6].

Previous research has identified knots of varying complexity for complete graphs, but has not explored lower bounds. For example, it is known that $K_{972}$ always contains a knot more complex the trefoil, but it is not known if 972 is the smallest $n$ for a $K_n$ that satisfies this condition [7].

**Theorem 3.3** (Flapan 2002 [8]). *If a graph contains a 3-component link, it must contain a 2-component link with linking number at least 2.*

Theorem 3.3 was used to prove that $K_{10}$ is intrinsically linked with a 2-component link of linking number at least 2. This theorem identifies identities between different types of linking structures that are relevant for us in determining when graphs contain links with linking number at least 2.

**Theorem 3.4** (Flapan, Naimi, Pommersheim 2001 [9]). *$K_{10}$ is intrinsically triple linked, and $K_9$ is not intrinsically triple linked.*

**Remark 3.1.** *A triple linked graph is a graph that contains a 3-component link.*

The proof of this theorem provided a drawing of $K_9$ with no 3-component links. Because a 3-component link in a graph implies that there exists a 2-component link with linking number $\geq 2$, this embedding of $K_9$ is not known for sure to possess 2-component links with linking number $\geq 2$ without further analysis. This embedding will be a subject of investigation later in the paper.

**Theorem 3.5** (Flapan 2002 [8])**.** *Every embedding of $K_{10}$ in $\mathbb{R}^3$ contains a 2-component link $L = L_1 \cup J_1$ such that we have $|lk(L_1, J_1)| \geq 2$.*

Note that Theorem 3.4 identifies $K_{10}$ as intrinsically linked with a 3-component link, and by Theorem 3.3 if a graph contains a 3-component link, it must contain a 2-component link with linking number $\geq 2$. Theorem 3.5 identifies that all embeddings of $K_{10}$ possess links of higher complexity than the Hopf link. However, Theorem 3.5 does not identify whether or not $K_{10}$ is the least possible complete graph that suits the condition. In our research, we employ a computational approach to define new upper and lower bounds of the smallest complete graph that intrinsically possesses disjoint cycles of linking number at least 2.

By Theorem 3.1, $K_6$ is intrinsically linked. Clearly $K_5$ cannot be intrinsically linked, as it does not possess two pairs of disjoint cycles. Thus, $K_6$ is the current greatest lower bound for the least $n$ for a $K_n$ that possesses at least one pair of disjoint cycles with linking number at least two. The current best upper bound for this problem is $n = 10$, as indicated by Theorem 3.5. In our research, we use a computational analysis of embeddings of complete graphs in order to refine the upper and lower bound of the size of complete graphs that are intrinsically linked with linking number at least 2.

**Theorem 3.6** (Naimi, Pavelescu 2014 [10])**.** *Every linear embedding of $K_9$ in $\mathbb{R}^3$ contains a link with three components.*

Theorem 3.6 shows that all linear embeddings of $K_9$ contain a 3-component link; because all graphs containing 3-component links contain 2-component links with linking number $\geq 2$, all linear embeddings of $K_9$ contain 2-component links with linking number $\geq 2$. As is the case with Theorem 3.5, it is not known whether this is the smallest $n$ satisfies this condition. Our research is concerned with discovering the greatest $n$ of linear embeddings of $K_n$ that must possess linking number $\geq 2$ in order to define a new lower bound of the least $K_n$ that satisfies this condition.

**Theorem 3.7** (Guy 1972 [11])**.** *For $n \leq 12$, the crossing number of a complete graph $K_n$ is bounded by $cr(K_n) \leq \frac{1}{4}\lfloor\frac{n}{2}\rfloor\lfloor\frac{n-1}{2}\rfloor\lfloor\frac{n-2}{2}\rfloor\lfloor\frac{n-3}{2}\rfloor$.*

Guy quantifies the upper limit for the minimal crossing number of a graph $K_n$. Because crossings are required for the presence of links and knots in nonplanar graphs, it is often useful to attempt to construct a graph with a relatively small crossing number when attempting to prove by counterexample that a graph is not intrinsically knotted or linked with a particular type of knot or link, for this lowers the likelihood that a link or knot is present. Theorem 3.7 is important to consider when constructing embeddings of a complete graph.

# 4 Results

## 4.1 Computational Approach

We use Java programming to computationally analyze the links contained in complete graphs. We construct crossing matrices and graphs in order to accomplish this.

**Definition 4.1.** The *signed crossing matrix $C$* of a graph $K_n$ is a square $\binom{n}{2}$ by $\binom{n}{2}$ matrix such that the rows and columns enumerate and represent each distinct edge in the graph. Let $C_{ij} = C_{ji}$ be the signed crossing number of edge $i$ and $j$, valuing the respective element of the matrix as 0 if the edges do not cross. If the edges do cross, then $C_{ij} = C_{ji}$ is the sum of the signs of the total number of crossings with those same two edges.

**Remark 4.1.** *The expression for determining linking number based off of values in the matrix is*

$$lk((a_1, \ldots, a_p), (b_1, \ldots, b_q)) = \sum_{i=1,2\ldots,p;\ j=1,2,\ldots,q} C_{a_i a_{i+1} b_j b_{j+1}} \operatorname{sgn}(a_{i+1} - a_i) \operatorname{sgn}(b_{j+1} - b_j).$$

Note that $C = C^T$ and that the majority of the elements in $C$ are 0. Additionally, the signed crossing matrix only represents a particular embedding of a graph, not all embeddings of the graph.

Further explanation of the code can be found in Appendix A.1.

## 4.2 General Spatial Embeddings of Complete Graphs

Although Theorem 3.5 establishes that $K_{10}$ possesses at least one pair of links with linking number greater than two, there was no investigation as to whether this may be the least possible $n$.

By Theorem 3.1, $K_6$ is intrinsically linked. By Theorem 3.7, the minimal crossing number of an embedding of $K_5$ must be 1. Because linking number is defined as half of the signed crossing number, and linking number is an integer, the sum of signed crossing numbers for the graph must be at least 2 to have a link. Therefore $K_5$ can not be intrinsically linked. Consequently, $K_6$ is the least possible bound for being intrinsically linked with a link of linking number at least 2.

Using the computational method articulated in Appendix A.1, embeddings of $K_6$, $K_7$, and $K_8$ were found that did not contain any pair of linked cycles with the magnitude of linking number greater than 1. Because of this, it must be the case that neither $K_7$ nor $K_8$ can possess links with a greater complexity than the Hopf link in every embedding. Thus, neither $K_7$ nor $K_8$ are intrinsically linked with a 2-component link that has linking number of at least 2. Embeddings of $K_6$, $K_7$ and $K_8$ that contain maximum linking number 1 between their cycles are identified in Figure 6, Figure 7, and Figure 8, respectively.

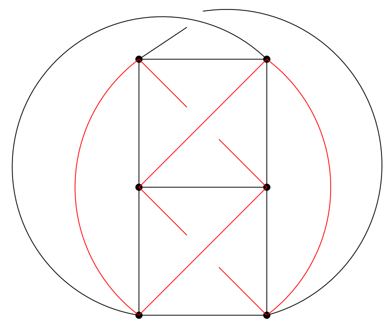

Figure 6: Embedding of $K_6$ with the red line representing the pair of linked cycles. Note they have linking number 1 [5].

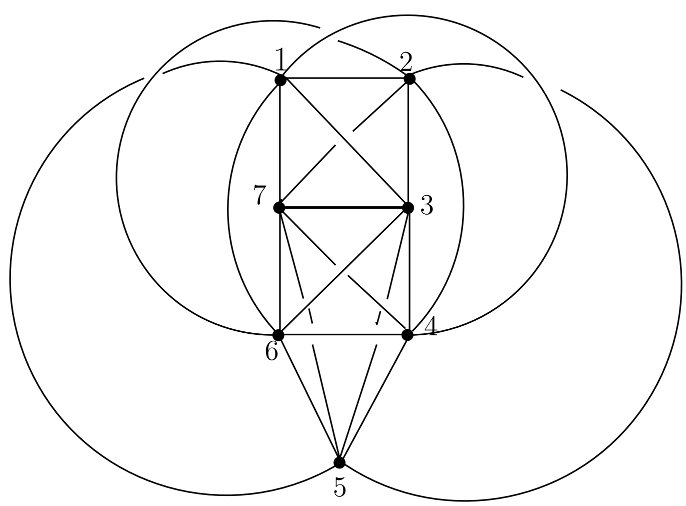


Figure 7: Embedding of $K_7$ with no pair of links with linking number greater than 1.

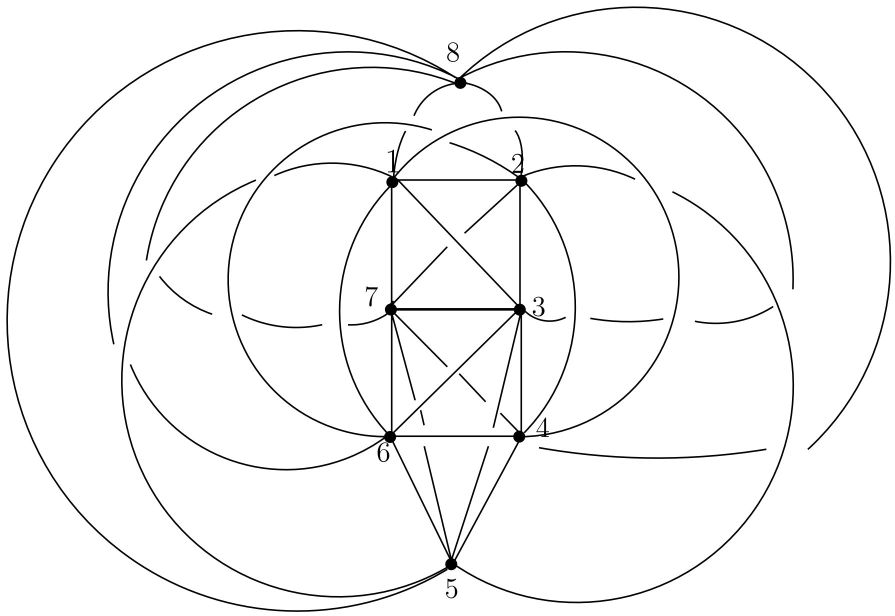


Figure 8: Embedding of $K_8$ with no pair of links with linking number than 1.

These two embeddings of $K_7$ and $K_8$ were generated by constructing general embeddings of $K_7$ and $K_8$ with either minimal or close to the minimal crossing number calculated from Theorem 3.5, then running computations with the computer program to check if any cycles with linking number of at least 2 were present. If these cycles were present, crossings were changed, the changes would be logged into the code and the code would recompute. For example, note that difference between the $K_7$ subgraph of the provided $K_8$ formed by the vertices $v_1, v_2, \ldots, v_7$ and the $K_7$ provided is the crossing between edges $(v_4, v_7)$ and $(v_3, v_5)$.

## 4.3 Linear Embeddings of Complete Graphs

We begin by considering the $K_7$ diagram in Figure 7 that does not contain any 2-component links with linking number $\geq 2$ and attempt to show a similar embedding as linear. Choose three separate vertices to lie on the same plane. We can then characterize all of the points by assigning heights in conjunction with a standardized projection on the

$xy$-plane in order to determine whether or not the points are able to exist as straight lines as depicted in the embedding without affecting the crossings contained within the graph. We found a linear embedding of $K_7$ using this method in Figure 9. A $K_6$ without linking number $\geq 2$ can be formed from removing $v_5$ and its corresponding edges from $K_7$ in Figure 9.

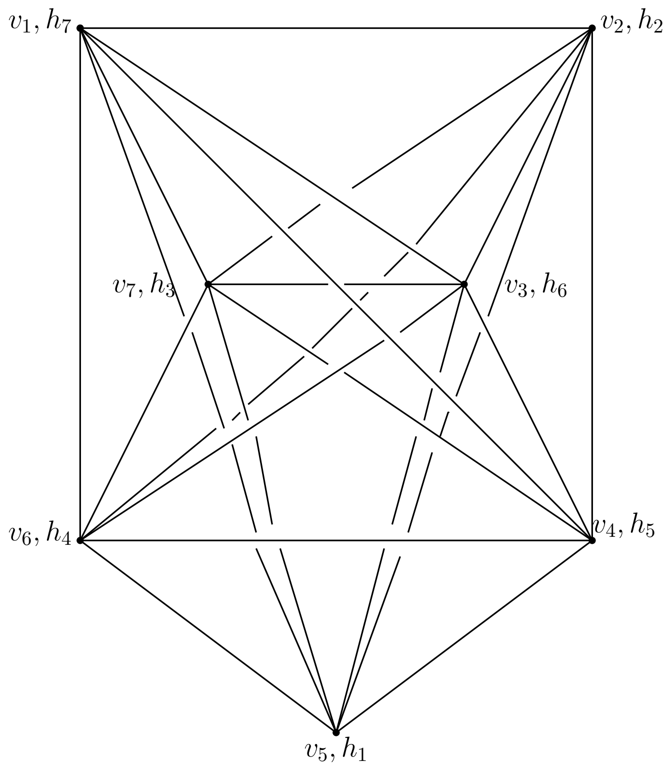


Figure 9: Linear embedding of $K_7$ with no pair of links with linking number greater than 1.

In the $x$-$y$-projection of the linear embedding in Figure 9, the only relationships given regarding how the embedding exists in space are which edges lie above or below others, given by crossings, and where they are relative to each other. If we are able to assign heights to every vertex so that for each crossing, the $x$ and $y$ coordinates of different edges at that point are identical but the heights, or $z$ coordinates, reflect the over and under crossings of the strands, then that diagram is able to exist as a linear embedding in $\mathbb{R}^3$. Beginning with a projection of $K_7$ on an $x$-$y$ plane with straight edges, if we are able to assign $z$ coordinates to all vertices that reflect the crossing information of that graph, we know that there exists that embedding of the graph in space.

In Figure 9, heights are associated for each vertex such that for each $h$, if $i < i + 1$ for some $i$, $h_i < h_{i+1}$, with the exception of $h_4 = h_5 = h_6 = 0$. In order to generate this diagram, we begin by choosing 3 vertices, $v_6, v_4$, and $v_3$ to be on the same plane. We then place $v_1$, connecting it to all other existing vertices so that all edges containing $v_1$ lie above the remaining edges. We follow the same procedure with $v_7$, but place it under all other vertices. Next we place $v_2$ below the construction, resulting in edges that lie below all other edges, and lastly repeating the same procedure for $v_5$. This procedure, because it is similar to the spatial embedding of $K_7$'s construction, also yields a complete graph with no links of linking number at least 2.

By Theorem 3.6, $K_9$ is known to have a 3-component link in all linear embeddings, and consequently $K_9$ must have a 2 component link with linking number $\geq 2$ by Theorem 3.3 [8]. Thus, by providing a linear embedding of $K_7$ that does not contain a link with linking number $\geq 2$, showing $K_9$ is a lower bound becomes a matter of whether $K_8$ is intrinsically linked with linking number at least 2.

## 4.4 $K_9$ Embedding Without Triple Link

Recall that Theorem 3.4 stated that $K_9$ is not intrinsically linked with a 3-component link. As an example of a $K_9$ embedding with no 3-component links, the embedding of $K_9$ illustrated in Figure 10 was given.

Because we know that a 3-component link in a graph suggests the presence of a 2-component link with linking number $\geq 2$ as well, this graph is interesting to analyze because it may suggest a small amount of 2-component links with linking number $\geq 2$.

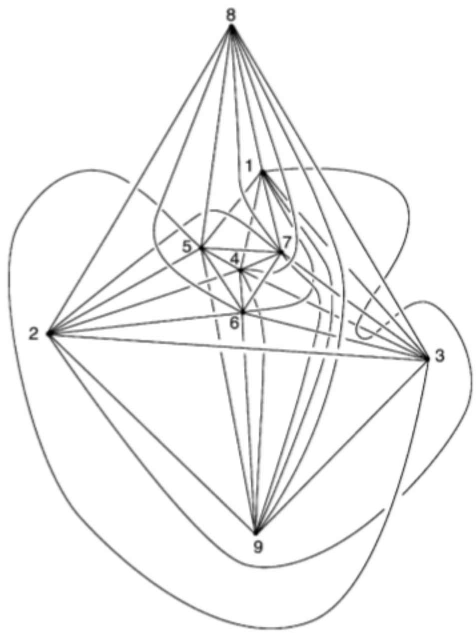


Figure 10: Embedding of $K_9$ with no 3-component link [9]

Although a minimal crossing number might seem as if it would prevent more complex structures from arising, the crossing number of the $K_9$ in Figure 10 is 56, which is significantly larger than 36, the minimal crossing number of $K_9$ generated by Theorem 3.7. Calculating the signs of all of the crossings from the convention articulated in Section 4.1 yields a total of 321 cycles of linking number $\geq 2$ in Figure 10. This is abnormal because it shows that the complexity of crossings required for possessing many 2-links with linking number $\geq 2$ also can lead to no 3-links.

As a means of further analyzing the relationship between crossings of a graph and the structures they form within a graph, we can create representations of crossings and edges through forms such as the signed crossing matrix, as mentioned in Section 4.1, or a crossing graph. Crossing relationships can be shown when considering the graph $G$'$= (E, C)$, where for a graph $G = (V, E)$, $E \subseteq V \times V$ and $C \subseteq E \times E$. $E$, or the edges in the original graph, correspond to vertices in $G$' while $C$ represents the set of crossings and corresponds to edges in $G$'. This graph has an adjacency matrix of the crossing matrix defined in Definition 4.1. In $G$', edges are connected if there is a crossing between them. Consider $G$' for the embedding of $K_9$ in Figure 11.

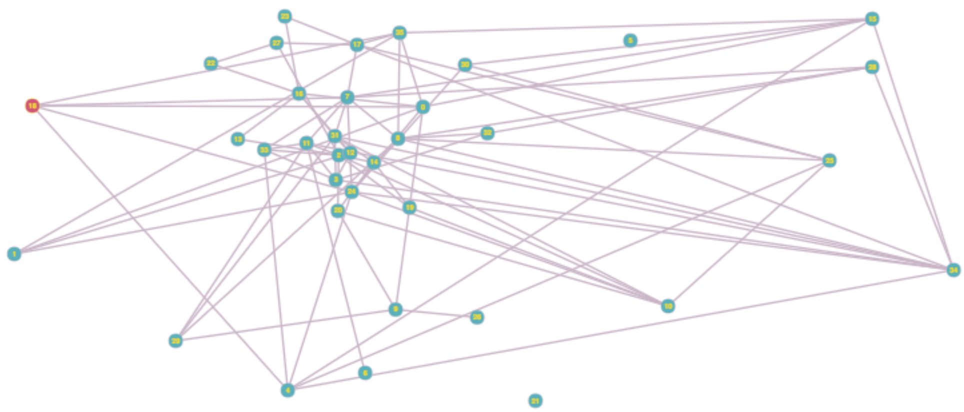

Figure 11: Crossing graph of a $K_9$ with no 3-component link [12]

The abundance of connections in the graph $G$' suggests presence of non-trivial topological structures that may reflect properties of the original graph. $G$' is a useful tool for analyzing the crossings within a graph and the corresponding structures they create within the graph.

# 5 Applications

Our results are an instance of an interplay between topology and geometry — the topology of the abstract graph and the geometry of its spatial embeddings. As indicated by our results, there is much space for geometric approaches in knot theory and topological graph theory.

These relationships between topology and geometry in the representations of graphs hold significance beyond mathematics as well. In physical networks, graph theory in addition to the topology and geometry that characterizes them are essential for modeling, such as for modeling the means of transporting nutrients in plants. Additionally, the structures possessed in graphs and knots formed by DNA polymers on a molecular level reflect physical properties and characteristics of those polymers.

In the study of transport networks, especially with respect to leaves, the presence or absence of loops can still reflect the optimization of the network for certain features. Venations in leaves are the transport networks for nutrients within the leaves. Katifori, Szöllősi, and Magnasco found that although loopless leaf venation has the benefit for efficiency, the presence of loops can allow for the optimal state of a transport network because high loop density can handle both resilience to damage and fluctuations in load [2]. Because our results are concerned with characterizations of topological representations of graphs, these types of concepts can be utilized in the study of efficient networks in biological and physical systems.

Topological analysis of graphs also is present in molecular level analysis of polymers as well as their physical properties. The Coulomb energy is the energy required to charge a conductive object and relates to the size and shape of the object. Vargas-Lara et al. found that these Coulomb energies are directly proportional to the knot crossing number of DNA[3]. By allowing for detections of similar structures, our results can allow for the simplification of analyses of the topological objects contained in complex structures, which can consequently be used to determine physical properties of these structures.

# 6 Discussion and Future Work

The results are primarily concerned with providing upper and lower bounds for the $n$ of $K_n$ graphs that are intrinsically linked with a structure of linking number $\geq 2$. We found a linear embedding of $K_7$ that did not contain links with linking number $\geq 2$, demonstrating that $K_7$ is not intrinsically linked with these structures. Because it was previously proven that $K_9$ is intrinsically linked with a 3-component link, applying Theorem 3.3 reveals that $K_9$ contains a 2-component link with linking number $\geq 2$. However, it is unknown whether

$K_9$ is the smallest $K_9$ with this structure because it is unknown whether $K_8$ contains a link with linking number at least 2. We do, however, show that for linear embeddings of complete graphs the bound for the least $n$ of a $K_n$ intrinsically linked with linking number at least 2 is $8 \leq n \leq 9$.

We also exhibit spatial embeddings of $K_7$ and $K_8$ that do not possess links with linking number $\geq 2$, proving that $K_7$ and $K_8$ are not intrinsically linked with such structures. It is known that $K_{10}$ is intrinsically linked with this structure in all spatial embeddings, yet it is not known if $n = 10$ is the least possible $K_n$ that satisfies this condition. We show that the least $n$ for a complete graph $K_n$ containing linking number at least 2 in spatial embeddings must satisfy $9 \leq n \leq 10$.

The most direct future work of this project would be concerned with investigating the intrinsic linkedness with linking number $\geq 2$ of $K_9$ in spatial embeddings as well as $K_8$ in linear embeddings— if $K_9$ is intrinsically linked with this structure, then $K_9$ is the least complete graph with this property, and if it does not then $K_{10}$ is. Similarly, if $K_8$ is intrinsically linked with a linking number $\geq 2$ in linear embeddings then it is the least $K_n$ with this property; otherwise, $K_9$ is. Our investigation into this problem found that even in embeddings of $K_9$ with no 3-component links, there were hundreds of 2-component links with linking number $\geq 2$. Further research can investigate the prevalence of different types of links or knots in various spatial embeddings of complete graphs as well as linear embeddings.

The significance of the crossing matrix and crossing graph in describing structures formed within the graphs they are derived from could provide interesting results as well. The crossing matrix and crossing graph could provide new avenues for investigating linkedness and knottedness in their parent graph.

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# A Computational Methodology

## A.1 Computational Approach

Within the program[1], the standard direction of signed crossing number for the intents and purposes of the code was chosen by considering the direction of the edge going away from the vertex labelled with a lower number and towards the vertex labelled with the higher number. Note that this choice is arbitrary.

The program generates the signed crossing matrix of a given embedding as a parameter through user input, utilizing crossing information. The program also enumerates all pairs of vertices as rows and columns of the signed crossing matrix. All possible cycles of all possible lengths 3 through $n-3$ are generated. By looping through all pairs of lengths of possible cycles of a graph $K_n$, e.g.

$$(3,3),(3,4),\ldots,(3,n-3),(4,4),\ldots,$$

the program can run through all pairs of disjoint cycles possible in the particular embedding of a graph in an organized fashion. During this process, linking numbers are noted, allowing the program to display the complexity of links in the given embedding of a graph. The program would reverse signs of certain edges composing directed cycles when the direction of multiple edges were conflicting. For example, consider the cycle $(v_1v_4v_3)$. The direction of the first edge, $(v_1, v_4)$, in this 3-cycle matches the signing convention of going from the lesser numbered vertex to the greater number, while the next two edges $(v_4, v_3)$ and $(v_3, v_1)$ do not. In this scenario, the program would "switch" the sign of the crossing numbers of these edges in the linking number computation.

User input consists of the edges that cross and the sign of their crossing. From this input, the sign crossing matrix $C$ is generated. Independent of the crossing matrix, several intermediate steps are required for the generation of disjoint cycles. The cycle information

[1]The code is publicly available at www.github.com/sabzer/IntrinsicLinkedness

for the complete graph was mainly stored in arrays The largest array is constructed with elements that are arrays of cycle types. Suppose one is characterizing the cycles of $K_n$ in code, for some given $n$. This array is be of length $n-5$, and contain in its first element all 3-cycles in $K_n$, in its second element 4-cycles in $K_n$, etc. up to $n-3$-cycles in $K_n$. Each of these subarrays of cycles of length $i$ are created by first choosing a combination of $i$-cycles, and then running permutations that generate the remaining cycles of that length. Identical cycles that may have been overcounted in permutations are avoided by checking to see if permutations are either backwards or backwards and shifted; by fixing the first position in a permutation we avoid wasting significant computational power in deleting cycles.

Once all cycles are generated, cycles of a given length $i$ are considered in conjunction with all other cycles of length $i, i+1, \ldots, n-i$. A check is run to see if the cycles are disjoint by comparing the vertices present in the cycles; if they are not, the linking number calculation for that pair of cycles is abandoned. If they are, then the linking number between them is calculated and noted.

To calculate linking number solely from the computational information stored by the crossing matrix, recall that we develop the expression

$$lk((a_1, \ldots, a_p), (b_1, \ldots, b_q)) = \sum_{i=1,2\ldots,p_{j=1,2,\ldots,q}} C_{a_i a_{i+1} b_j b_{j+1}} \operatorname{sgn}(a_{i+1} - a_i) \operatorname{sgn}(b_{j+1} - b_j),$$

where $\operatorname{sgn}(x)$ is the signum function, and the sequences of vertices represent cycles. With this expression, the program can run through all pairs of disjoint cycles of a given embedding of a complete graph and note non-zero crossing numbers, as well as the cycles that lead to these crossing numbers. This allows the program to quickly determine whether an embedding of $K_7$, $K_8$, or $K_9$ possess at least one pair of disjoint cycles in an embedding that does not contain a link with linking number $\geq 2$ when generating counterexamples to prove that a graph is not intrinsically embedded with that link. If it does find a pair, we are uncertain whether that graph is intrinsically embedded with a linking number $\geq 2$ and continue searching for a counterexample.

### A.1.1 Computational Complexity

We proceed with an investigation into the computational complexity of the program used to determine the linking numbers of all pairs of disjoint cycles of a given $K_n$.

Take some $K_n$. The possible lengths of cycles are 3 to $n-3$. The code searches through all possible pairs of cycles of lengths that add up to at most $n$ and calculate their linking numbers only if the cycles are disjoint. We will investigate how many linking number computations the program must undergo when running a complete analysis on the linking number between all pairs of disjoint cycles in a graph.

Consider two possible cycle lengths $a$ and $b$ in a $K_n$, where $a \geq 3,\ b \geq 3,$ and $a+b \leq n$. Then the number of disjoint cycles of the given $K_n$ that the code would have to run through is equal to

$$\binom{n}{a}\frac{(a-1)!}{2}\binom{n-a}{b}\frac{(b-1)!}{2},$$

if $a \neq b$, and

$$\frac{1}{2}\binom{n}{a}\frac{(a-1)!}{2}\binom{n-a}{a}\frac{(a-1)!}{2}$$

if $a = b$. Thus, for a given $K_n$, the total quantity of cycles for which the linking number is calculated in the code is

$$\sum_{a=3}^{\lfloor \frac{n}{2} \rfloor}\sum_{b=a+1}^{n-a}\binom{n}{a}\frac{(a-1)!}{2}\binom{n-a}{b}\frac{(b-1)!}{2}+\sum_{a=3}^{\lfloor \frac{n}{2} \rfloor}\binom{n}{a}\frac{(a-1)!}{2}\binom{n-a}{b}\frac{(b-1)!}{2}. \tag{1}$$

Now consider the possible cycle lengths in $K_{2n}$ and the corresponding quantities of disjoint cycles. A case analysis reveals that the disjoint cycles of length $n$ dominate the computation by a significant margin. To approximate the computational weight of this case we can employ Stirling's approximation. From Equation 1, the number of disjoint pairs of $n$-cycles is $\frac{1}{2}\binom{2n}{n}(\frac{(n-1)!}{2})^2$. Factorization and reorganization yields the expression $\frac{1}{8n^3} \times \frac{(2n)!}{n!}$. We then apply Stirling's approximation for large factorials, which states that for large $n$,

$n! \sim (\frac{n}{e})^n\sqrt{2\pi n}$.

Utilizing Stirling's approximation on Equation 1 yields

$$\frac{(2n)!}{n!} \sim \frac{\left(\frac{2n}{e}\right)^{(2n)}\sqrt{4\pi n}}{\left(\frac{n}{e}\right)^{n}\sqrt{2\pi n}} = \sqrt{2} \times 2^{2n} \times n^n \times e^{-n} \sim \left(\frac{4n}{e}\right)^n$$

for large $n$.

Here we can see that as $2n$ gets increasingly large, the computations that the computer has to go through increases around the order of $n^n$. Because in $K_{2n}$ the $n$-cycle case dominates the computation, this is a sufficient expression with which to determine computational complexity.