The Modular Representation Theory of Cyclic Groups of Prime Power Order

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Abstract

It is well-known that representations of a group G over a field K are in bijection with modules over the group algebra $K[G]$—this is the basis for the field of modular representation theory. We study the modular representation theory of cyclic groups with prime power order C_{p^k} over finite fields $\mathbb{F}_p$. There is a broad background in the literature on the representation theory of C_{p^k} over a finite field $\mathbb{F}_n$ when $n \nmid p^k$, but little is known when $n \mid p^k$, which is the case we study. We find a basis for the representation ring of $\mathbb{F}_p[C_{p^k}]$, which allows us to find a much simpler structure to describe the representation ring. Our results help us better understand the representation theory of cyclic groups, which have applications in Number Theory and the Langlands Program.

Summary

Abstract algebra is a field of math that deals with studying abstract objects and their properties. However, solving problems and studying the properties of these abstract objects can be very difficult, and it is much easier to study concrete, linear objects. This is the motivation for the field of representation theory, which represents abstract objects as matrices, which are very easy to deal with. We study the representation theory of one object in particular, called a cyclic group, and we describe the underlying structure of how its representations interact with each other. Our results have applications in several fields of math, including the Langlands Program, which seeks to connect two fields of study: number theory and geometry.

1 Introduction

Representation theory is a method of transforming problems in abstract algebra into problems in linear algebra by representing elements of more complicated algebraic structures as matrices. This allows us to use the many powerful tools of linear algebra that are traditionally applied to matrices to tell us about the properties of groups, rings, fields, and other algebraic objects. Additionally, representation theory has applications in a vast range of mathematical fields, as well as theoretical physics, making it an extremely useful field of study.

One of the fundamental results in representation theory is the following theorem, given by Maschke in 1898.

Theorem 1.1. *Let V be a representation of the finite group G over a field F in which $|G|$ is invertible. Let W be an invariant subspace of V. Then there exists an invariant subspace W_1 of V such that $V = W \oplus W_1$ as representations [1].*

Maschke's Theorem implies that if the characteristic of the field F does not divide the order of the group G, all subrepresentations split. This implies that indecomposable representations are always irreducible. An object satisfying this property that all indecomposable representations are irreducible is called semisimple. Mashke's theorem says that the representation theory of a group over a field with characteristic not dividing the order of the group is semisimple. Many techniques exist to study semisimple representation theory and the majority of the work in the field focuses on this case. Character theory provides strong restrictions on the possible dimensions of irreducible representations. Once the irreducible representations are determined, character theory provides an efficient algorithm for decomposing representations.

When the characteristic of the field divides the order of the group, this is known as modular representation theory. Modular representation theory is not semisimple and far less is known. The most fundamental theorem in nonsemisimple representation theory is the Jordan form, which says that the indecomposible representations of $\mathbb{Z}$ over an algebraically closed field k correspond to pairs $\lambda \in k$ an eigenvalue and d a dimension. If a group has representation theory at most as complicated as the representation theory of $\mathbb{Z}$, the representation theory is said to be *tame*, otherwise it is said to be *wild*. An immediate consequence of the Jordan form is that every cyclic group has tame modular representation theory. The only noncyclic p-group with tame modular representation theory is $\mathbb{Z}/2 \times \mathbb{Z}/2$.

In the tame case, the collection of indecomposable representations is well understood but the behavior of any of the standard functors on representations are poorly understood. However, when the characteristic of the field divides the group, much less is known. Because finite fields must have prime power order, we study groups of prime power order, specifically, cyclic groups of prime power order. We study these groups from the approach of modular representation theory, which studies modules instead of representations. We find a concise description of the representation ring of C_{p^k} over the field $\mathbb{F}_p$ as the quotient of a polynomial ring with some relations.

2 Background

2.1 Representations

Definition 2.1. A *representation* of a group G over a field K is a pair (V, ρ) of a K-vector space V and a homomorphism $\rho : G \to \mathrm{GL}(V)$ where $\mathrm{GL}(V)$ is the group of linear automorphisms of V [2].

For a simple example, consider $C_n = \langle x \mid x^n = 1 \rangle$ the cyclic group of order n. If G is some other group, a homomorphism $f \colon C_n \to G$ is determined by $f(x)$ because f is a homomorphism so $f(x^k) = f(x)^k$. Because $f(g)^n = f(g^n) = f(1) = 1$, the image of f must be an element of G of order n. Thus homomorphisms $C_n \to G$ are in bijection with elements of G of order n. In particular, a representation of C_n on a K vector space of dimension d is specified by a $d \times d$ matrix M such that $M^n = Id$. For instance, the one-dimensional representations of C_n over a field K are given by nth roots of unity in K.

Definition 2.2. A *group action* of a group G over a set X is a map $G \times X \to X : (g, x) \mapsto gx$ satisfying

1. $(gh)x = g(hx)$

2. $1_G x = x$

for all $g, h \in G$ and $x \in X$.

A group action is linear if X is a vector space and the elements of G act linearly on X. Because a representation maps from G to $GL(V)$, each element in G is represented by a linear automorphism on V. Thus, we can completely define a representation by how its elements act on V, which implies that a representation is completely equivalent to a linear group action on a vector space.

2.2 Modules

Definition 2.3. A left R-module M over the ring R consists of an abelian group $(M, +)$ and an operation $R \times M \to M$ (called scalar multiplication) such that for all $r, s \in R$ and for all $x, y \in M$, we have

1. $r(x + y) = rx + ry$

2. $(r + s)x = rx + sx$

3. $(rs)x = r(sx)$

4. $1_R x = x$ where 1_R is the multiplicative identity in R.

A right R-module is defined analogously.

Definition 2.3 is the standard definition of a module, but Definition 2.4 is an equivalent, more useful definition in the context of representation theory.

Definition 2.4. A *module* over a K-algebra R is a pair (V, ρ) of a K-vector space V and a map $\rho\colon R \to \mathrm{End}(V)$ from R to the ring of linear endomorphisms of V.

If R is a field, then an R-module is a vector space over R, thus modules over a ring are a generalization of vector spaces over a field. For any ring R, an example of an R-module is the Cartesian product R^n, where scalar multiplication defined by multiplying component-wise. The ring $\mathrm{End}(R^n)$ is given by $n \times n$ matrices of elements of R, and the map $\rho\colon R \to R^n$ embeds R as the diagonal matrices. Any ring homomorphism $R \to S$ makes S into an R-module. Modules are useful in representation theory because they provide another way of looking at group representations.

Definition 2.5. Given a group G and a field K, the *group algebra* $K[G]$ is defined as a vector space with basis G, so that a typical element $r \in K[G]$ is given by $\sum_{g \in G} a_g[g]$, along with a bilinear operation $K[G] \times K[G] \to K[G]$. This operation is given by setting $[g][h] = [gh]$ and linearly extending.

We study the representation theory of $\mathbb{F}_p[C_{p^k}]$, which is an example of a group algebra. There is an isomorphism from

$$\mathbb{F}_p[C_{p^k}] \quad \text{to} \quad \mathbb{F}_p[x]/(x^{p^k} - 1),$$

given by $g \mapsto x$, where g is a generator. A typical element in $\mathbb{F}_p[C_{p^k}]$ looks like a polynomial in one variable over a finite field, and takes the form

$$\sum_{i=1}^{p} a_i g^{i-1}.$$

Lemma 2.1. *Modules over $K[G]$ are in bijection with representations of G over K.*

Lemma 2.1 follows almost directly from Definition 2.1 and Definition 2.4, with a few rigorous steps involved, because modules and representations are almost definitionally equivalent. The bijection between modules over $K[G]$ and representations of G over K can be used to study the representation theory of cyclic groups C_n. Representations of C_n over $\mathbb{F}_p$ are equivalent to modules over $\mathbb{F}_p[C_n]$, and it is simpler to study these modules instead of representations, because modules encompass all of the structure of representations in one object, rather than the ordered pair (V, ρ) from Definition 2.1.

2.3 Jordan Normal Form

Representation theory is motivated by a desire to apply the tools of linear algebra to abstract algebra, and in our case, we use the Jordan normal form of a matrix, which is a powerful result from linear algebra. The Jordan normal form of a matrix A is a matrix J composed of Jordan blocks that satisfies $J = P^{-1}AP$, so J is similar to A. The Jordan blocks which compose the Jordan normal form have the form

$$\mathcal{J}_n = \begin{pmatrix} \lambda & 1 & 0 & \cdots & 0 \\ 0 & \lambda & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \lambda & 1 \\ 0 & 0 & 0 & 0 & \lambda \end{pmatrix}.$$

The main focus of this paper is studying the cyclic group of order p^k, C_{p^k}, along with vector spaces over $\mathbb{F}_p$. For all $g \in C_{p^k}$, we have $g^{p^k} = 1_G$, and because the map ρ is a homomorphism, we must also have the matrix representation $M = \rho(g)$ satisfy $M^{p^k} = id$. Therefore, when we consider Jordan blocks as matrix representations of elements of C_{p^k}, we must have $\mathcal{J}_n^{p^k} = id$.

This implies $\lambda^{p^k} = 1$, so λ is a p^k-th root of unity. However, because $\mathcal{J}_n \in \mathbb{F}_p$, we have that $x^{p^k} - 1 = (x-1)^{p^k}$, and so all roots of unity are 1, hence, $\lambda = 1$. So, all Jordan blocks have the form

$$\mathcal{J}_n = \begin{pmatrix} 1 & 1 & 0 & \cdots & 0 \\ 0 & 1 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

where $\mathcal{J}_n$ is n-dimensional. A matrix in Jordan normal form will be composed of Jordan blocks as shown below

$$\mathcal{J} = \begin{pmatrix} \mathcal{J}_{i_1} & 0 & 0 & \cdots & 0 \\ 0 & \mathcal{J}_{i_2} & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \mathcal{J}_{i_{n-1}} & 0 \\ 0 & 0 & 0 & 0 & \mathcal{J}_{i_n} \end{pmatrix}.$$

2.4 Indecomposable Representations

The set of all representations of a group G over a field K forms a ring $\mathrm{Rep}^{\otimes}(K[G])$ when one considers representations as $K[G]$-modules. It is a common problem in representation theory to study the indecomposable representations in $\mathrm{Rep}^{\otimes}(K[G])$.

Given some representation ρ, we call ρ *indecomposable* if ρ cannot be written as the direct sum of two non-zero representations [3]. Understanding the direct sum of representations is simplest when considering modules: the direct sum of two modules V_i and V_j is the module $V_i \oplus V_j$ with basis given by appending the basis of V_j to the basis of V_i.

If we choose some $g \in G$, g will be represented as a matrix T in $GL(V)$ through the map $\rho(g) = T$. Because we are concerned with cyclic groups C_{p^k}, every other matrix representation is determined by T if g is a generator. Therefore, we can associate a matrix to ρ: the matrix of T with respect to some basis of $K[G]$, with g the generator in C_{p^k}. Then, ρ is indecomposable if the Jordan normal form is only composed of one Jordan block. This is equivalent to ρ not being the direct sum of two non-zero representations because the direct sum of two matrices $\mathbf{A}$ and $\mathbf{B}$ is the block diagonal matrix

$$\mathbf{A} \oplus \mathbf{B} = \begin{pmatrix} \mathbf{A} & \mathbf{0} \\ \mathbf{0} & \mathbf{B} \end{pmatrix} = \begin{pmatrix} a_{11} & \cdots & a_{1n} & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} & 0 & \cdots & 0 \\ 0 & \cdots & 0 & b_{11} & \cdots & b_{1q} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & b_{p1} & \cdots & b_{pq} \end{pmatrix},$$

so if the Jordan normal form has more than one Jordan block, it is the direct sum of those Jordan blocks, and thus decomposable. One can use this connection between decomposing modules and the Jordan normal form of a matrix to prove the follwoing well-known result about modular representations of cyclic groups that will be useful later on:

Lemma 2.2. *For C_{p^k} over a field K, it is well-known that there are precisely p^k classes of indecomposable $K[C_{p^k}]$-modules. The i-th module M_i, where $1 \leq i \leq p^k$, has dimension i.*

From now on, we denote the n-dimensional indecomposable module by V_n. The matrix associated with each V_n is the $n \times n$ Jordan block with $\lambda = 1$. Multiplication by this matrix corresponds to the group action by the generator g of G. Now that we know what all indecomposable representations of C_{p^k} look like, we can compute direct sums and tensor products through operations on matrices, and we can start to study the representation ring.

We are interested in determining the decomposition of a general element of $\mathrm{Rep}^{\otimes}(K[G])$ into the indecomposable modules V_n. The Jordan normal form proves to be extremely useful for computing such decompositions. For example, if we have the element $V_2 \otimes V_3$ over $\mathbb{F}_5[C_5]$, we can compute $\mathcal{J}_2 \otimes \mathcal{J}_3$, where $\mathcal{J}_n$ is the n-dimensional Jordan block, then take the Jordan normal form of the resulting matrix:

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \otimes \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

In this case, we find that the Jordan normal form is composed of 2 Jordan blocks of size 4 and 2. Therefore, $V_2 \otimes V_3 = V_4 \oplus V_2$.

3 Representation Rings of Cyclic Groups

The structure of the representation ring $\mathrm{Rep}^{\otimes}(G)$ of a group G completely specifies how all the representations of G interact with each other, so we are interested in finding a simple way to describe its structure. Theorem 3.2 is our main result, and it does precisely this. To prove Theorem 3.2, we need Definition 3.1 and Lemma 3.1.

Definition 3.1. Given a linear endomorphism $A : V \to V$, a Jordan chain of length k in a vector space V is a sequence of non-zero vectors $v_1, \ldots, v_k \in V$ that satisfies

$$Av_k = \lambda v_k, \quad Av_{i-1} = \lambda v_i, \quad 2 \leq i \leq k.$$

Lemma 3.1. *Denote $w_j \otimes w_k$ by w_{jk}. When starting at w_{00}, the number of Jordan chains ending at w_{jk} in $W_j \otimes W_k$ is given by $\binom{j+k}{j} \mod p$, which corresponds to the number of Jordan blocks of maximal dimension in the decomposition of $W_j \otimes W_k$.*

Proof. It is well-known that the matrix of some linear transformation $T : V \to V$ is a Jordan block of dimension n with respect to a basis $\{v_1, \dots, v_n\}$ if and only if $\{v_1, \dots, v_n\}$ is a Jordan chain for T. In representation theory, we consider T as the matrix representation of a group action g, and we can use this fact to find Jordan blocks in a tensor product of modules. To simplify computation, we consider the group action $g - 1$ instead of g. We can do this because g acts on the basis of W_n as follows: $w_0 \mapsto w_1 \mapsto w_2 \mapsto \dots \mapsto w_n$ (this is because $\lambda = 1$), so we have that $g(w_{jk}) = w_{j,k+1} + w_{j+1,k} + w_{jk}$ and $(g-1)(w_{jk}) = w_{j,k+1} + w_{j+1,k}$. When repeatedly applying a group action to find a Jordan chain, we only care about the highest degree terms in each step, so we can disregard the w_{jk} term by using $g - 1$ instead of g.

The number of Jordan chains ending with each basis element w_{ab}, which we will denote as $J(w_{ab})$, will be $J(w_{a-1,b}) + J(w_{a,b-1})$. Notice that this relation is equivalent to the relation in Pascal's Triangle, and in fact, because we know $J(w_{00}) = \binom{0}{0} = 1$, the indices match as well, so we have that $J(w_{ab}) = \binom{a+b}{b} \mod p$. □

Theorem 3.2. *Consider the representation ring $Rep^{\otimes}(\mathbb{F}_p[C_{p^n}])$. This ring has two equivalent bases given by $\{V_i \mid 0 < i \leq p^k\}$ and $\{\bigotimes V_{p^i+1}^{a_i} \mid 0 \leq i < k\}$.*

Proof. The basis $\{V_i \mid 0 < i \leq p^n\}$ trivially spans $\text{Rep}^{\otimes}(\mathbb{F}_p[C_{p^n}])$, and because it has p^k elements, it is a basis. Thus, we must show that $M = \{\bigotimes V_{p^i+1}^{a_i} \mid 0 \leq i < n\}$ is a basis. Notice that M has p^n elements, as there are k choices for i and p choices for a_i. Therefore, it remains to prove that M spans $\text{Rep}^{\otimes}(\mathbb{F}_p[C_{p^n}])$, which composes the remainder of the proof.

For simplicity, let $W_i = V_{i+1}$. We will show that one can find values of i and a_i such that the equation

$$\bigotimes W_{p^i}^{a_i} = W_k + \bigoplus_{i=0}^{k-1} b_i W_i \tag{1}$$

holds true for each k, because if it is true, then we can substitute an element generated from M for each W_i where $i < k$, which expresses W_k in terms of elements in M. We will proceed by induction. Assume that $\bigotimes W_{p^i}^{\bar{a}_i} = W_{k-p^j} + \bigoplus_{i=0}^{k-p^j-1} b_i W_i$, where $\bar{a}_i = a_i$ if $i \neq j$ and $\bar{a}_j = a_j - 1$. Tensoring both sides of the equation by W_{p^j}, we obtain

$$\bigotimes W_{p^i}^{a_i} = W_{p^j} \otimes W_{k-p^j} + W_{p^j} \otimes \bigoplus_{i=0}^{k-p^j-1} b_i W_i. \tag{2}$$

We will show that the right side of Equation (2) contains a nonzero W_k term.

The length of a maximal Jordan chain in $W_m \otimes W_n$ is $m+n$, so if we can find a maximal Jordan chain in $W_{p^i} \otimes W_{k-p^i}$, we will have a Jordan block of dimension k in the decomposition of the tensor product $W_{p^i} \otimes W_{k-p^i}$, and we will be done. Because we are looking for a maximal Jordan chain, we only have one starting point in the basis of $W_{p^i} \otimes W_{k-p^i}$, w_{00}. By Lemma 3.1, there are $\binom{p^i+k-p^i}{p^i} \equiv \binom{k}{p^i} \mod p$ Jordan chains that start with w_{00} and end with w_{k,p^i-k}, which will all be maximal. Therefore, we must show that $\binom{k}{p^i} \neq 0 \mod p$. By Lucas' Theorem, this is true if and only if every digit in the base p representation of k is greater than or equal to the corresponding digit in the base p representation of p^i. However, the greatest digit in the base p representation of p^i is always 1, so for every nonzero value of k, it is possible to choose an i such that the same digit in the base p representation of k is at least 1.

Finally, we must show that we can actually construct every value of k from 1 to p^k with this induction, because our increment on k is not necessarily by 1. We will prove this by doing induction on each value for i. Denote the base p representation of n by n_p. In general, we can construct the number $k = (c+1)p^i$, where $0 < c+1 < p^{n-i}$ and $p \nmid c+1$, by incrementing cp^i by p^i, which is possible because $((c+1)p^i)_p$ has a digit of $c+1 \mod p \geq 1$ and $(p^i)_p$ has a digit of 1. The base case for each value of i is $c = 0 \implies k = p^i$, which is automatically true, because p^i is already in the monomial basis M. Therefore, we can construct every value of k by finding the maximum i such that $p^i \mid k$, then incrementing $k - p^i$ by p^i. $\square$

Theorem 3.1 is interesting to us not only because it describes an alternative basis of the representation ring $\text{Rep}(\mathbb{F}_p[C_{p^k}])$, but because it gives us a way to describe the underlying structure of the representation ring as the quotient of a well-understood ring with some relations. For each W_{p^i}, $W_{p^i}^p = W_0$, where W_0 is the trivial representation, or the multiplicative identity in the representation ring. Additionally, we can decompose tensor products into direct sums using the Jordan normal form strategy. Thus, by writing $W_{p^i}^p$ as a direct sum of lower-dimensional terms, we obtain a relation where $W_0 = \bigoplus W_k$. We can repeat this for each W_{p^i}, until we have k relations. Finally, we can write

$$\text{Rep}(\mathbb{F}_p[C_{p^k}]) \cong \mathbb{Z}[W_1, W_p, \cdots, W_p^k] / \left(\bigoplus_{i=1}^{p-1} W_i = W_0, \bigoplus_{j=1}^{p^2-1} W_j = W_0, \cdots \right),$$

where the W_{k_i} and W_{k_j} terms represent the direct sum decomposition of each $W_{p^i}^p$.

4 Future Work

One feature of representations that we used frequently throughout this paper is the relative ease with which we can decompose tensor products of representations into direct sums. In fact, besides finding such decompositions algorithmically using Jordan normal form, many formulas are known for decomposing tensor products of representations into direct sums, such as the following formulas, given by Hughes and Kemper [4]

$$V_2 \otimes V_n \cong \begin{cases} V_{n-1} \oplus V_{n+1} & \text{if } p \nmid n, \\ V_n \oplus V_n & \text{if } p \mid n. \end{cases} \tag{3}$$

and

$$V_{p-1} \otimes V_i = V_{p-i} \oplus (i-1)V_p. \tag{4}$$

However, the decompositions of symmetric and exterior powers of modules are much less-studied. Himstedt and Symonds recently made progress on this problem by proving the following equation for computing exterior powers [5]:

$$\Lambda^r(V_{2^{n-1}+s}) \cong \bigoplus_{2i+j=r} \Omega_{2^n}^{i+j}(\Lambda^i(V_s) \otimes (\Lambda^j(V_{2^{n-1}-s})) \oplus tV_{2^n} \tag{5}$$

Their results hold only in cyclic groups of order 2^n over the field $\mathbb{F}_2$, so it is interesting to attempt to generalize their formula to odd primes as well.

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References

[1] P. Webb. A course in finite group representation theory. http://www-users.math.umn.edu/~webb/RepBook/RepBookLatex.pdf. Accessed 27 July 2018.

[2] N. Dupré. Representation theory workshop. https://www.dpmms.cam.ac.uk/~nd332/rep_workshop.pdf, 2015. Accessed 5 July 2018.

[3] A. I. Shtern. Indecomposable representation. http://www.encyclopediaofmath.org/index.php?title=Indecomposable_representation&oldid=17010. Accessed 24 July 2018.

[4] I. Hughes and G. Kemper. Symmetric powers of modular representations, hilbert series and degree bounds. *Communications in Algebra*, 6 2007.

[5] F. Himstedt and P. Symonds. Exterior and symmetric powers of modules for cyclic 2-groups. *Journal of Algebra*, 7 2014.