A combinatorial interpretation of the h- and γ-vectors of the cyclohedron

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July 28, 2015

Abstract

We give an overview of the theory of convex polytopes and focus our attention on the family of graph-associahedra, constructed from connected graphs. We present a combinatorial interpretation of the γ-vector of the cyclohedron and conjecture a descent-preserving bijection between the $\mathcal{B}$-trees of the cycle graph on $n+1$ vertices and all lattice paths on the $n \times n$ grid.

Summary

The mathematical area of combinatorics has a long history, and many branches of research have emerged from it. We study a connection between graph theory and geometry which has proven to be the source of many interesting phenomena. From a collection of vertices joined by edges we follow a known procedure to construct its associated polyhedral shape: it is the main focus of this type of research to find the number of faces of the generated shape. Every collection of vertices and edges will generate a shape with certain properties, and we point toward an interpretation for the number of faces of a particular kind of shape.

1 Introduction

Polytopes are the modern representation of convex shapes in an arbitrary number of dimensions. The study of geometry is one of the oldest branches of mathematics, but the framework in which polytopes are studied today was not fully introduced until 1967 [1]. The modern approach to classical geometry provides new insight into old questions, while allowing the pursuit of new paths of research.

A notable example of the relationship between polytopes and a seemingly unrelated area of mathematics is the realization of a very interesting subset of polytopes: graph-associahedra. These polytopes are a particular type of generalized permutohedra, as defined by Postnikov [2], and their study relies on the analysis of combinatorial properties of graphs, to provide insight into the combinatorial structure of the associated polytopes.

For well-understood graphs, results regarding the structure of their associated polytopes are abundant (see [3, 2, 4]). For example, the path graph on n-vertices yields the associahedron with vertices corresponding to the triangulations of the $(n+2)$-gon, and the complete graph on n-vertices yields the permutohedron with vertices corresponding to permutations of the vector $(1, \dots, n)$. It is known that the number of k-dimensional faces of these polytopes is defined as a function of *Narayana* and *Eulerian* numbers respectively, and similarly beautiful results have been found for other popular graphs, such as the cycle on n vertices (see [4] for a comprehensive survey). The study of the combinatorial structure of graph-associahedra relies heavily on the symmetry of the graph chosen; removing an edge from the complete graph or adding an edge between two vertices of the cycle dramatically increases the difficulty of finding elegant formulas to describe the associated polytopes.

In Section 2 we introduce the theory of convex polytopes and graph-associahedra. In Section 3 we construct a bijection between equivalence classes of the $\mathcal{B}$-trees of the cycle on $n+1$ vertices and lattice paths on the $n \times n$ grid. We state a generalized conjecture of our bijection and present a combinatorial interpretation of the γ-vector of the graph-associahedra of the cycle graph.

2 Definitions

We begin by defining the essential concepts used thorough this paper.

An n-dimensional *polytope* P is the convex hull of a finite set of points $S \subset \mathbb{R}^n$. It is equivalently defined as the bounded intersection of closed halfspaces (see Ziegler [1] for a detailed explanation).

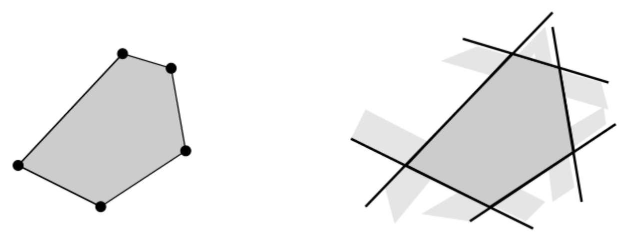

Figure 1: Equivalent realizations of a polytope.

We call an n-dimensional polytope *simple* if each of its vertices is adjacent to n edges. P^* is *dual* or *polar* to P if there is an inclusion-reversing bijection between the faces of the two, that is, vertices of P correspond to facets of P^*, and more specifically, k-dimensional faces correspond to $(n-k-1)$-dimensional faces. The dual of a simple polytope is *simplicial*; all of its facets are simplices. A polytope P is *flag* if any set of pairwise intersecting facets of P has non-empty intersection.

For a given n-dimensional polytope P, we encode its combinatorial structure in several different ways. First, we define $f_i(P)$ to be the number of i-dimensional faces of P, with $f_0(P)$ the number of vertices, $f_1(P)$ the number of edges joining vertices, and so on. Then we define $(f_0(P), ..., f_n(P))$ to be the *f-vector*, and $f_P(t) = \sum_{i=0}^{n} f_i(P)t^i$ the *f-polynomial.*

For example, for the usual cube in three dimensions, its f-vector is $f(\mathrm{Cube}_3) = (8, 12, 6, 1)$ and its f-polynomial is $f_{\mathrm{Cube}_3}(t) = 8 + 12t + 6t^2 + t^3$.

When P is simple, it is convenient to define the *h-vector* as the coefficient vector of the h-polynomial, constructed from the relation $f_P(t) = h_P(t+1)$. From Sommerville [5], we know that the h-vector satisfies the *Dehn-Sommerville* symmetry $h_i(P) = h_{n-i}(P)$. Furthermore, Gal [6, Section 2] showed that if P is flag as well, the f and h-vectors can be equivalently encoded in a shorter γ-vector $(\gamma_0(P), ..., \gamma_{\lfloor \frac{n}{2} \rfloor}(P))$ and its corresponding γ-polynomial, determined by the relation

$$h_P(t) = \sum_{i=0}^{\lfloor \frac{n}{2} \rfloor} \gamma_i(P) t^i (1+t)^{n-2i},$$

which follows from the expansion of the palindromic h-polynomial.

Thus, by finding the value of the very compact γ-vector of P, we can derive the value of its f-vector. Gal [6] conjectured that all entries in the γ-vector of flag polytopes are positive. Buchstaber et al. [7] proved the non-negativity of the γ-vector of all graph-associahedra. More particular proofs have been described for some collections of polytopes (see Erokhovets [8], for example), but the general question of γ-positivity remains open.

For a given connected graph G with labeled nodes $\{1, ..., n\}$ we study its associated polytope, the aptly named *graph-associahedron*.

Definition 2.1. [4, Definition 6.1] A *building set* $\mathcal{B}$ on S is a collection of subsets of S that satisfy the following conditions:

1. If $I, J \in \mathcal{B}$ and $I \cap J \neq \emptyset$, then $I \cup J \in \mathcal{B}$.

2. $\{i\} \in \mathcal{B}$ for all $i \in S$.

We specialize the building set to our connected graph G and its vertex set S. Given a subset of vertices I, let $G|_I$ be the *induced subgraph* of G obtained by considering only vertices $i \in I$. Note that $\mathcal{B}(G)$ consists of all subsets $I \subset S$ such that the induced subgraph

$G|_I$ restricted by I is connected. The set of inclusion-maximal subsets of $\mathcal{B}(G)$ is simply $\mathcal{B}_{\max}(G) = \{S\}$, because S is connected by definition. For every building set, we define a geometrical process that will generate its associated polytope.

Definition 2.2. [4, Definition 6.3] For every $I \in \mathcal{B}$, let Δ_I be the convex hull of the set of standard basis vectors $\{e_i : i \in I\}$ in $\mathbb{R}^n$. Define the *nestohedron* $P_{\mathcal{B}}$ to be the Minkowski sum of the simplices Δ_I:

$$P_{\mathcal{B}} := \sum_{I \in \mathcal{B}} \Delta_I.$$

We specialize again the definition to the building set of our connected graph, and call the associated polytope the *graph-associahedron* of G. Here follows a definition that allows us to relate the structure of the building set of a graph to the combinatorial structure of its associated nestohedra.

Definition 2.3. [2, Definition 7.3] Given a building set $\mathcal{B}$, a subset $N \subset \mathcal{B} \setminus \mathcal{B}_{\max}$ is a *nested set* if it satisfies the conditions:

1. For $I, J \in N$, either $I \subset J$ or $J \subset I$ or $I \cap J = \emptyset$.
2. For any collection of $k \geq 2$ disjoint sets $I_1, ..., I_k \in N$, the union $I_1 \cup ... \cup I_k$ is not in $\mathcal{B}$.

We define the *nested set complex* $\Delta_{\mathcal{B}}$ as the collection of all nested sets of $\mathcal{B}$. We finally describe the important relationship between $\Delta_{\mathcal{B}}$ and $P_{\mathcal{B}}$:

Theorem 2.1. *[2, Theorem 7.4] For a building set $\mathcal{B}$, the associated nestohedron $P_{\mathcal{B}}$ is a simple polytope of dimension $n - |\mathcal{B}_{max}|$. Its dual simplicial complex is isomorphic to the nested set complex $\Delta_{\mathcal{B}}$.*

Specializing to the case of a connected graph on n vertices, we note that its graph-associahedron is $(n-1)$-dimensional.

It was further proven by Postnikov [2] that the vertices of $P_{\mathcal{B}}$ are the inclusion-maximal nested sets of $\Delta_{\mathcal{B}}$, and the dimension of a given face in $P_{\mathcal{B}}$ has dimension $n - |N| - |\mathcal{B}_{\max}|$, where N is its associated nested set.

We can define lower and upper bounds for the combinatorial information of nestohedra. The largest possible building set is the set of all subsets of S; equivalently, we construct the graphical building set $\mathcal{B}(K_n) = 2^{[n]} \setminus \{\emptyset\}$ from the complete graph and call it the *complete building set*. Its f- and h-vectors define then the upper bounds for any nestohedra. The smallest possible connected building set is the set of all singletons of $[n] = \{1, 2, \ldots, n\}$, $\mathcal{B} = \{\{1\}, ..., \{n\}\}$, which yields a nestohedron equivalent to the $(n-1)$-simplex, defining the lower bound for the f- and h-vectors.

Let us define the two constructions which allow Postnikov [4] to redefine the h-vector as a function of the building set. For a rooted labeled tree T, let us define $T_{\leq i}$ as the set of nodes that are descendants of i. Two nodes are *incomparable* if neither $i \in T_{\leq j}$ nor $j \in T_{\leq i}$.

Definition 2.4. [4, Definition 8.1] For a connected building set $\mathcal{B}$ on $[n]$, define a $\mathcal{B}$-*tree* to be a rooted tree T on the node set $[n]$ which satisfies the following conditions:

1. For all $i \in [n]$, $T_{\leq i} \in \mathcal{B}$.

2. For $k \geq 2$ incomparable nodes $i_1, ..., i_k \in [n]$, $\bigcup_{j=1}^{k} T_{\leq i_j} \notin \mathcal{B}$,

We interpret the construction of a $\mathcal{B}$-tree of a graphical building set as follows: pick a vertex i in G and consider it the root of the tree T. For each connected component of $G \setminus \{i\}$, generate a subtree and join it from its root to i by an edge. Repeat this process for every possible subtree sequence.

To help illustrate the process, consider the path graph Path_3 on three vertices labeled $\{1, 2, 3\}$ with edges $\{(1, 2), (2, 3)\}$. To construct the $\mathcal{B}$-trees of Path_3, pick the vertex 1 and generate all possible trees rooted on 1. We remove 1 from G, leaving us with a single connected component $\{2, 3\}$: we can now construct two subtrees, either $2 \to 3$ or $3 \to 2$. So all trees rooted on 1 are $\{1 : [2 : [3]]\}$ and $\{1 : [3 : [2]]\}$. We repeat the same process but with root 2: remove it from the graph and we get two disconnected components, $\{1\}$ and $\{3\}$. Since there are no possible subtrees for any of the components, we find that the only tree rooted on 2 is just $\{2 : [1, 3]\}$. Finally, choose 3 as the root and we get that all trees rooted on 3 are $\{3 : [1 : [2]]\}$ and $\{3 : [2 : [1]]\}$. Figure 2 shows the generated trees described in this construction process.

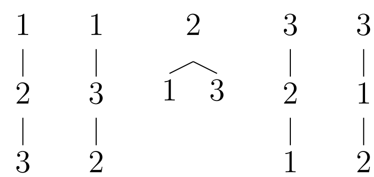

Figure 2: Decomposition of Path_3 in $\mathcal{B}$-trees.

Figure 3 shows the process of constructing a single $\mathcal{B}$-tree from the complete graph K_4, illustrating the concept of connectivity.

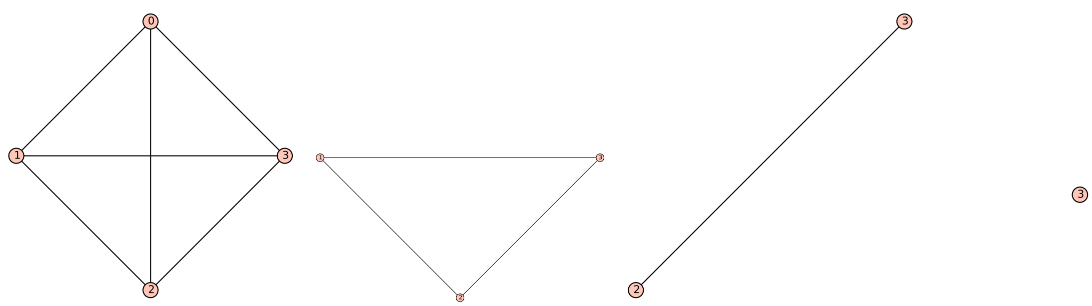

Figure 3: The process of constructing a $\mathcal{B}$-tree of K_n never disconnects it, therefore all nodes of the tree have at most one child.

Definition 2.5. [4, Definition 8.7] For a building set $\mathcal{B}$ on $[n]$, define the set $S_n(\mathcal{B}) \subset S_n$ of $\mathcal{B}$-*permutations* as the set of permutations $w \in S_n$ such that for any $i \in [n]$, $w(i)$ and $\max\{w(1), ..., w(i)\}$ are in the same connected component of $\mathcal{B}|_{\{w(1),...,w(i)\}}$.

Finally, we define a statistic that is used to describe the h- and γ-polynomials. The *descent number* $\mathrm{des}(w)$ of a permutation $w \in S_n$ is the number of indices $i \in [n]$ such that

$w(i) > w(i+1)$. Similarly, we define the *descent number of a tree* $\mathrm{des}(T)$ as the number of edges joining the vertex j to its parent i, satisfying $i > j$. Let $\hat{S}_n$ be the set of n-permutations with no two consecutive descents $w(i-1) > w(i) > w(i+1)$ and no final descents $w(n-1) > w(n)$.

Postnikov [2] showed how to describe the h-vector of a building set $\mathcal{B}$ as the descent-generating function of its $\mathcal{B}$-trees.

Corollary 2.2. *[2, Corollary 8.4] Given a connected building set $\mathcal{B}$, the h-polynomial of $P_{\mathcal{B}}$ is given by*

$$h_{\mathcal{B}}(t) = \sum_{T \in \mathcal{B}\text{-}trees} t^{\mathrm{des}(T)}.$$

Let a connected building set $\mathcal{B}$ be *chordal* if for any $I = \{i_1 < ... < i_r\} \in \mathcal{B}$, the set $\{i_s, i_{s+1}, ..., i_r\}$ for all $s \leq r$ is also an element of $\mathcal{B}$. Postnikov [4, Proposition 8.10] proved that $\mathcal{B}$-trees, $\mathcal{B}$-permutations and the set of vertices of $P_{\mathcal{B}}$ are all in bijective correspondence. Furthermore, they proved the following:

Theorem 2.3. *[4, Theorem 1.1] For a connected chordal building set $\mathcal{B}$ on $[n]$, the h-vector of the nestohedron $P_{\mathcal{B}}$ is given by*

$$\sum_i h_i t^i = \sum_{w \in S_n(\mathcal{B})} t^{\mathrm{des}(w)},$$

and the γ-vector of $P_{\mathcal{B}}$ is given by

$$\sum_i \gamma_i t^i = \sum_{w \in S_n(\mathcal{B}) \cap \hat{S}_n} t^{\mathrm{des}(w)}.$$

We specialize the result to graphical chordal building sets, constructed from *chordal graphs*; graphs in which every induced cycle has at most three vertices. Theorem 2.3 says that for any chordal graph, the h- and γ-polynomials of its graph-associahedron can be described from a subset of permutations equivalent to the $\mathcal{B}$-trees.

To describe the structure of $P_{\mathcal{B}(G)}$ for a connected graph G, we must understand and describe the structure of either its associated set of $\mathcal{B}$-trees or the set of $\mathcal{B}$-permutations.

We present the known h-polynomials of classical graph-associahedra. We shall abuse notation by referring to the h-polynomial of the graph-associahedron of G as $h_G(t)$ instead of $h_{\mathcal{B}(G)}(t)$.

The *path graph* Path_n on n-vertices generates all $C_n = \frac{1}{n+1}\binom{2n}{n}$ binary trees on n vertices, labeled by depth-first search. The descent of a tree is the number of right edges, and it is counted by the Narayana numbers $N(n,k) = \frac{1}{n}\binom{n}{k}\binom{n}{k-1}$. Thus, the h-polynomial is

$$h_{\mathrm{Path}_n}(t) = \sum_{k=0}^{n-1} N(n, k+1) \cdot t^k.$$

The *complete graph* on n-vertices K_n generates all linear trees, easily realized as permutations on n words. The vertices of its graph-associahedron are exactly all permutations of the vector $(1, ..., n)$, and thus the descent of the trees is the usual Eulerian polynomial

$$h_{K_n}(t) = \sum_{k=0}^{n-1} A(n, k+1) \cdot t^k.$$

3 Combinatorics of the cyclohedron

The graph-associahedron of the cycle graph Cycle_n is known as the *cyclohedron*, with its vertices corresponding to the triangulations of the $(2n+2)$-gon; see Simion [9].

Using the known h-polynomial of $P_{\mathrm{Cycle}_{(n+1)}}$, we derive a combinatorial interpretation for its γ-polynomial. The previous derivation by Reiner et al. [10] used hypergeometric series transformations.

3.1 Conjectural bijection between lattice paths and $\mathcal{B}$-trees

Consider the set of $\mathcal{B}$-trees of the $\mathrm{Cycle}_{(n+1)}$ graph. To construct a tree T, we pick a vertex $i \in [n+1]$ and set it as the root of the T. When we remove i from $\mathrm{Cycle}_{(n+1)}$, we are left with a Path_n with labels shifted cyclically. Therefore, there are $(n+1)C_n$ $\mathcal{B}$-trees, C_n of these rooted at $n+1$ such that $T \setminus \{n+1\}$ is the set of binary trees on n vertices. We define an operation which characterizes this cyclic construction.

Definition 3.1. Let the operation $\mathrm{Shift}^k(T)$ exchange the labels of T from i to $i-k \pmod{n+1}$, for a $\mathcal{B}$-tree T of $\mathrm{Cycle}_{(n+1)}$.

Lemma 3.1. *The operation* $\mathrm{Shift}^k(T)$ *defines a partition of the set of* $\mathcal{B}$*-trees of the* $\mathrm{Cycle}_{(n+1)}$ *graph into* C_n *equivalence classes of size* $n+1$*.*

Proof. Let T be a $\mathcal{B}$-tree of $\mathrm{Cycle}_{(n+1)}$ rooted on $n+1$. The operation $\mathrm{Shift}^k(T)$ generates a set of $n+1$ trees with decreasing root labels when applied $n+1$ times. We then define an equivalence relation on $\mathcal{B}$-trees. Consider all C_n binary trees rooted at $n+1$. Note that for a pair of non-equal such trees T_a and T_b, it is impossible to have $T_a = \mathrm{Shift}^k(T_b)$ for some k, since the root of T_a has to be $n+1$, but it is generated from $(n+1)-k \equiv n+1 \pmod{n+1}$ which implies that $k=0$, and thus $T_a = T_b$, a contradiction. Similarly, it is impossible that $\mathrm{Shift}^k(T_a) = \mathrm{Shift}^{k'}(T_b)$ for some $k > k'$, because then $\mathrm{Shift}^{k-k'}(T_a) = T_b$, implying that $T_a = T_b$.

Two trees T and T' are congruent under our equivalence relation if there exists some $k \in [n+1]$ such that $T = \mathrm{Shift}^k(T')$. Reflexivity is obvious, simply let $k=0$. For symmetry, if $T = \mathrm{Shift}^k(T')$, then $\mathrm{Shift}^{n-1-k}(T) = \mathrm{Shift}^{k+n-1-k}(T') = \mathrm{Shift}^0(T') = T'$. For transitivity, if $T = \mathrm{Shift}^k(T')$ and $T' = \mathrm{Shift}^{k'}(T'')$, then $T = \mathrm{Shift}^k(\mathrm{Shift}^{k'}(T'')) = \mathrm{Shift}^{k+k'}(T'')$. □

We take the representative tree of an equivalence class to be the binary tree rooted at $n+1$. There is a natural bijection between binary trees on n vertices and lattice paths on the

$n \times n$ grid. A binary tree can be expressed as a Dyck word, a string of balanced parenthesis. Denote $\text{rep}(i)$ as the Dyck word representation of a binary tree node. Given a node i, if it has no children then $\text{rep}(i) = ()$. If it has one children j such that $j > i$, then $\text{rep}(i) = ()(\text{rep}(j))$. If it has one children j such that $j < i$, then $\text{rep}(i) = (\text{rep}(i))$. If it has two children j, j' such that $j > i > j'$, then $\text{rep}(i) = (\text{rep}(j'))(\text{rep}(j))$.

A Dyck word also encodes a Dyck path, a type of path which does not cross the diagonal, with opening parenthesis representing horizontal edges, and closing parenthesis representing vertical edges. Note that a sequence "$\ldots)(\ldots$" in a Dyck word is equivalent to a peak in a Dyck path, and to a descent in a binary tree. We thus have a map ϕ from all C_n binary trees on n vertices rooted at $n+1$ to all Dyck paths on the $n \times n$ grid, preserving the descent number of the trees as the number of peaks of the paths.

To make this section self-contained, we include the following definition and lemma from Chen [11].

Definition 3.2. The operation $\text{Shift}^k(P)$ generates a lattice path on the $n \times n$ grid as follows: let e be the last horizontal edge in P stemming from the diagonal, let A be the subset of edges below e and let B be the subset of edges above e. Then $\text{Shift}^1(T)$ is the path obtained by moving the first edge of B to $(0, 0)$, adding e at its last edge and completing the path by adding A at e. $\text{Shift}^k(P)$ repeats the operation $\text{Shift}^1(P)$ k times.

Lemma 3.2. *[11, Theorem 0.1] The set of lattice paths on the $n \times n$ grid from $(0, 0)$ to (n, n) is partitioned into C_n equivalence classes of size $n + 1$ under the $\text{Shift}^k(P)$ operation.*

By applying $\text{Shift}^1(P)$ to a Dyck path P, we increase by one the number of vertical edges above the diagonal. We take the representative lattice path of an equivalence class to be the unique Dyck path contained in it, with no vertical edges above the diagonal.

We conjecture that the equivalence class $[T] = \{\text{Shift}^k(T) : k \in [n]\}$ of a binary tree on n nodes rooted at $n + 1$ defines the same descent-generating function as the equivalence class

$[P] = \{\mathrm{Shift}^k(P) : k \in [n]\}$ of the Dyck path $P = \phi(T)$.

Conjecture 3.3. *For every binary tree T on n nodes rooted on $n+1$ and its equivalent Dyck path $P = \phi(T)$, we have*

$$\sum_{T' \in [T]} t^{\mathrm{des}(T')} = \sum_{P' \in [P]} t^{\mathrm{des}(P')}.$$

We verified Conjecture 3.3 for all binary trees on n nodes for $n \in \{1, 2, \ldots, 13\}$.

As a conditional corollary assuming Conjecture 3.3, it follows that all the $\mathcal{B}$-trees of the cycle graph $\mathrm{Cycle}_{(n+1)}$ are in bijection with all lattice paths on the $n \times n$ grid, preserving the descent number.

3.2 Combinatorial interpretation of the γ-vector of the cyclohedron

The h-polynomial of the cyclohedron was computed by Simion [9]:

$$h_{\mathrm{Cycle}_{(n+1)}} = \sum_{k=0}^{n} \binom{n}{k}^2.$$

Reiner [10] derived the γ-polynomial using hypergeometric series transformations on the h-polynomial. We find a combinatorial description by following the approach first used in Postnikov [4] to compute the γ-polynomials of chordal nestohedra.

Theorem 3.4. *The γ-polynomial of the cyclohedron is given by*

$$\gamma_{\mathrm{Cycle}_{(n+1)}}(t) = \sum_{r=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{r, r, n-2r} t^r.$$

Recall that for a lattice path P with vertical edges $V \subset P$ and horizontal edges $H \subset P$, the descent number of P is defined as the number of peaks $\mathrm{peaks}(P) = |\{i \in V : i+1 \notin V\}|$.

Consider the set $S(k) = \{P : \text{peaks}(P) = k\}$ of all lattice paths with k descents. Points on the grid representing descents cannot be of the form $(i, 0)$, (n, i) nor (n, n) for any $i \in [n]$. Given a set of k descent coordinates, there is only one path with the specified descent points, and therefore $|S(k)| = \binom{n}{k}^2$. Summing over all possible number of descents, we get the total of $\sum_{k=0}^{n} \binom{n}{k}^2 = \binom{2n}{n}$ paths.

We separate the coordinates of the peaks in $S(k)$ in two sets of x- and y-coordinates $R(P), Q(P) \subset [n]$ respectively, for a given lattice path P. We abuse notation by referring to $R(P)$ and $Q(P)$ as R and Q. Let us define the following subsets of $[n]$:

- $A(P) = R \setminus Q$,
- $B(P) = Q \setminus R$,
- $C(P) = R \cap Q$ and
- $D(P) = [n] \setminus (R \cup Q)$.

We similarly refer to $A(P)$ as A, and so on. The four subsets defined partition $[n]$, therefore $A \cup B \cup C \cup D = [n]$ and $|A| + |B| + |C| + |D| = n$. Since there is the same number of x- and y-coordinates, let $r := |A| = |B|$ and thus $2 \cdot |A| + |C| + |D| = n$.

We define the operation $\text{Hop}_m(P)$ for certain $m \in [n]$ as follows. If $m \in A$ or $m \in B$, the operation is undefined. If $m \in C$, then $\text{Hop}_m(P)$ removes m from R and Q. If $m \in D$, then $\text{Hop}_m(P)$ adds m to R and Q.

The following technical properties of our operation ensure that we can use the method discovered by Postnikov [2] to describe the γ-polynomial of a nestohedron.

Lemma 3.5. *The operation* $\text{Hop}_m(P)$ *has the following properties:*

(1) $\text{Hop}_m(P)$ *is commutative.*

(2) $\text{Hop}_m(P) \circ \text{Hop}_m(P)$ *is the identity operation.*

(3) If $\text{peaks}(P) = k$*, we can generate a path* P' *such that* $\text{peaks}(P') = n - k$ *through a sequence of* $\text{Hop}_m(P)$ *operations.*

Proof. For (1), applying $\text{Hop}_m(P)$ for a valid $m \in [n]$ has no effect on a valid $m' \in [n]$. For (2), say that $m \in C$. Then $\text{Hop}_m(P)$ removes m from R and Q, and applying it again moves m back to R and Q. The same argument holds if $m \in D$. For (3), first note that $\text{peaks}(P) = |R| = |Q|$. At most, we have that $|D| = n - r$ when $R = Q$. At least, we have that $|D| = r$ when $|C|$. By applying $\text{Hop}_m(P)$, the number of peaks of the path generated fluctuates in that range. $\square$

Proof of Theorem 3.4. We define an equivalence relation under the $\text{Hop}_m(P)$ operation which partitions the set of all lattice paths on the $n \times n$ grid. Two lattice paths are congruent under the Hop operation if we can generate one from the other through a sequence of $\text{Hop}_m(P)$ for some $m \in [n]$. By taking the representative path of each equivalence class such that $|D| = r$, we find that there are $\binom{n}{r,r,n-2r}$ possible elements in the equivalence class $[P]$.

From Postnikov [2], it follows that an operation that partitions the set of $\mathcal{B}$-trees and satisfies Lemma 3.5 fully defines the γ-polynomial. $\square$

4 Conclusion

We presented a combinatorial interpretation for the γ-polynomial of the graph-associahedra of the cycle graph, based on the study of lattice paths on the $n \times n$ grid. We also presented a conjecture which directly relates equivalence classes of the lattice paths on the $n \times n$ grid with equivalence classes of the $\mathcal{B}$-trees of the cycle graph on $n+1$ vertices. Finding a complete bijection between lattice paths and $\mathcal{B}$-trees is an interesting problem for future research.

5 Acknowledgments

I would like to thank my mentor Pavel Galashin, graduate student at MIT, for helping me conduct this research. Thanks also go to everyone involved in the MIT Mathematics department; Prof. Jerison and Prof. Moitra. I thank Dr. Rickert and Dr. Khovanova as well for their recommendations regarding the writing and formatting of this paper. I thank Professor Alexander Postnikov for suggesting this project.

I am thankful to the Center for Excellence in Education, the Research Science Institute and the Massachusetts Institute of Technology for providing the opportunity to spend a summer doing research, as well as the Fundació Catalunya–La Pedrera for sponsoring me.

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