Strang Splitting for the Variable-coefficient Burgers' Equation

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Abstract

In this paper, we use Strang splitting to solve the variable-coefficient Burgers' equation $u_t = a(t)u_{xx} + b(x,t)uu_x$. We prove by showing the stability and consistency of Strang splitting that if the equation is well-posed, then the Strang splitting method has first-order convergence. The convergence of Strang splitting allows us to numerically solve the variable-coefficient Burgers' equation by solving its two constituent equations: the heat equation and the inviscid Burgers' equation. We have good numerical methods for these two equations respectively.

Summary

In this paper, we apply a method called Strang splitting to solve a partial differential equation called the variable-coefficient Burgers' equation, which has applications in acoustics. Strang splitting is an algorithm for computers to approximate the exact solution of the equation and it can be useful when finding exact solutions is difficult. We prove that the solution obtained by Strang splitting converges to the exact solution when we devote enough computing power.

1 Introduction

One well-studied PDE that has applications in areas such as fluid mechanics and traffic flow is Burgers' equation [1]. In one-dimensional space, Burgers' equation takes the form

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = \nu\frac{\partial^2 u}{\partial x^2}, \tag{1}$$

where $u(x,t)$ is a scalar function of position x and time t, and the constant ν is called the diffusive constant [2]. A generalization of the 1D equation is the variable-coefficient Burgers' equation

$$\frac{\partial u}{\partial t} = a(t)\frac{\partial^2 u}{\partial x^2} + b(x,t)u\frac{\partial u}{\partial x}, \tag{2}$$

where $a(t)$ is a functions of time and $b(x,t)$ is a function of position and time. Equation (2) is useful in nonlinear acoustics where the two coefficients are physical parameters of the medium. Analytically, Equation (2) can no longer be solved using the Cole-Hopf transformation used for the regular Burgers' equation [3], but requires transformations using Lie groups [4].

Our study aims to solve the variable-coefficient Burgers' equation numerically. Numerical methods are algorithms that approximate the solutions of the equation. The main difficulty in numerically solving Equation (2) is that Burgers' equation is a mixed equation, because it has both the term uu_x, which belongs to the hyperbolic inviscid Burgers' equation, and the term u_{xx}, which belongs to the parabolic heat equation [5]. For parabolic equations, the finite difference method usually works well, while for nonlinear hyperbolic equations, such as the inviscid Burgers' equation, the finite volume method is more suitable. For a mixed equation, standard methods that apply to either hyperbolic or parabolic equations may not be stable or converge.

There has been much research about numerical methods for Equation (1). For example, Aref and Daripa [6] used the finite difference method that discretizes space only to solve Equation (1). In 2011, Holden, Lubich, and Risebro [7] applied the operator splitting method to Equation (1) and proved second-order convergence for the method. Intuitively, the operator splitting method solves a mixed equation by solving its constitutive equations. In the case of Burgers' equation, this means we can use a combination of standard methods that solve the heat equation and the inviscid Burgers' equation to get a correct approximation. In this paper, we focus on a type of operating splitting method called Strang splitting and prove that the numerical solution obtained from Strang splitting has first-order convergence to the exact solution of the variable-coefficient Burgers' equation. The technique of using consistency and stability to prove convergence is modeled after the proof in [7].

Section 2 includes a description of important definitions and theorems the proof requires. Section 3 shows the convergence theorem we proved. Section 4 and Section 5 show results in the stability and consistency of the Strang splitting method, which are necessary for the proof of convergence in Section 6. Lastly, in Section 7, we discuss possible directions for future research.

2 Preliminaries

Definition 2.1 (Weak derivative)**.** On $\mathbb{R}$, $g(x)$ is a weak derivative of $f(x)$ if for all $\phi \in C_c^\infty(\mathbb{R})$, we have

$$\int_{\mathbb{R}} f\phi' \, dx = -\int_{\mathbb{R}} g\phi \, dx.$$

In this case, we write $g = f'$ or ∂f.

Definition 2.2 (H^m norm)**.** The H^m norm of a function $f(x)$ on $\mathbb{R}$ is defined as

$$\|f\|_{H^m} = \left(\sum_{k=0}^{m} \int_{\mathbb{R}} (\partial_x^k f)^2 \, dx \right)^{\frac{1}{2}}.$$

where $\frac{\partial}{\partial x}$ is a weak derivative in the sense of Definition 2.1.

Definition 2.3 (Well-posedness)**.** In this paper, Burgers' equation and its variable-coefficient versions are well-posed in H^s if for a given time T, for any initial condition $\|u_0\|_{H^s} \leq R$, there is a unique strong solution $u \in C([0,T];H^s)$ that is Lipschitz continuous, which means that there exists a Lipschitz bound $K(R,T)$ such that

$$\|\bar{u}(t_2) - u(t_2)\|_{H^s} \leq K\|\bar{u}(t_1) - u(t_1)\|_{H^s}. \tag{3}$$

for all $[t_1,t_2] \subset [0,T]$ where $\bar{u}(t_1)$ and $u(t_1)$ are arbitrary solutions to initial data $u_0, \overline{u}_0$ at time t_1 that have $\|\bar{u}(t_1)\|_{H^s} \leq R$ and $\|u(t_1)\|_{H^s} \leq R$.

Definition 2.4 (Fréchet Derivative for Operators)**.** Let $T : H^m \to H^s$ be an operator acting on Sobolev spaces. The Fréchet derivative of T at $f \in H^m$ is another operator, denoted as $dT(f)$ such that for all $h \in H^m$, we have

$$\lim_{\epsilon\to 0} \frac{\|T(f+\epsilon h) - Tf - dT(f)[\epsilon h]\|_{H^s}}{\epsilon\|h\|_{H^m}} = 0.$$

Theorem 2.1 (Sobolev embedding [8])**.** *For integers $s, k \geq 0$ such that $s > 1/2 + k$, if $f(x) \in H^s$, then $f \in C^k(\mathbb{R})$ and in fact, there exists a constant $C > 0$ such that*

$$\sum_{j=0}^{k} \|\partial_x^j f\|_{L^\infty} \leq C\|f\|_{H^s}.$$

Theorem 2.2 (Moser's Inequality [9])**.** *For functions $f(x)$, $g(x) \in H^{m+2}((-\infty,\infty))$, in particular $f, g \in L^\infty(\mathbb{R})$ by the theorem above, we have the Moser's inequality*

$$\sum_{k=0}^{m+2} \|\partial_x^k(fg) - f\partial_x^k g\|_{L^2} \leq C(\|f_x\|_{L^\infty}\|\partial_x^{m+1} g\|_{L^2} + \|\partial_x^{m+2} f\|_{L^2}\|g\|_{L^\infty}),$$

where C is a positive constant that only depends on $m+2$.

Theorem 2.3 (Algebra structure [9])**.** *Given $f(x)$, $g(x) \in H^s$ where $s > 1/2$, there exists a constant C such that*

$$\|fg\|_{H^s} \leq C\|f\|_{H^s}\|g\|_{H^s}. \tag{4}$$

Theorem 2.4 (Grönwall's Inequality [8])**.** *Given continuous functions $u(t)$ and $f(t)$ on interval $[a,b]$, if*

$$u' \leq f(t)u(t),$$

then for $t \in [a,b]$

$$u(t) \leq e^{\int_a^t f(s)ds} u(a).$$

Theorem 2.5 (Well-posedness of $u_t = u_{xx} + uu_x$ [10]). *The constant-coefficient Burgers' equation $u_t = u_{xx} + uu_x$ is well-posed in H^s for all $s \geq 0$.*

Theorem 2.6 (Well-posedness of $u_t = a(t)u_{xx} + b(t)uu_x$). *The space-independent-variable-coefficient Burgers' equation $u_t = a(t)u_{xx} + b(t)uu_x$ is well-posed in H^s for all $s \geq 0$.*

The proof is similar to the constant coefficient equation in the literature; the only difference is that in the proof we need

$$\int_0^T \|u_x(t,\cdot)\|_{L_x^2}^2 \, dt < \infty$$

To ensure this, we need $a(t) \geq c > 0$ for some constant c and all $t \in [0, T]$. But this is automatically true if we assume $a > 0$ and continuous.

Strang Splitting

To give a precise definition of Strang splitting, we introduce some key notations. Let the heat operator A be defined by $A : u \mapsto u_{xx}$ and the Burgers operator be defined by $B : u \mapsto uu_x$. For brevity, we define the operator $A_t = a(t)A$ and $B_t = b(x,t)B$. Using this notation, the variable-coefficient Burgers' equation can be written as

$$u_t = A_t u + B_t u, \tag{5}$$

Given the solution of Equation (5) $u(t_1)$ at time t_1, we can express the exact solution of Equation (5) at time t_2 as

$$u(t_2) = e^{\int_{t_1}^{t_2} A_t + B_t dt} u(t_1).$$

We use $\Phi_{A+B}^{t_1,t_2}$ to denote the solution operator of $e^{\int_{t_1}^{t_2} A_t + B_t dt}$. Similarly, for the heat equation

$$u_t = A_t u, \tag{6}$$

we use $\Phi_A^{t_1,t_2}$ to denote heat equation solution operator $e^{\int_{t_1}^{t_2} A_t dt}$. For the inviscid Burgers' equation

$$u_t = B_t u, \tag{7}$$

we use $\Phi_B^{t_1,t_2}$ to denote its solution operator $e^{\int_{t_1}^{t_2} B_t dt}$.

Discretizing the time domain $[0,T]$ into time steps $0 = t_0 < t_1 < \cdots < t_N = T$ with uniform mesh size $\Delta t = T/N$, we use u_n to denote the numerical solution at time $t_n \in \{t_1, t_2, \ldots, t_N\}$. Using the Strang splitting method to solve the variable-coefficient Burgers' equation, given the numerical solution u_n, the solution at the time step $(n+1)\Delta t$ by

$$u_{n+1} = \Phi_A^{(n+\frac{1}{2})\Delta t,(n+1)\Delta t} \Phi_B^{n\Delta t,(n+1)\Delta t} \Phi_A^{n\Delta t,(n+\frac{1}{2})\Delta t} u_n. \tag{8}$$

We use the shorthand $\Psi^{n\Delta t,(n+1)\Delta t} = \Phi_A^{(n+\frac{1}{2})\Delta t,(n+1)\Delta t} \Phi_B^{n\Delta t,(n+1)\Delta t} \Phi_A^{n\Delta t,(n+\frac{1}{2})\Delta t}$ for convenience. The initial condition is denoted by $u_0 = u(0)$. Given an initial condition, the numerical solution u_n is therefore produced by

$$u_n = \Psi^{(n-1)\Delta t,n\Delta t} \Psi^{(n-2)\Delta t,(n-1)\Delta t}, \ldots, \Psi^{\Delta t,0} u_0.$$

3 Main Result

We prove the convergence of Strang splitting under the following conditions. Let integer $m \geq 2$. The value of m is kept fixed in the rest of the paper. We assume $b(x,t) \in C\big([0,T]; C^{m+2}(\mathbb{R})\big)$, $a(t) > 0$, and $a(t) \in C^1([0,T])$. We consider the solution of the equation on the domain $(-\infty, \infty) \times [0,T]$ with the vanishing at infinity boundary condition: $\lim_{x\to\infty} \partial_x^j u(x,t) = 0$, $\lim_{x\to-\infty} \partial_x^j u(x,t) = 0$ for all $j \in \{0, 1, 2, \ldots, m+2\}$. Moreover, given time T, we consider solutions with the bound

$$\|u(t)\|_{H^{m+2}} \leq \rho < R,$$

for all $t \in [0,T]$. In particular, the initial data satisfies

$$\|u_0\|_{H^{m+2}} \leq \rho < R.$$

The main convergence theorem is the following.

Theorem 3.1 (First-order convergence). *Given $a(t)$ and $b(x,t)$, if $u_t = a(t)u_{xx} + b(x,t)uu_x$ is well-posed in H^m, and hence in H^{m+2} in the sense of Definition 2.3, then there exists a $\overline{\Delta t}$ such that for any $\Delta t \leq \overline{\Delta t}$ and $t_n = n\Delta t \leq T$,*

$$\|u_n - u(t_n)\|_{H^m} \leq \gamma \Delta t, \tag{9}$$

where $\overline{\Delta t}$, γ depend only on T, R, and ρ.

Because Theorem 2.6 ensures the well-posedness of the space-independent-variable-coefficient Burgers' equation

$$u_t = a(t)u_{xx} + b(t)uu_x, \tag{10}$$

we know that Strang splitting converges when solving this equation. However, well-posedness for space-dependent coefficients is harder and beyond the scope of the paper.

4 Stability of the Inviscid Burgers' Operator

In this section, we show that the solution to the variable-coefficient inviscid Burgers' equation $u_t = b(x,t)uu_x$ is bounded in H^{m+2} norm. This result is needed for proving consistency in Section 5 and convergence in Section 6.

Lemma 4.1. *Given $b(x,t)$, if $\|u(t)\|_{H^2} \leq \kappa$ for all $t \in [0,T]$, then the solution $u(t_1)$ to the variable-coefficient inviscid Burgers' equation at time t_1 has*

$$\|u(t_1)\|_{H^{m+2}} \leq e^{c\kappa(t_1-t_0)}\|u(t_0)\|_{H^{m+2}},$$

where c depends only on T and $u(t_0)$ is the solution at time t_0 for any $[t_0,t_1] \subseteq [0,T]$.

Proof. We bound $\|u(t)\|_{H^{m+2}}$ by first bounding the derivative of its square $\frac{d}{dt}\|u(t)\|^2_{H^{m+2}}$. Differentiating Equation (7) j times and taking the integral over $\mathbb{R}$ gives us

$$\frac{1}{2}\frac{d}{dt}\|\partial_x^j u(t)\|^2_{L^2} = \int_{-\infty}^{\infty} \partial_x^j u\, \partial_x^j(buu_x)\; dx.$$

Summing the terms for $j \in \{0,1,2,\ldots,m+2\}$ produces

$$\frac{1}{2}\frac{d}{dt}\|u(t)\|^2_{H^{m+2}} = \sum_{j=0}^{m+2}\int_{-\infty}^{\infty} \partial_x^j u\, \partial_x^j(buu_x)\; dx.$$

Applying the product rule on $\partial_x^k(b(x,t)uu_x)$ produces

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|u(t)\|_{H^{m+2}}^2 &= \sum_{j=0}^{m+2}\sum_{k=0}^{j}\binom{j}{k}\int_{-\infty}^{\infty}\partial_x^j u\,\partial_x^{j-k}b\,\partial_x^k(uu_x)\;dx\\ &= \underbrace{\sum_{j=0}^{m+2}\sum_{k=0}^{j}\binom{j}{k}\int_{-\infty}^{\infty}\partial_x^j u\,\partial_x^{j-k}b\,(\partial_x^k(uu_x)-u\partial_x^{k+1}u)\;dx}_{\text{I}}\\ &\quad+\underbrace{\sum_{j=0}^{m+2}\sum_{k=0}^{j}\binom{j}{k}\int_{-\infty}^{\infty}\partial_x^j u\,\partial_x^{j-k}b\,u\partial_x^{k+1}u\;dx}_{\text{II}}\end{aligned}$$

Let μ be the maximum binomial coefficient, which means $\binom{m+2}{k}\le\mu$ for all k, and applying the Cauchy-Schwarz inequality to I, we have

$$\text{I}\le\mu b_{max}\sum_{j=0}^{m+2}\sum_{k=0}^{j}\|\partial_x^j u\|_{L^2}\|\partial_x^k(uu_x)-u\partial_x^{k+1}u\|_{L^2},$$

where $b_{max}=\max_{j\in\{0,1,2,\ldots,m+2\}}\|\partial_x^j b\|_{L^\infty_{x,t}}$. Moser's inequality produces

$$\begin{aligned}\text{I}\le&\mu b_{max}\sum_{j=0}^{m+2}\sum_{k=0}^{j}2C\|u_x\|_{L^\infty}\|\partial_x^j u\|_{L^2}^2\\ \le&\mu b_{max}p\|u_x\|_{L^\infty}\|u\|_{H^{m+2}}^2\\ \text{(Theorem 2.1)}\quad\le&\mu b_{max}p\|u\|_{H^2}\|u\|_{H^{m+2}}^2.\end{aligned}$$

To bound II, we write II as a sum of two expressions produced by taking $k=j$ and $k<j$

$$\text{II}=\underbrace{\sum_{j=0}^{m+2}\int_{-\infty}^{\infty}\partial_x^j u\,\partial_x^{j+1}u\,bu\;dx}_{\text{II}_1}+\underbrace{\sum_{j=0}^{m+2}\sum_{k=0}^{j-1}\binom{j}{k}\int_{-\infty}^{\infty}\partial_x^j u\,\partial_x^{k+1}u\,\partial_x^{j-k}b\,u\;dx}_{\text{II}_2}.$$

From the chain rule, we have $\text{II}_1=\sum_{j=0}^{m+2}\int_{-\infty}^{\infty}\frac{1}{2}\partial_x(\partial_x^j u)^2bu\;dx$. We apply integration by parts to II_1 and because u vanishes at the boundary, we get

$$\text{II}_1=\sum_{j=0}^{m+2}-\frac{1}{2}\int_{-\infty}^{\infty}b(\partial_x^j u)^2u_x\;dx\le\sum_{j=0}^{m+2}\frac{1}{2}b_{max}\|u_x\|_{L^\infty}\|\partial_x^j u\|_{L^2}^2=\frac{1}{2}b_{max}\|u\|_{H^2}\|u\|_{H^{m+2}}^2$$

We continue to bound II_2

$$\begin{aligned}\text{II}_2&\le\sum_{j=0}^{m+2}\sum_{k=0}^{j-1}\mu b_{max}\|u\|_{L^\infty}\int_{-\infty}^{\infty}|\partial_x^j u|\,|\partial_x^{k+1}u|\;dx\\ &\le\sum_{j=0}^{m+2}\sum_{k=0}^{j-1}\mu b_{max}\|u\|_{L^\infty}\|\partial_x^j u\|_{L^2}\|\partial_x^{k+1}u\|_{L^2}.\end{aligned}$$

Because $k+1 \leq j$, from Sobolev embedding, we get

$$\mathrm{II}_2 \leq \sum_{j=0}^{m+2} \sum_{k=0}^{j-1} \mu b_{max} \|u\|_{L^\infty} \|\partial_x^j u\|_{L^2}^2 \leq \mu b_{max} p \|u\|_{H^2} \|u\|_{H^{m+2}}^2.$$

Because μ only depends on $m+2$, which is fixed, and b_{max} only depends on time T with $b(x,t)$ given, summing up I and II gives us

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|u(t)\|_{H^{m+2}}^2 &\leq c(T)\|u\|_{H^2}\|u\|_{H^{m+2}}^2 \\ \frac{d}{dt}\|u(t)\|_{H^{m+2}} &\leq c(T)\|u\|_{H^2}\|u\|_{H^{m+2}}.\end{aligned}$$

Applying Grönwall's Lemma bounds $\|u(t)\|_{H^{m+2}}$ by the exponential growth in Equation (4.1) and the proof is therefore complete. □

Lemma 4.2. *Given $b(x,t)$ and $t_0 \in [0,T]$, there exists a constant Δt_c with $t_0 + \Delta t_c \in [0,T]$ that depends only T and $\|u(t_0)\|_{H^m}$ such that for all $\Delta t \in [0, \Delta t_c]$ we have*

$$\|u(t_0 + \Delta t)\|_{H^m} \leq 2\|u(t_0)\|_{H^m}.$$

Proof. Replacing $m+2$ with m in the proof of Lemma 4.1 gives us

$$\frac{d}{dt}\|u(t)\|_{H^m} \leq c(T)\|u\|_{H^2}\|u\|_{H^m}.$$

Applying Sobolev inequality to $\|u_x\|_{L^\infty}$ produces

$$\frac{d}{dt}\|u(t)\|_{H^m} \leq c(T)\|u\|_{H^m}^2.$$

Because we have an inequality in the form of $y' \leq ky^2$, where $y(t) = \|u(t)\|_{H^m}$. Integrating gives us

$$y(t_0 + \Delta t) \leq \frac{y(t_0)}{1 - c(T)\Delta t y(t_0)}.$$

This bound for $\|u(t)\|_{H^m}$ shows that lemma is proven. □

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5 Consistency of Strang Splitting

This section proves that Strang splitting is second-order consistent, which informally means that the local error at a single time step is proportional to $(\Delta t)^2$. This bound for local error is necessary to prove the convergence of Strang splitting in Section 6.

Lemma 5.1. *Given functions $a(t)$ and $b(x,t)$, if $\|u_0\|_{H^{m+2}} < R$ and $\Delta t \le T$, then the local error of the Strang splitting in H^m norm satisfies*

$$\|\Psi^{0,\Delta t}(u_0) - \Phi^{0,\Delta t}_{A+B}(u_0)\|_{H^m} \le c_1(R,T)\Delta t^2, \tag{11}$$

where c_1 is a constant that depends only on R and T.

Proof. In this proof, we use the shorthand $b(t) = b(x,t)$ and $b'(t) = \partial_t b(x,t)$ for convenience. Let $\phi(s)$ be $e^{\int_s^t A_\beta d\beta} u(s)$, we have $\phi'(s) = e^{\int_s^t A_\beta d\beta} B_t u$. The equation $\phi(t) = \phi(0) + \int_0^t \phi' \, ds$ from the variation-of-constants formula therefore gives us an expression for the exact solution to the variable-coefficient Burgers' equation at time t

$$u(t) = e^{\int_0^t A_\beta d\beta} u_0 + \int_0^t e^{\int_s^t A_\beta d\beta} B_t u(s) \, ds. \tag{12}$$

Let $\phi(\sigma) = e^{\int_\sigma^s A_\beta d\beta} u(\sigma)$, then $\frac{d}{d\sigma} B_t(\phi(\sigma)) = b'(\sigma)B(\phi) + b(\sigma) dB(\phi)[\phi']$. Here, $dB(u)[v]$ is the notation for the Fréchet derivative of the operator B at u acting on v and $dB(u)[v] = u_x v + u v_x$. Hence

$$B_t(u(s)) = B_t(e^{\int_0^s A_\beta d\beta} u_0) + \int_0^s b'(\sigma)B(\phi) + b(\sigma) dB(\phi)[\phi'] \, d\sigma, \tag{13}$$

where $\phi'(\sigma) = e^{\int_\sigma^s A_\beta d\beta} B_t u(\sigma)$. Using equation (12) and (13) and having $t = \Delta t$, we have

$$\Phi^{0,\Delta t}_{A+B}(u_0) = u(\Delta t) = e^{\int_0^{\Delta t} A_\beta d\beta} u_0 + \int_0^{\Delta t} e^{\int_s^{\Delta t} A_\beta d\beta} B_t(e^{\int_0^s A_\beta d\beta} u_0) \, ds + e_1, \tag{14}$$

where

$$e_1 = \int_0^{\Delta t} \int_0^s e^{\int_s^{\Delta t} A_\beta d\beta} (b'(\sigma)B(\phi(\sigma)) + b(\sigma) dB(\phi(\sigma))[\phi'(\sigma)]) \, d\sigma ds.$$

Let v_0 be the initial condition for the inviscid Burgers' equation $v_t = B_t v$, the solution at time Δt can be written as a Taylor series with an integral remainder

$$\begin{aligned} v(\Delta t) &= v_0 + \Delta t b(0) B v_0 + \Delta t^2 \int_0^1 (1-\theta)(b'(\Delta t\theta) B v + b(\Delta t\theta) dB(v)[v']) \, d\theta \\ &= v_0 + \Delta t b(\frac{\Delta t}{2}) B v_0 + \Delta t^2 \int_0^1 (1-\theta)(b'(\Delta t\theta) B v + b(\Delta t\theta) dB(v)[v']) \, d\theta \\ &\quad - \Delta t^2 \frac{B v_0}{2} \int_0^1 b'\left(\frac{\Delta t}{2}\theta\right) \, d\theta. \end{aligned}$$

We therefore get an expression for the numerical solution $\Psi^{0,\Delta t}(u_0)$ produced by Strang splitting

$$\Psi^{0,\Delta t}(u_0) = e^{\int_0^{\Delta t} A_\beta d\beta} u_0 + \Delta t e^{\int_{\frac{\Delta t}{2}}^{\Delta t} A_\beta d\beta} b\left(\frac{\Delta t}{2}\theta\right) B(e^{\int_0^{\frac{\Delta t}{2}} A_\beta d\beta} u_0) + e_2, \tag{15}$$

where

$$e_2 = \underbrace{\Delta t^2 \int_0^1 (1-\theta) e^{\int_{\frac{\Delta t}{2}}^{\Delta t} A_\beta d\beta} (b'(\Delta t\theta) Bv + b(\Delta t\theta)\, dB(v)[v'])\ d\theta}_{e_{21}}$$
$$\underbrace{-\frac{\Delta t^2}{2} e^{\int_{\frac{\Delta t}{2}}^{\Delta t} A_\beta d\beta} Bv_0 \int_0^1 b'\left(\frac{\Delta t}{2}\theta\right)\ d\theta}_{e_{22}}.$$

Using Equation (15) and (14), the local error in Equation (11) can then be written as

$$\Psi^{\Delta t}(u_0) - \Phi^{\Delta t}_{A+B}(u_0) = \Delta t e^{\int_{\frac{\Delta t}{2}}^{\Delta t} A_\beta d\beta} b\left(\frac{\Delta t}{2}\theta\right) B(e^{\int_0^{\frac{\Delta t}{2}} A_\beta d\beta} u_0) \tag{16}$$
$$- \int_0^{\Delta t} e^{\int_s^{\Delta t} A_\beta d\beta} B_t(e^{\int_0^s A_\beta d\beta} u_0)\ ds + e_2 - e_1 \tag{17}$$
$$= e_0 + e_2 - e_1. \tag{18}$$

Now, we show that this error term is second order with respect to Δt. The principle error term, e_0, can be considered as the quadrature error of the midpoint rule for the integral $\int_0^{\Delta t} f(s)\ ds$, where

$$f(s) = e^{\int_s^{\Delta t} A_\beta d\beta} b(s) B(e^{\int_0^s A_\beta d\beta} u_0).$$

The principle error term can therefore be written as

$$e_0 = \Delta t f\left(\frac{\Delta t}{2}\right) - \int_0^{\Delta t} f(s)\ ds = \Delta t^2 \int_0^1 k(\theta) f'(\Delta t\theta)\ d\theta$$

where $k(\theta)$ is the Peano kernel [7]. To bound e_0, we expand $f'(s)$ and get

$$\begin{aligned} f'(s) =& e^{\int_s^{\Delta t} A_\beta d\beta} b'(s) B(e^{\int_0^s A_\beta d\beta} u_0) \\ & - e^{\int_s^{\Delta t} A_\beta d\beta} (A_s(b(s) B(e^{\int_0^s A_\beta d\beta} u_0)) - b(s) dB(e^{\int_0^s A_\beta d\beta} u_0)[A_s e^{\int_0^s A_\beta d\beta} u_0]) \\ =& \underbrace{e^{\int_s^{\Delta t} A_\beta d\beta} b'(s) B(e^{\int_0^s A_\beta d\beta} u_0)}_{e_{01}} \\ & \underbrace{- a(s) e^{\int_s^{\Delta t} A_\beta d\beta} (b_{xx} B(e^{\int_0^s A_\beta d\beta} u_0) + 2 b_x(x,s) (B(e^{\int_0^s A_\beta d\beta} u_0))_x)}_{e_{02}} \\ & \underbrace{- a(s) e^{\int_s^{\Delta t} A_\beta d\beta} (b(s) [A,B] (e^{\int_0^s A_\beta d\beta} u_0)}_{e_{03}}, \end{aligned}$$

where $[A,B]$ is the Lie commutator defined by $[A,B](v) = dA(v)[Bv] - dB(v)[Av]$. Because the operator $e^{\int_s^{\Delta t} A_\beta d\beta}$ decreases H^m norm we get

$$\begin{aligned}\|e_{01}\|_{H^m} &\leq \|b'\|_{L^\infty}\|B(e^{\int_0^s A_\beta d\beta}u_0)\|_{H^m}\\ &\leq \|b'\|_{L^\infty}C\|e^{\int_0^s A_\beta d\beta}u_0\|_{H^m}\|e^{\int_0^s A_\beta d\beta}u_0\|_{H^{m+1}}\\ &\leq C\|b'\|_{L^\infty}\|u_0\|_{H^m}\|u_0\|_{H^{m+1}}\end{aligned}$$

For e_{02} we compute

$$\|e_{02}\|_{H^m} \leq \|a\|_{L^\infty}(\|b_{xx}\|_{L^\infty}C_1\|u_0\|_{H^m}\|u_0\|_{Hm+1} + 2\|b_x\|_{L^\infty}\|u_0\|_{H^{m+1}}\|u_0\|_{Hm+2})$$

From [7], we have $\|[A,B](v)\|_{H^m} \leq C\|v\|^2_{H^{m+2}}$, which allows us to compute the bound for e_{03}

$$\|e_{03}\|_{H^m} \leq C\|b(s)\|_{L^\infty}\|a(s)\|_{L^\infty}\|u_0\|^2_{H^{m+2}}. \tag{19}$$

Because the Peano kernel $k(\theta)$ is bounded, we have $\|e_0\|_{H^m} \leq C\Delta t^2$, where C only depends on $\|b'\|_{L^\infty}$, $\|b_{xx}\|_{L^\infty}$, $\|b_x\|_{L^\infty}$, $\|b(s)\|_{L^\infty}$, and $\|a(s)\|_{L^\infty}$. Next we compute the bound for e_1, plugging in $\phi(\sigma) = e^{\int_\sigma^s A_\beta d\beta}u(\sigma)$, we have

$$\|e_1\|_{H^m} \leq \int_0^{\Delta t}\int_0^s \|b'(\sigma)B(e^{\int_\sigma^s A_\beta d\beta}u(\sigma)) + b(\sigma)dB(e^{\int_\sigma^s A_\beta d\beta}u(\sigma))[e^{\int_\sigma^s A_\beta d\beta}B_\sigma u(\sigma)]\|_{H^m}\, d\sigma ds.$$

Because $dB(u)[v] = (uv)_x$, we get

$$\begin{aligned}\|e_1\|_{H^m} \leq& C_1\|b'(\sigma)\|_{L^\infty}\int_0^{\Delta t}\int_0^s \|u(\sigma)\|_{H^m}\|u(\sigma)\|_{H^{m+1}}\, d\sigma ds\\ &+ \|b(\sigma)\|_{L^\infty}\int_0^{\Delta t}\int_0^s \|((e^{\int_\sigma^s A_\beta d\beta}u(\sigma))(e^{\int_\sigma^s A_\beta d\beta}B_\sigma u(\sigma)))_x\|_{H^m}\, d\sigma ds\\ \text{(Theorem 2.3)}\quad \leq& C_1\|b'(\sigma)\|_{L^\infty}\int_0^{\Delta t}\int_0^s \|u(\sigma)\|^2_{H^{m+2}}\, d\sigma ds\\ &+ C_2\|b(\sigma)\|_{L^\infty}\int_0^{\Delta t}\int_0^s \|u(\sigma)\|_{H^{m+1}}\|b(\sigma)Bu(\sigma)\|_{H^{m+1}}\, d\sigma ds\\ \text{(Theorem 2.3)}\quad \leq& C_1\|b'(\sigma)\|_{L^\infty}\int_0^{\Delta t}\int_0^s \|u(\sigma)\|^2_{H^{m+2}}\, d\sigma ds\\ &+ C_2\|b(\sigma)\|^2_{L^\infty}\int_0^{\Delta t}\int_0^s \|u(\sigma)\|^2_{H^{m+1}}\|u(\sigma)\|_{H^{m+2}}\, d\sigma ds\\ \leq& C(R,b)\Delta t^2.\end{aligned}$$

With e_0 and e_1 bounded, we are left with e_2. Separating e_{21} into two terms, we get

$$\begin{aligned}
\|e_{21}\|_{H^m} \leq & \Delta t^2 \Big(\int_0^1 \|b'(\Delta t\theta) B v\|_{H^m} \, d\theta + \int_0^1 \|b\,(\Delta t\theta)\, dB(v)[v']\|_{H^m} \, d\theta \Big) \\
\text{(Theorem 2.3)} \quad \leq & \Delta t^2 \Big(C_1 \|b'\|_{L^\infty} \int_0^1 \|v(\Delta t\theta)\|_{H^m} \|v(\Delta t\theta)\|_{H^{m+1}} \, d\theta \\
& + \|b\|_{L^\infty} \int_0^1 \|v(\Delta t\theta) B_t(v(\Delta t\theta))\|_{H^{m+1}} \Big) \\
\text{(Theorem 2.3)} \quad \leq & \Delta t^2 \Big(C_1 \|b'\|_{L^\infty} \int_0^1 \|v(\Delta t\theta)\|_{H^m} \|v(\Delta t\theta)\|_{H^{m+1}} \, d\theta \\
& + C_2 \|b\|_{L^\infty}^2 \int_0^1 \|v(\Delta t\theta)\|_{H^{m+1}}^2 \|v(\Delta t\theta)\|_{H^{m+2}} \, d\theta \Big).
\end{aligned}$$

Because $v(\Delta t\theta) = \Phi_B^{0,\theta\Delta t} e^{\int_0^{\frac{\Delta t}{2}} A_\beta d\beta} u_0$, and Lemma 4.2 has $\|\Phi_B^{\theta\Delta t} u_0\|_{H^{m+2}} \leq 2R$ when Δt is sufficiently small, we have $\|e_{21}\|_{H^m} \leq C\Delta t^2$, where C only depends on R, $\|b'\|_{L^\infty}$, and $\|b\|_{L^\infty}$. Lastly, we look at e_{22} and get

$$\|e_{22}\|_{H^m} \leq \Delta t^2 \frac{\|b'\|_{L^\infty}}{2} \|u_0\|_{H^m} \|u_0\|_{H^{m+1}}.$$

Summing up norms of e_0, e_1, and e_3, we see that $\|\Psi^{\Delta t}(u_0) - \Phi_{A+B}^{\Delta t}(u_0)\|_{H^m} \leq c_1 \Delta t^2$, where c_1 depends on $\|a(t)\|_{L^\infty}$ for $t \in [0, T]$, and $\|b(x,t)\|_{L^\infty}$, $\|\partial_t b(x,t)\|_{L^\infty}$, $\|\partial_x b(x,t)\|_{L^\infty}$, and $\|\partial_x^2 b(x,t)\|_{L^\infty}$for $t \in [0,T]$ and $x \in [0,1]$. With a and b given, the L^∞ norms only depend on T. The proof is complete. □

6 Proof of Convergence

Proof. The proof for Theorem 3.1 is similar to the proof in [7] and uses the same strong induction. Assuming $n\Delta t \leq T$, the induction hypothesis is that for $k \leq n-1$, we have

$$\begin{aligned}
\|u_k\|_{H^m} \leq & R \\
\|u_k\|_{H^{m+2}} \leq & e^{2cRk\Delta t} \|u_0\|_{H^{m+2}} \leq C_0 \\
\|u_k - u(t_k)\|_{H^m} \leq & \gamma \Delta t,
\end{aligned}$$

where $C_0 = e^{2cRT} \|u_0\|_{H^{m+2}}$ with c from Lemma 4.1, and $\gamma = K(R,T) c_1(C_0, T) T$, where c_1 is the constant from Lemma 5.1. We prove that these conditions hold for $k = n$. For simplicity, we use the shorthand $\Phi^{k,n}_{A+B} = \Phi^{k\Delta t, n\Delta t}_{A+B}$. We adopt the notation u^k_n from [7], which is defined by

$$u^k_n = \Phi^{k,n}_{A+B}(u_k).$$

where u_k is the numerical solution at time $k\Delta t$. The global error $\|u_n - u(t_n)\|_{H^m}$ is bounded by

$$\begin{aligned}\|u_n - u(t_n)\|_{H^m} &\leq \sum_{k=0}^{n-1} \|u^{k+1}_n - u^k_n\|_{H^m} \\ &\leq \sum_{k=0}^{n-1} \|\Phi^{k+1,n}_{A+B}(\Psi^{k,k+1}u_k) - \Phi^{k+1,n}_{A+B}(\Phi^{k,k+1}_{A+B}u_k)\|_{H^m}.\end{aligned}$$

Because of the Lipschitz continuity of the well-posedness assumption, we can find $K(R, \Delta t)$ such that

$$\|\bar{u}(t + \Delta t) - u(t + \Delta t)\|_{H^m} \leq K(R, \Delta t)\|\bar{u}(t) - u(t)\|_{H^m}$$

for any $t \in [0, T]$ and any $\|\bar{u}(t)\|_{H^m} \leq R$, $\|u(t)\|_{H^m} \leq R$. We look at the bound of the H^m norm of $\Phi^{k,k+1}_{A+B}u$ using $K(R, \Delta t)$ and get

$$\begin{aligned}\|\Phi^{k,k+1}_{A+B}u_k\|_{H^m} \leq& \|\Phi^{k,k+1}_{A+B}u_k - \Phi^{k,k+1}_{A+B}u(t_k)\|_{H^m} + \|u(t_{k+1})\|_{H^m} \\ \leq& K(R, \Delta t)\gamma\Delta t + \rho.\end{aligned}$$

Because $\rho < R$, this expression shows that $\|\Phi^{k,k+1}_{A+B}u_k\|_{H^m} < R$ for Δt sufficiently small. Because the inductive hypothesis has $\|\Phi^{k,k+1}_{A+B}u_k\| \leq R$, we have

$$\|u_n - u(t_n)\|_{H^m} \leq \sum_{k=0}^{n-1} K(R, T)\|\Psi^{k,k+1}u_k - \Phi^{k,k+1}_{A+B}u_k\|_{H^m}.$$

For each k, we define $a_k(t) = a(t+k\Delta t)$ and $b_k(x, t) = b(x, t+k\Delta t)$. Then, $\Psi^{k,k+1}u_k - \Phi^{k,k+1}_{A+B}u_k$ is equal to the local error produced by applying one Strang splitting to the equation $u_t = a_k u_{xx} + b_k u u_x$ with u_k as initial data. Thus, by Lemma 5.1, we still have second-order consistency for each k, which gives us

$$\begin{aligned}\|u_n - u(t_n)\|_{H^m} \leq& \sum_{k=0}^{n-1} K(R, T)c_1(C_0, T)\Delta t^2 \\ \leq& K(R, T)c_1(C_0, T)T\Delta t.\end{aligned}$$

Beccause $\gamma = K(R,T)c_1(C_0,T)T$, the third inductive hypothesis is now extended to $k = n$. Using the triangle inequality, we have

$$\begin{aligned}\|u_n\|_{H^m} \leq & \|u_n - u(t_n)\|_{H^m} + \|u(t_n)\|_{H^m} \\ \leq & \gamma\Delta t + \rho.\end{aligned}$$

Hence, we have $\|u_n\|_{H^m} \leq R$ when Δt is sufficiently small and the second inductive hypothesis is finished. For $\theta \in [0,1]$, by Lemma 4.2 we have

$$\|(\Phi_B^{k,k+\theta}\Phi_A^{k,k+\frac{1}{2}}u_{n-1})_x\|_{L^\infty} \leq 2\|\Phi_A^{k,k+\frac{1}{2}}u_{n-1}\|_{H^m} \leq 2R. \tag{20}$$

Using the result in Equation (20) and Lemma 4.1, we have

$$\|u_n\|_{H^{m+2}} \leq e^{2cR\Delta t}\|u_{n-1}\|_{H^{m+2}} \leq e^{2cRn\Delta t}\|u_0\|_{H^{m+2}}.$$

The induction is finished and the first order convergence of Strang splitting is proven.

□

7 Future Work

The results in this paper suggest several directions for future research. One can try to prove second-order convergence for Strang splitting. This would require proving the third-order consistency, which can be difficult. A natural extension of our result is trying to relax the conditions described in Section 3. However, this results in the inability to use many inequalities and one needs to make modifications to the proof. Another possible direction is looking at variable-coefficient PDEs with similar structure such as $u_t = au_{xx} + bf(u)u_x$ where f is a function of u or $u_t = ap(\partial_x)u + buu_x$ where $p(\partial_x)$ is a polynomial differential operator.

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