Finding α–Hölder Continuous Curves through Points in the Unit Square

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August 2, 2016

Abstract

The Erdős–Szekeres theorem states that given a sequence of N real numbers, there exists a monotonic subsequence of length at least $N^{1/2}$. By applying this theorem to N points in the unit square, we observe that there exists a function f which passes through at least $N^{1/2}$ of the given points that is Lipschitz continuous, or that has a bounded first derivative. We extend this result by determining the maximum number of the given points $N^{\beta}, 0 < \beta < 1$, that a function can contain while maintaining α–Hölder continuity, a more general form of Lipschitz continuity.

Summary

Often, in computer science and statistics, we must construct a curve for a set of points, nodes, or data values to satisfy some parameter. Consider a function that travels through a subset of given points in the unit square. If the absolute slope between any two points on the function is bounded above by some constant, the function satisfies a strict form of continuity, 1–Hölder continuity. In our problem, we wish to construct an α–Hölder continuous curve that travels through a point subset of a given size. In particular, we ask how large the proportion of points on the curve can be before any function traveling through a subset of given size cannot be α–Hölder continuous. By defining new functions and considering a grid configuration of points within the unit square, we determine bounds on the maximum proportion of given points any α–Hölder continuous function may pass through.

1 Introduction

In many instances, we wish to determine the greatest number of points a curve can contain while maintaining a set of desired properties; this situation is apparent in statistics and curve fitting, where curves that best fit a given set of data points are constructed. In computer science and mathematics, this situation also manifests itself through problems such as the Travelling-Salesman Problem, in which we wish to construct a Hamiltonian cycle through a given set of points of the least total distance [3]. In our paper, we explore a specific property of a function containing some subset of N points in a unit square.

This property is Lipschitz continuity, which exists for a function if the slope between every pair of points in the function's domain has absolute value bounded above by some constant, known as the Lipschitz constant. That is, if the function has a bounded first derivative, it is Lipschitz continuous, and the bound itself is the Lipschitz constant. We can extend the idea of Lipschitz continuity to the more general notion of α–Hölder continuity, which quantifies the function's rate of change through parametrization of the denominator by α.

Results regarding Lipschitz continuity can be derived from the celebrated Erdős–Szekeres theorem, which states that given any sequence length N of real numbers, there exists a monotonic subsequence of length at at least $N^{1/2}$ [2]. First published in 1935, this result quickly attracted mathematicians and generated alternative proofs using techniques from the Pigeonhole Principle to the greedy algorithm [6]. We examine this theorem in context of N points in the unit square, where given N points, there exists a nonincreasing or nondecreasing function that contains at least $N^{1/2}$ points.

In general, various results have been found concerning α–Hölder mappings to the unit square. In 1996, Buckley [1] determined the bound on α for which there existed Peano curves that were α–Hölder continuous and found α to be at most $\frac{1}{2}$. A year later, Matoušek [4] used the Erdős–Szekeres theorem to prove that for any subset of a plane with positive Lebesgue measure, there exists a Lipschitz mapping onto the unit square. In this work, we determine the maximum proportion of given points in the unit square a function can pass through while still maintaining α–Hölder continuity.

Our paper proceeds as follows: In Section 2, we define the basic terms surrounding our problem. Then in Section 3.1, we start with a result derived from the Erdős–Szekeres theorem by Matoušek [4]:

Theorem (Matoušek)**.** *Given N random points in the unit square, there exists a function f passing through $N^{1/2}$ points that has Lipschitz constant satisfying $||f||_{C^1} < 1$.*

Next in Section 3, we expand to a more general result, determining and proving the tightness of the upper bound for the value of the Lipschitz constant. In Section 4 we prove several important properties of a more flexible function ϕ, including its Lipschitz and α–Hölder constants when the points are chosen from a grid in the unit square. We introduce a grid point configuration to show our main result, bounds on the maximum proportion of given points the function can pass through to guarantee α–Hölder continuity. Lastly, we generalize one of the bounds to any arbitrary configuration of points given.

2 Preliminaries

Given some function, the Lipschitz constant is the upper bound of the absolute slope between any two points on the function. More formally, the Lipschitz constant $||f||_{C^1}$ can be expressed as the minimum such that the inequality $\big|f(x_1) - f(x_2)\big| \leq ||f||_{C^1}|x_1 - x_2|$ holds for all pairs x_1, x_2 in the domain of f. We further expand this definition to a general α to examine the α–Hölder constant $||f||_{C^\alpha}$, formally defined below:

Definition. *For a function f containing a given set of points, the* ***α–Hölder constant***

$||f||_{C^\alpha}$ is the minimum value such that the expression

$$\big|f(x_1) - f(x_2)\big| \leq ||f||_{C^\alpha}|x_1 - x_2|^\alpha \tag{1}$$

*is satisfied for all x_1, x_2 in the domain of f, where $\alpha \in (0, 1]$. The **Lipschitz constant** $||f||_{C^1}$ is a specific case of the α–Hölder constant where $\alpha = 1$. If a function has a Lipschitz or α–Hölder constant, then it is **Lipschitz continuous** or α–**Hölder continuous**, respectively.*

Given N points in the unit square and α where we can choose orthonormal coordinates, we aim to find bounds on β for which there exists an α–Hölder continuous function f through N^β points with $0 < \beta < 1$. To increase flexibility in our analysis, we introduce another function ϕ: $[0, 1] \to [0, 1]^2$, where the quantity $\big|\phi(x_i) - \phi(x_j)\big|$ is the Euclidean distance between the images of the mappings onto the unit square, and $|x_i - x_j|$ is the distance between the two inputs on the unit interval. Thus, the Lipschitz constant for ϕ can be interpreted as the maximum stretch factor for any segment mapped from the unit interval onto the unit square. Note that since the expression for ϕ relies on Euclidean distances, the freedom to choose orthonormal coordinates has no effect on $||\phi||_{C^\alpha}$.

3 Lipschitz Continuity

We begin by considering $f : [0, 1] \to [0, 1]$, a continuous and differentiable function within the unit square. Specifically, we analyze the Lipschitz constant $||f||_{C^1}$ which occurs when $\alpha = 1$, or when $||f||_{C^1}$ represents the maximum absolute value of slope on f.

3.1 Results from the Erdős–Szekeres theorem

Using the Erdős–Szekeres theorem, we prove a bound on $||f||_{C^1}$ to give the same result as Matoušek [4]:

Theorem 3.1 (Matoušek)**.** *Given N random points in the unit square, there exists a function f passing through $N^{1/2}$ points that has Lipschitz constant satisfying $||f||_{C^1} < 1$.*

Proof. Set initial coordinates such that no two points share an x-coordinate or y-coordinate. By the Erdős-Szekeres theorem, for any N points in the unit square, it is possible to choose $N^{1/2}$ monotonic points. Assume without loss of generality that these points are monotonically increasing. Thus the segments which connect consecutive points of increasing abscissae form $N^{1/2}-1$ segments. The slope of each segment is nonnegative, so the angle from the horizontal lies in the interval $(0, \frac{\pi}{2})$.

By rotating the coordinate system counterclockwise $\frac{\pi}{4}$ about the origin, the angle from the horizontal now lies in the interval $(-\frac{\pi}{4}, \frac{\pi}{4})$, and thus the slope of each segment has absolute value less than 1, as seen in Figure 1.

With this coordinate system, we can construct f with Lipschitz constant less than 1 that passes through the $N^{1/2}$ initially monotonically increasing points. A similar argument exists if the points are monotonically decreasing. Therefore the Lipschitz constant $||f||_{C^1} < 1$. $\square$

Next, we extend our result to the α–Hölder constant for f.

Corollary 3.1. *For N random points in the unit square, there exists a function f passing through $N^{1/2}$ points that has α–Hölder constant $||f||_{C^\alpha} < \sqrt{2}^{1-\alpha}$ for any given $\alpha \in (0, 1]$.*

Proof. We begin by rearranging the α–Hölder expression $\big|f(x_1) - f(x_2)\big| \leq ||f||_{C^\alpha}|x_1 - x_2|^\alpha$ as

$$|\tan\theta|\Big(|x_1 - x_2|\Big)^{1-\alpha} \leq ||f||_{C^\alpha}$$

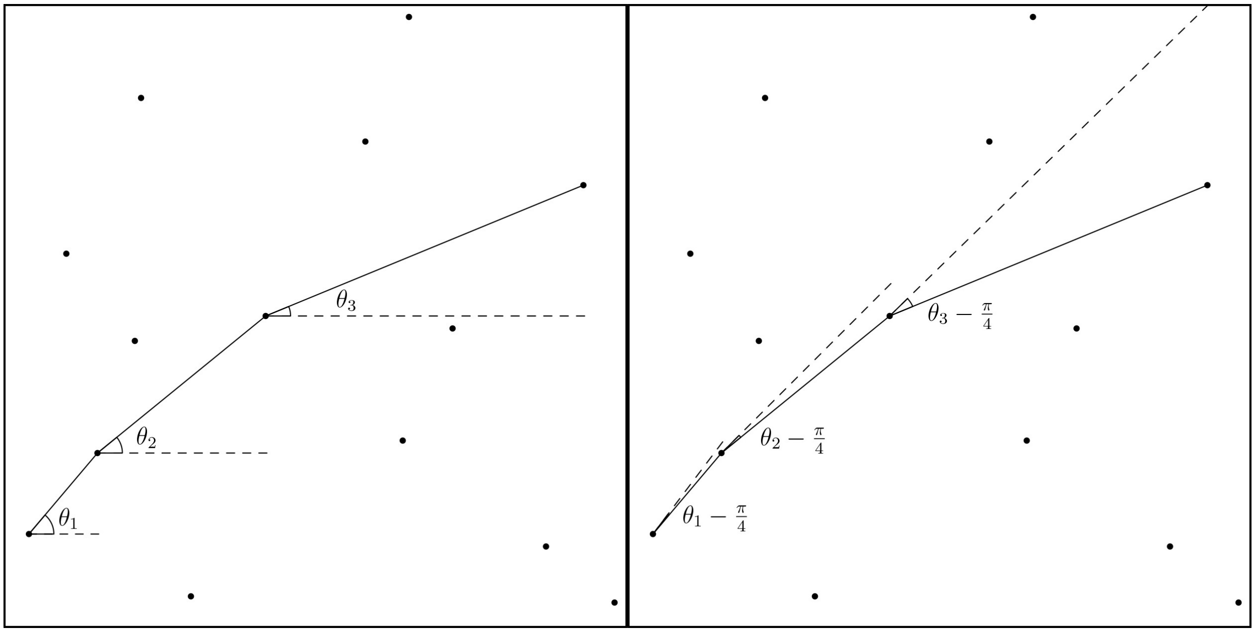

Figure 1: Counterclockwise rotation by $\frac{\pi}{4}$ for $N = 14$ random points

for all x_1, x_2, and θ such that $\tan\theta$ is the slope between the points with abscissae x_1 and x_2. As proven in Theorem 3.1, we know that for N random points and $\beta \leq \frac{1}{2}$, $\max|\tan\theta| < 1$ with a $\frac{\pi}{4}$ rotation, and it follows that $\max|x_1 - x_2| = \sqrt{2}$. Thus we obtain

$$||f||_{C^\alpha} < \sqrt{2}^{1-\alpha}.$$

It follows that for f, Lipschitz continuity implies α–Hölder continuity. $\square$

Thus, we find that for any given set of N points, there will always exist an α–Hölder continuous function through at least $N^{1/2}$ points for all $\alpha \in (0, 1]$.

3.2 Tightness of Lipschitz Bound

From Theorem 3.1, we know that there exists a Lipschitz continuous function ($||f||_{C^1} < 1$) through $N^{1/2}$ of the points. We wish to see if a function f passing through more than $N^{1/2}$ points can be Lipschitz continuous. To verify the strictness of the Lipschitz bound $||f||_{C^1} < 1$

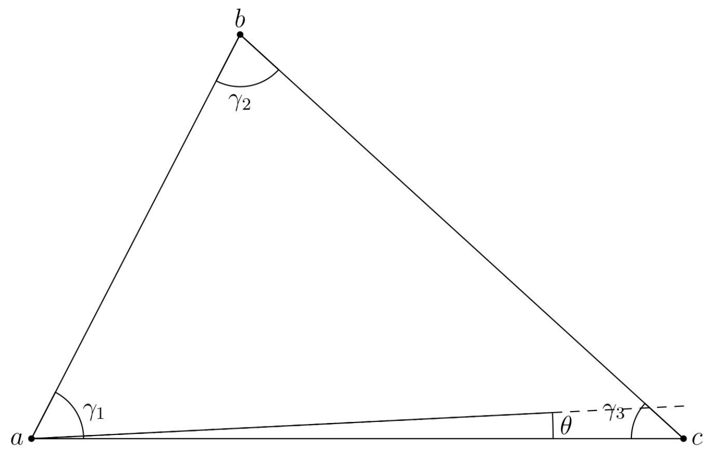

Figure 2: Rotation by θ in a triangle

for $\beta \leq \frac{1}{2}$, we first prove the following lemma:

Lemma 3.1. *Let there be 3 points a, b, c, and let the angles formed by the connecting segments be γ_1, γ_2, and γ_3. Then for any function f through a, b, and c, the Lipschitz constant* $||f||_{C^1} = \min_{i=1,2,3} \cot\left(\dfrac{\gamma_i}{2}\right)$.

Proof. Suppose without loss of generality that the initial ordering by increasing abscissae is a, b, c, as depicted in Figure 2, and the set of initial absolute slopes is $S_1 = \{\tan\gamma_1, 0, \tan\gamma_3\}$. Rotate by an angle θ such that the order by abscissae remains the same. The set of new absolute slopes is thus $S_2 = \{\tan(\gamma_1 - \theta)$, $\tan\theta$, and $\tan(\gamma_3 + \theta)\}$. Note that the Lipschitz constant is interpreted as the minimum-maximum absolute slope over all pairs of points on the domain of the function. The minimum of $\max S_2$ occurs when $\tan(\gamma_1 - \theta) = \tan(\gamma_3 + \theta)$, so $||f||_{C^1} = \tan\frac{\gamma_1+\gamma_3}{2} = \cot\frac{\gamma_2}{2}$. Applying this over all possible orderings of a, b, and c, we find that $||f||_{C^1} = \min_{i=1,2,3} \cot\left(\dfrac{\gamma_i}{2}\right)$. □

We apply this result to special cases involving right angles, where $||f||_{C^1} = 1$:

Corollary 3.2. *If f passes through any three points which form a right triangle or any 2 pairs of points which form orthogonal segments, then* $||f||_{C^1} = \cot\frac{\pi}{4} = 1$.

In the next theorem, we introduce function g to determine whether our upper bound, $||f||_{C^1} < 1$ is tight for $\beta \leq \frac{1}{2}$.

Theorem 3.2. *Let $g(n)$ be the largest possible N such that there exists a configuration of N points where Lipschitz constant $||f||_{C^1} \geq 1$ for any subset of n points. Then if $n > 2$, we have $g(n) = (n-1)^2$.*

Proof. Consider the grid arrangement of $(n-1)\times(n-1)$ points within the unit square, where we select subset size n. By the pigeonhole principle, there must be at least 2 points in one column and 2 points in one row. Thus there exist 2 orthogonal segments, and our result from Corollary 3.2 gives us $||f_{C^1}|| \geq 1$. By Theorem 3.1, for $(n-1)^2+1$ points, $||f_{C^1}|| < 1$ for at least one subset of n points. Thus $g(n) < (n-1)^2+1$, and it follows that $g(n) = (n-1)^2$. $\square$

Therefore, our Lipschitz bound $||f||_{C^1} < 1$ for $\beta \leq \frac{1}{2}$ is tight. That is, a function f which passes through more than $N^{1/2}$ points is not guaranteed to be Lipschitz continuous or has Lipschitz constant greater than 1. Later in Section 4.1, we show tightness of the bound $\beta \leq \frac{1}{2}$ for which any greater value does not guarantee Lipschitz continuity for f.

4 α–Hölder continuity

In this section, we determine bounds on β which guarantee an α–Hölder continuous function. Through Corollary 3.1, we see that given N points, there exists an α–Hölder continuous function f through some $N^{1/2}$ points. We wish to see if we can incorporate more than $N^{1/2}$ points on both f and a new function ϕ. Specifically, we consider a grid configuration of $k \times k$ points within the unit square where, by finding bounds on β for which α–Hölder continuity exists for a function through the grid, we can generalize the bounds on β for an α–Hölder continuous function through any configuration of points.

4.1 Properties of the curve ϕ

Aside from function $f : [0,1] \to [0,1]$ we examine another more flexible function $\phi : [0,1] \to [0,1]^2$ which maps from the unit interval onto the unit square. First, we show that ϕ is α–Hölder continuous for all $\beta \le \frac{1}{2}$. We then determine whether we can incorporate more than $N^{1/2}$ points onto ϕ while still preserving α–Hölder continuity, thus showing tightness of our bound on β from Section 3.

Theorem 4.1. *Given N points in $[0,1]^2$, there exists some ϕ containing at least $N^{1/2}$ points such that $||\phi||_{C^\alpha} < 2$.*

Proof. Divide the unit square into $\sqrt{N}$ congruent, nonoverlapping vertical strips of width $\dfrac{1}{\sqrt{N}}$. By the pigeonhole principle, at least $\sqrt{N}$ points must fall in one strip. Let us examine the path within this strip that passes through all $\sqrt{N}$ points in order from bottom to top. By the triangle inequality, the maximum length of this path is less than

$$\left(\frac{1}{\sqrt{N}}\right)(\sqrt{N}-1)+1=2.$$

We then have $||\phi||_{C^\alpha} < 2$. $\square$

Thus, we know that for all $\beta \le \frac{1}{2}$, the function ϕ is α–Hölder continuous. Corollary 4.1 immediately follows:

Corollary 4.1. *Given N points in $[0,1]^2$, there exists ϕ passing through at least $N^{1/2}$ points such that $||\phi||_{C^1} < 2$.*

Since $||\phi||_{C^1}$ can be interpreted as the maximum stretch factor for any segment from the unit interval to the unit square, we see that when $\phi = (x\sqrt{2}, f(x\sqrt{2}))$, the length of the segment can only be stretched by a maximum factor of 2 for its corresponding image in the unit square.

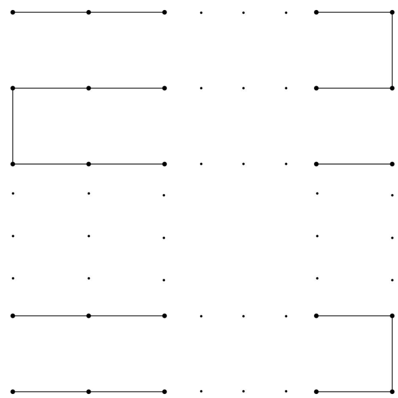

Figure 3: Example path $\operatorname{Im}\phi_1$ through all points in the $(s+1)\times(s+1)$ grid

4.2 Bounds for α–Hölder continuity

Using the grid configuration, we now generalize bounds on β for which α–Hölder continuity is guaranteed for a function ϕ passing through any N^β points.

First, we develop reasoning necessary for Theorem 4.2 by examining the lower bound of $||\phi_1||_{C^\alpha}$ in the grid configuration, where ϕ_1 passes through all $(s+1)^2$ points.

Lemma 4.1. *For all ϕ which passing through all $(s+1)^2$ grid points, we have $||\phi||_{C^\alpha} \geq s^{2\alpha-1}$.*

Proof. The minimum Euclidean distance between any pair of points in the grid is $\frac{1}{s}$. Further, for every ϕ, by the averaging principle, the length of a subinterval between some x_i, x_{i+1} must be of length of order at least $\frac{1}{s^2}$. Thus the lower bound for the α–Hölder constant is $\frac{\frac{1}{s}}{\left(\frac{1}{s^2}\right)^\alpha} = s^{2\alpha-1}$. □

Next, we determine whether ϕ can contain more than $N^{1/2}$ points while remaining α–Hölder continuous, showing in the context of the $k\times k$ unit square grid that as $k^2 = N$ approaches infinity, such a function becomes α–Hölder discontinuous.

Theorem 4.2. *For any $\alpha \in (0,1]$ and $k\times k$ unit square grid where $k^2 = N$, if $\beta > \frac{1}{2\alpha}$, then any function ϕ that passes through more than N^β points has α–Hölder constant at least $N^{2\alpha\beta-1}$.*

Proof. For any $N = k^2$ points, we wish to see if there exists a α–Hölder continuous function ϕ that passes through any $k^{1+\epsilon}$ points where $\epsilon > 0$. Consider a grid of k^2 points. Then, with the same reasoning used for Lemma 4.1, we obtain

$$||\phi||_{C^\alpha} \geq \frac{\frac{1}{k}}{\left(\frac{1}{k^{1+\epsilon}}\right)^\alpha} = k^{\alpha+\alpha\epsilon-1}. \tag{2}$$

Then because $k^2 = N$, we can see that $1 + \epsilon = 2\beta$. Note that if $\alpha + \alpha\epsilon - 1 > 0$, the function ϕ is not α–Hölder continuous as N becomes sufficiently large. Thus we get

$$\alpha(1 + \epsilon) - 1 > 0$$
$$\beta > \frac{1}{2\alpha},$$

where $||\phi||_{C^\alpha} \geq N^{2\alpha\beta-1}$. □

Because the exponent of the lower bound α–Hölder constant is positive when $\beta > \frac{1}{2\alpha}$, for sufficiently large N arbitrarily configured points, any ϕ passing through more than $N^{\frac{1}{2\alpha}}$ points is not guaranteed to be α–Hölder continuous. From this result, we obtain the following corollary for the Lipschitz case:

Corollary 4.2. *Given any unit square grid of N points where N is sufficiently large, there does not exist ϕ which contains more than $N^{1/2}$ points and is Lipschitz continuous.*

Since any function f corresponds to a function ϕ through $\phi(x) = (x, f(x))$, we deduce the following result from Corollary 4.3:

Corollary 4.3. *Given any unit square grid of N points where N is sufficiently large, there does not exist f which passes through more than $N^{\frac{1}{2\alpha}}$ points and is Lipschitz continuous. In particular, there does not exist Lipschitz continuous f which passes through more than $N^{1/2}$ points.*

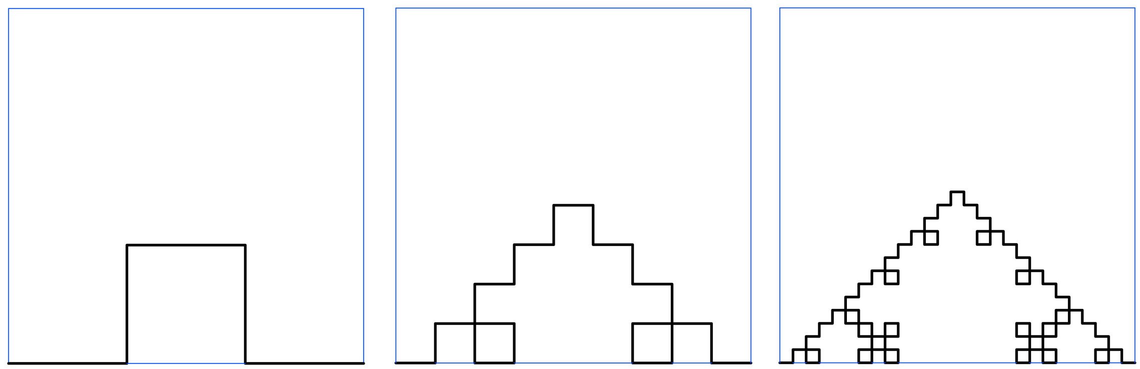

Figure 4: S_1, S_2, S_3 of fractal

Thus by the grid configuration, Corollary 4.3 proves the tightness of our bound $\beta \leq \frac{1}{2}$ from Section 3.2 that guarantees existence of a Lipschitz continuous function f through N^β points.

Next, we wish to determine the strictness of bound $\beta < \frac{1}{2\alpha}$ from Theorem 4.2. In the following result, we construct a fractal within the grid to show that a α–Hölder continuous function can still exist for $\beta = \frac{1}{2\alpha}$. We arrive at the following theorem:

Theorem 4.3. *For a unit square grid of N points where N is sufficiently large and $\alpha = \log_5 3$, there exists ϕ that passes through $N^{\frac{1}{2\alpha}}$ points and is α–Hölder continuous.*

Proof. By constructing a fractal within the unit square, we obtain a curve which passes through more than $N^{\frac{1}{2\alpha}}$ points and is α–Hölder continuous. Construction of the fractal (Figure 4) is as follows: The base case S_1 consists of a square containing 5 segments of length $\frac{1}{3}$, with orientation right, up, right, down, right. For each successive stage S_i, construct a square of side length $\frac{1}{3^i}$ within the middle of each segment from the previous stage.

We assume that the fractal consists of a large number of iterations and thus fits in a large grid, where a point lies at each 90 degree turn. First, we note at S_i where i is sufficiently large, the fractal passes through 5^i points out of 9^i total points, so $N^\beta = \left(\frac{5}{9}\right)^i$, and $\beta = \log_9 5 > \frac{1}{2}$. To satisfy $\beta = \frac{1}{2\alpha}$, we set $\alpha = \log_5 3$ and show that the fractal is $\log_5 3$–Hölder continuous.

We now consider any pair of points on the fractal and denote the difference in horizontal and vertical distances as h and v, respectively. Observe that for any two points separated by a horizontal distance of $\frac{1}{3^n}$, the path between passes through at least $\big(\dfrac{1}{5^n}\big)(5^i) = 5^{i-n}$ points. Then by definition, $h = \frac{1}{3^n}$, and substitution gives $5^{i-n} = 5^{i+\log_3 h}$. So given the horizontal distance h between two points, the number of points on the path between them is at least $5^{i+\log_3 h}$. We can apply similar reasoning to v, and we obtain the following result for our α–Hölder constant:

$$||\phi||_{C^\alpha} \leq \frac{\sqrt{h^2+v^2}}{\left(\frac{\max\{5^{i+\log_3 h}, 5^{i+\log_3 v}\}}{5^i}\right)^\alpha} = \frac{\sqrt{h^2+v^2}}{\big(\max\{5^{\log_3 h}, 5^{\log_3 v}\}\big)^\alpha}.$$

Note that $0 \leq h \leq 1$, and $0 \leq v \leq \frac{1}{2}$. Assuming that $h \geq v$, we have

$$||\phi||_{C^\alpha} \leq \frac{\sqrt{h^2+v^2}}{\big(5^{\log_3 h}\big)^{\log_5 3}} \leq \frac{2h}{h} = 2.$$

Assuming $h < v$ gives the same bound.

Now, we consider the case where two points are horizontally separated by $h = \frac{r}{3^n}$ and r is an integer greater than 1. Since $r > 1$, we are still guaranteed at least $5^{i+\log_3 h}$ points along the curve between the two points, so our original bound holds. Thus our fractal exemplifies a ϕ that passes through $N^{\frac{1}{2\alpha}}$ points and is α–Hölder continuous. □

Thus α–Hölder continuity can still be achieved when $\beta = \frac{1}{2\alpha}$, more specifically when $\alpha = \log_5 3$. By considering a second curve through the grid, we deduce a lower bound for the maximum number of points an α–Hölder continuous function can pass through, with the bound specific to a grid configuration of N points. Thus we arrive at our main result:

Theorem 4.4. *Let $h_\alpha(N)$ be the maximum number of points that any α–Hölder continuous function ϕ can pass through in the grid with N points. Then the following bounds hold for $h_\alpha(N)$:*

$$\Omega\big(N^{1-\frac{\alpha}{2}}\big) = h_\alpha(N) = O\big(N^{\frac{1}{2\alpha}}\big). \tag{3}$$

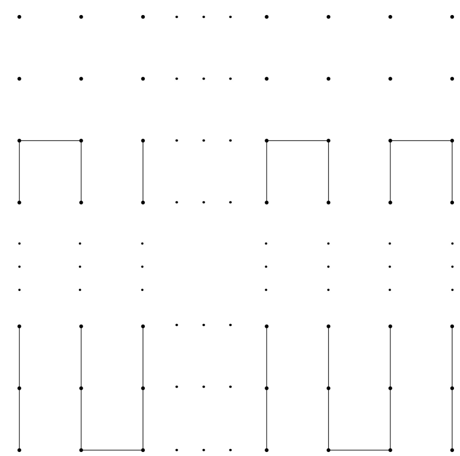

Figure 5: Function through $k^{\epsilon+1}$ points in a $k \times k$ grid

Proof. Consider a grid of size $k \times k$ within a unit square with vertices at $(0,0),(0,1),(1,1)$ and $(1,0)$, displayed in Figure 5. Begin at $(0,0)$, and move upward to point $(0, k^{\epsilon-1})$. Then move right to $(\frac{1}{k}, k^{\epsilon-1})$, and downward to $(\frac{1}{k}, 0)$. Continue snaking in this fashion, moving horizontally by $\frac{1}{k}$ at the end of every segment length $k^{\epsilon-1}$. Assuming that the distances between x_i, x_{i+1} for all $i \in \{1, 2, \ldots, k^2-1\}$ are equal on the unit interval, it is sufficient to consider the α–Hölder expression with respect to the specific point $(0,0)$ and all other points on the curve.

We first consider the value of the α–Hölder expression between the two points connected by the longest straight segment, which intuitively gives a relatively large α–Hölder value. Thus we examine $(0,0)$ and $(0, k^{\epsilon-1})$, and we find that $\dfrac{|\phi(x_1) - \phi(x_{k^\epsilon})|}{x_1 - x_{k^\epsilon}} = k^{\alpha+\epsilon-1}$. If for the remaining cases the α–Hölder values are lower or $\alpha = 1-\epsilon$ guarantees α–Hölder continuity, we are done.

First, introduce i and j such that $i \in \{1, 2, \ldots, k^\epsilon\}$, and j is a positive odd integer at most $k-1$.

Case 1: Point has coordinates $(0, \frac{i}{k})$.

The α–Hölder value for $(0,0)$ and any point of the form $(0, \frac{i}{k})$ reduces to $i^{1-\alpha}\big(k^{\alpha\epsilon+\alpha-1}\big)$. Note, however, that this expression cannot exceed $k^{\alpha+\epsilon-1}$ for any $\alpha \in (0,1]$.

Case 2: Point has coordinates $(\frac{j+1}{k}, \frac{i}{k})$.

Let $j' = j+1$, and assume that $\alpha = 1-\epsilon$. Then our α–Hölder expression is as follows:

$$\frac{\sqrt{\left(\frac{i}{k}\right)^2 + \left(\frac{j'}{k}\right)^2}}{\left(\frac{j'k^\epsilon + i}{k^{1+\epsilon}}\right)^\alpha} = \frac{\sqrt{i^2 + (j')^2}}{k\left(\frac{j'k^\epsilon + i}{k^{1+\epsilon}}\right)^\alpha} = \frac{\sqrt{i^2 + (j')^2}}{k^\epsilon\left(j' + \frac{i}{k^\epsilon}\right)^\alpha} \leq \frac{i + j'}{k^\epsilon\left(j' + \frac{i}{k^\epsilon}\right)^\alpha}.$$

Note that $i \leq k^\epsilon$, and $j' = (j')^\epsilon (j')^\alpha$, where $(j')^\epsilon < k^\epsilon$ and $(j')^\alpha < (j' + \frac{i}{k^\epsilon})^\alpha$. Thus,

$$\frac{i + j'}{k^\epsilon\left(j' + \frac{i}{k^\epsilon}\right)^\alpha} \leq 2$$

and the α–Hölder value is bounded by a constant.

Case 3: Point has coordinates $(\frac{j}{k}, \frac{i}{k})$.

Observe that the Euclidean distance between $(\frac{j-1}{k}, \frac{i}{k})$ and $(0,0)$ is shorter than that between $(\frac{j}{k}, \frac{i}{k})$ and $(0,0)$. Additionally, note that since the curve passes through $(\frac{j-1}{k}, \frac{i}{k})$ first, the distance between the inputs of $(\frac{j-1}{k}, \frac{i}{k})$ and $(0,0)$ is smaller than the distance between the inputs of $(\frac{j}{k}, \frac{i}{k})$ and $(0,0)$. It follows that the α–Hölder value for points $(\frac{j}{k}, \frac{i}{k})$ and $(0,0)$ is strictly lower than that for $(\frac{j-1}{k}, \frac{i}{k})$ and $(0,0)$.

Therefore, we see that $||\phi||_{C^\alpha} = k^{\alpha+\epsilon-1}$. In order for ϕ to be α–Hölder continuous, $\alpha + \epsilon - 1 \leq 0$, and since $\beta = \frac{1+\epsilon}{2}$, we have $\beta \leq 1 - \frac{\alpha}{2}$. Thus by this result and Theorem 4.2, we know that

$$\Omega\left(N^{1-\frac{\alpha}{2}}\right) = h_\alpha(N) = O\left(N^{\frac{1}{2\alpha}}\right).$$

$\square$

That is, for any given α and N points in a grid, as N becomes sufficiently large, the maximum number of given points an α–Hölder continuous function ϕ may contain is bounded between $N^{1-\frac{\alpha}{2}}$ and $N^{\frac{1}{2\alpha}}$. Note that $1-\frac{\alpha}{2} > \frac{1}{2}$, indicating that β exceeds the value of $\frac{1}{2}$ deduced from the Erdős–Szekeres theorem.

Finally, we interpret a result on path length within the unit square to give a bound for $||\phi||_{C^1}$ for any set of N points in the unit square. Because the Lipschitz constant $||\phi||_{C^1}$ represents the maximum stretch factor from the unit interval onto the unit square, we examine the bound proposed by Verblunsky [7] for the shortest path through N points in the unit square, determining a lower bound for $||\phi||_{C^1}$ for any arbitrary N points in the unit square.

Theorem 4.5 (Verblunsky)**.** *For any arbitrary N points in the unit square, the shortest path through the points has length less than $2 + (2.8N)^{1/2}$.*

Since all mappings are from the unit interval, the maximum value for the denominator of the Lipschitz expression is 1. The lower bound follows.

Corollary 4.4. *For ϕ that passes through N arbitrary points in the unit square, $||\phi||_{C^1} < 2 + (2.8N)^{1/2}$.*

Thus we obtain an upper bound for $||\phi||_{C^1}$ in terms of the number of given points it passes through, rather than β, the proportion of total given points.

5 Conclusion

Our problem, inspired by the Erdős–Szekeres theorem, asked under what conditions a curve through some subset of N given points in the unit square would be α–Hölder continuous. We determined that there always existed an α–Hölder continuous curve that passed through at least $N^{1/2}$ of the given points. However, we asked if we could do better than $\beta = \frac{1}{2}$; that is, whether there would always exist an α–Hölder continuous curve that passed through more than $N^{1/2}$ of the given points. We found that when the given points were from a grid, as N tended to infinity, the maximum β to guarantee α–Hölder continuity for our function ϕ was bounded below by $1 - \frac{\alpha}{2}$ and above by $\frac{1}{2\alpha}$. Thus, we deduced that β could exceed $\frac{1}{2}$ while still allowing for an α–Hölder continuous function ϕ. However, in general for any N

arbitrarily configured points in the unit square, as N tended to infinity, the maximum β remained bounded below by $\frac{1}{2}$ but also bounded above by $\frac{1}{2\alpha}$.

Future work includes proving the conjecture that the bounds $\Omega\left(N^{1-\frac{\alpha}{2}}\right) = h_\alpha(N) = O\left(N^{\frac{1}{2\alpha}}\right)$ defined in Theorem 4.4 hold for any configuration of N points in the unit square and not restricted to those arranged in a grid. Additional extensions include analyzing properties of smooth α–Hölder continuous curves through a calculus-based approach, as well as applying our results in context of probability theory.

6 Acknowledgments

I would like to thank my mentor Ms. Thao Do, graduate student at Massachusetts Institute of Technology, who supported and guided me through each step of the research process. Further thanks go to Dr. John Rickert and Dr. Tanya Khovanova for providing insight to writing, formatting, and presenting findings. In addition, I would like to thank Dr. Larry Guth for proposing this problem, providing background, and suggesting ideas for how to tackle this problem. More thanks go to Dr. Slava Gerovitch, Dr. David Jerison and Dr. Ankur Moitra, for their tireless efforts in advising, overseeing, and ensuring an enjoyable research experience. Lastly, great thanks go to my sponsors Mr. John Yochelson, Dr. Melissa Flagg, Dr. Robin Staffin, Mr. Kavish Gandhi, Dr. Wenchien Chou, Dr. Fengjen Luo, Dr. Qi Wang, and Ms. Xiaojun Zhang, as well as the Center for Excellence in Education, the Research Science Institute, and Massachusetts Institute of Technology Math Department, for providing me with the facilities and means to experience an unparalleled summer of research.

References

[1] S. M. Buckley. Space-filling curves and related functions. *Easter*, 36:9–19, 1996.

[2] P. Erdős and G. Szekeres. A combinatorial problem in geometry. *Compositio Mathematica*, 2:463–470, 1935.

[3] K. Hoffman, M. Padberg, and G. Rinaldi. Traveling salesman problem. *Encyclopedia of Operations Research and Management Science*, pages 1573–1578, 2016.

[4] J. Matoušek. On Lipschitz mappings onto a square. *The Mathematics of Paul Erdös*, 2:303–309, 1997.

[5] A. Seidenberg. A simple proof of a theorem of Erdős and Szekeres. *Journal of the London Mathematical Society*, 34:352, 1959.

[6] J. M. Steele. Variations on the monotone subsequence theme of Erdős and Szekeres. *IMA Volumes in Mathematics and its Applications*, 72:111–131, 1995.

[7] S. Verblunsky. On the shortest path through a number of points. *Proceedings of the American Mathematical Society*, 2:904–913, 1951.