Generalizing the Inversion Enumerator to G-Parking Functions

Dylan Hendrickson

under the direction of
Pavel Galashin
Massachusetts Institute of Technology
Department of Mathematics

Research Science Institute
July 28, 2015

Abstract

We consider a generalization of the inversion enumerator for G-parking functions, $I_G(q)$. We find several recurrences for $I_G(-1)$, including a recursive formula whenever G is the cone over a tree. We relate $I_G(-1)$ to the number of partial orientations of G. Using the connection between I_G and the Tutte Polynomial, we find another recursive formula for $I_G(-1)$. For any partial cone G over a tree T, we compute $I_G(-1)$ by counting partial orientations of T with a specific set of vertices with even in-degree.

Summary

Classical parking functions are defined in terms of a line of cars trying to park based on which parking space they prefer. A generalization of this concept can be defined on a collection of nodes connected by edges, called a graph. The parking functions on a graph have information about how the graph is connected. We study a polynomial defined by parking functions on a graph, and relate this polynomial to other important concepts in graph theory.

1 Introduction and Background

Parking functions were first considered by Konheim and Weiss [1], and have been extensively studied, most notably by Richard Stanley [2]. They are defined informally in terms of cars trying to park: suppose n cars, numbered $1, \ldots, n$, drive down a road with n parking spots, numbered $0, \ldots, n-1$. Each car has a preferred parking spot, in which it will try to park. If a car's preferred spot is occupied, it parks in the next empty spot. The sequence of preferences $(a_1, a_2, \ldots, a_n)$ is a *parking function* if all cars manage to park.

Stanley [2] observed that a classical parking function of length n is a sequence $(a_1, a_2, \ldots, a_n)$ of natural numbers that satisfies the following: for each $k \in \{1, \ldots, n\}$, at least k a_i's are less than k. In other words, at least k cars want to park in the first k parking spaces. Stanley also showed that there are $(n+1)^{n-1}$ parking functions of length n. This is also Cayley's Formula for the number of trees on $n+1$ labeled vertices [3]. Let $\mathcal{P}_n$ denote the set of classical parking functions of length n. The n-th *inversion enumerator* is defined as the polynomial

$$I_n(q) := \sum_{(a_1,\ldots,a_n)\in\mathcal{P}_n} q^{\binom{n+1}{2}-a_1-a_2-\cdots-a_n}.$$

One generalization of parking functions is thought of in terms of sections of a parking lot. Each section has a non-negative number of available parking spaces, and each car has a preferred section in which to park, but has no preference among parking spaces within a section. More technically, let $\vec{b} = (b_1, \ldots, b_n)$ be a non-decreasing sequence of positive integers. We say a sequence $(a_1, \ldots, a_n)$ is a *$\vec{b}$-parking function* if for each $k \in \{1, \ldots, n\}$, at least k a_i's are less than b_k. We think of b_i as the cumulative number of parking spaces, so $b_i - b_{i-1}$ is the number of spaces in the ith section. When $\vec{b} = (1, \ldots, n)$, $\vec{b}$-parking functions are exactly ordinary parking functions.

Chebikin and Postnikov [3] studied a generalization of the inversion enumerator to $\vec{b}$-

parking functions, defining the *sum enumerator* as

$$I_{\vec{b}}(q) := \sum_{(a_1,\ldots,a_n)\in\mathcal{P}_{\vec{b}}} q^{a_1+a_2+\cdots+a_n-n},$$

where $\mathcal{P}_{\vec{b}}$ is the set of $\vec{b}$-parking functions. They found a formula for $I_{\vec{b}}(-1)$ in terms of the number of permutations with a prescribed set of descents that nicely generalized the $\vec{b} = (1, \ldots, n)$ case. We study another generalization of the inversion enumerator applied to G-parking functions.

Let G be an undirected graph on vertices $\{0, 1, \ldots, n\}$, allowing multiple edges. A sequence $(a_1, a_2, \ldots, a_n)$ of natural numbers is a *G-parking function* if for each nonempty set $U \subseteq \{1, \ldots, n\}$, there is some vertex $v \in U$ such that the number of edges between v and vertices outside U is *greater than* a_v. G-parking function is equivalently a function from $G\backslash 0$ to $\mathbb{N}$. Let $\mathcal{P}_G$ be the set of all G-parking functions. If G is the complete graph K_{n+1}, then $\mathcal{P}_G = \mathcal{P}_n$. To see why this is the case, let $U_0 = \{1, \ldots, n\}$. There must be some $v_0 \in U_0$ with $a_{v_0} < 0$. Then let $U_1 = U_0\backslash v$. There must now be $v_1 \in U_1$ with $a_{v_1} < 1$, and so on. Throughout this paper, we use G to represent both a graph and the set of its vertices.

For a graph G, define the *sum enumerator* as follows:

$$I_G(q) := \sum_{(a_1,\ldots,a_n)\in\mathcal{P}_G} q^{a_1+a_2+\cdots+a_n-n}.$$

Clearly $I_G(1) = |\mathcal{P}_G|$, the number of G-parking functions. Chebikin and Pylyavskyy [4] found a family of bijections between $\mathcal{P}_G$ and $\mathcal{T}_G$, the set of spanning trees of G, reducing the problem of finding $I_G(1)$ to counting spanning trees. Because complete graphs yield ordinary parking functions, $I_{K_{n+1}}(-1) = E_n$, the number of alternating permutations of $\{1, \ldots, n\}$ [3].

In Sections 2 and 3, we are concerned with graphs in which every vertex has an edge to

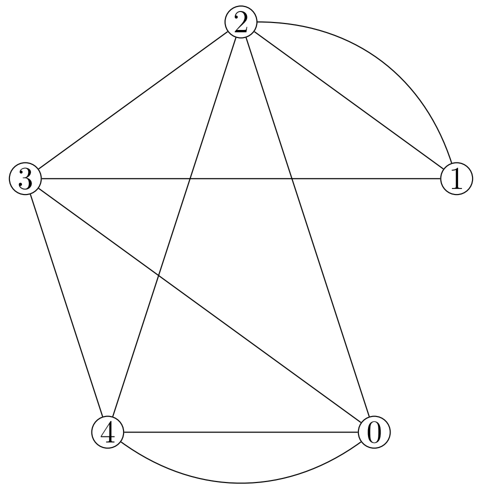

Figure 1: An example graph G with 5 vertices.

0. This is equivalent to the following definition of G-parking functions for a graph G on the vertices $\{1, \ldots, n\}$: $(a_1, \ldots, a_n)$ is a G-parking function if for each nonempty set U of vertices of G, there is some vertex $v \in U$ such that the number of edges between v and vertices outside U is *at least* a_v. A fact about G-parking functions using this definition translates into a fact about parking functions on the cone over G under the earlier definition. The cone over a graph G, denoted $\widehat{G}$ is the graph obtained by adding a vertex to G and adding an edge from the new vertex to each vertex in G. Because we consider graphs other than cones in Section 4, we use the earlier definition of G-parking function throughout the paper.

To illustrate these definitions, let G be the graph in Figure 1. It is easy to check that $(0, 4, 1, 1)$, $(1, 0, 3, 2)$, and $(2, 0, 0, 3)$ are G-parking functions. $(0, 4, 2, 1)$ is not a G-parking function because the set $U = \{1, 2, 3\}$ does not contain any vertices with enough edges to vertices outside of U. Upon counting the G-parking functions, we find that there are a total of 96, of which 49 have even sum and 47 have odd sum, so $I_G(-1) = 2$.

In Section 2, we prove recurrences allowing us to find $I_{\widehat{G}}(-1)$ for a graph by knowing its value for certain subgraphs. These recurrences give us a recursive method to find $I_{\widehat{G}}(-1)$ whenever G is a tree. In Section 3, we study partial orientations of graphs, and find that the number of partial orientations of G with even in-degrees is $\pm I_{\widehat{G}}(-1)$. In Section 4, we look at graphs where not all vertices have an edge to 0. We connect $I_G(-1)$ to the Tutte Polynomial, giving us another recurrence for $I_G(-1)$. For any partial cone G over a tree T, we find that $|I_G(-1)|$ is the number of partial orientations of T such that U is exactly the vertices with even in-degree.

2 Recursive Formulae for $I_G(-1)$

If we know the values of $I_{\widehat{G}}$ and $I_{\widehat{H}}$ for two graphs G and H, it is natural to ask what $I_{\widehat{G \cup H}}$ is, where $G \cup H$ is the disjoint union of G and H. Lemma 2.1 answers this question.

Lemma 2.1. *Let G and H be graphs on the vertices $\{g_1, \ldots, g_n\}$ and $\{h_1, \ldots, h_m\}$, respectively. Then $I_{\widehat{G \cup H}}(q) = I_{\widehat{G}}(q) I_{\widehat{H}}(q)$.*

Proof. Define $\mathcal{F} : \mathcal{P}_{\widehat{G}} \times \mathcal{P}_{\widehat{H}} \to \mathcal{P}_{\widehat{G \cup H}}$ by $\mathcal{F}((a_1, \ldots, a_n), (b_1 \ldots, b_m)) = (a_1, \ldots, a_n, b_1, \ldots, b_m)$. $\mathcal{F}$ is a bijection that preserves sums of parking functions, so

$$I_{\widehat{G \cup H}}(q) = \sum_{(a_1, \ldots, a_n, b_1, \ldots, b_m) \in \mathcal{P}_{\widehat{G \cup H}}} (q)^{a_1 + \cdots + a_n + b_1 + \cdots + b_m - (n+m)}$$

Because $\mathcal{F}((a_1, \ldots, a_n), (b_1, \ldots, b_m)) = (a_1, \ldots, a_n, b_1, \ldots, b_m)$,

$$\begin{aligned} &= \sum_{(a_1, \ldots, a_n) \in \mathcal{P}_{\widehat{G}}, (b_1, \ldots, b_m) \in \mathcal{P}_{\widehat{H}}} (q)^{a_1 + \cdots + a_n - n} (q)^{b_1 + \cdots + b_m - m} \\ &= \sum_{(a_1, \ldots, a_n) \in \mathcal{P}_{\widehat{G}}} (q)^{a_1 + \cdots + a_n - n} \sum_{(b_1, \ldots, b_m) \in \mathcal{P}_{\widehat{H}}} (q)^{b_1 + \cdots + b_m - m} \\ &= I_{\widehat{G}}(q) \times I_{\widehat{H}}(q). \end{aligned}$$

□

Applying Lemma 2.1 inductively shows that it also holds for graphs with more than two connected components. Because the sum enumerator of the cone over a graph can be understood by examining its connected components separately, we are most interested in finding $I_{\widehat{G}}(-1)$ for connected G.

The next graph decomposition for which we prove a recurrence is that of graphs centered around a star. Whenever a connected graph G has a vertex l such that removing l and its edges from the graph leaves a number of connected components equal to the degree of l in G, Theorem 2.3 can be used to reduce it to its components.

Definition 2.2. Suppose $G_1, \ldots, G_N$ are graphs, and that each G_i has a leaf l_i. Let v_i be the vertex connected to l_i. Let $\ast_{i\in\{1,\ldots,N\}} G_i$ be the graph formed by merging all l_i in $\bigcup_{i\in\{1,\ldots,N\}} G_i$ into a new vertex l. Let G'_i be the graph formed by removing l_i and its edge from G_i. See Figure 2 for an example.

Theorem 2.3. *Let $H = \ast_{i\in\{1,\ldots,N\}} G_i$. Then*

$$I_{\widehat{H}}(-1) = (-1)^{N-1} \sum_{\substack{U\subseteq\{1,\ldots,N\}\\ |U| \text{ even}}} \prod_{i\notin U} I_{\widehat{G}_i}(-1) \prod_{i\in U} I_{\widehat{G}'_i}(-1).$$

Proof. We assume that l is the first vertex of H, followed by v_1 through v_N, and that l_i and v_i are the first and second vertices of G_i, respectively. We also assume that H has n vertices and G_i has n_i vertices. To proceed, we need to define *partial $\widehat{G}$-parking function* and related terms. We use $\oplus$ to indicate sequence concatenation.

Definition 2.4. For any graph $\widehat{G}$, a *partial $\widehat{G}$-parking function* is the restriction of a $\widehat{G}$-parking function to the vertices of G except for the first vertex. Let $\mathcal{P}^*_{\widehat{G}}$ be the set of partial $\widehat{G}$-parking functions. We say a partial $\widehat{G}$-parking function $\vec{a}_i^*$ is *maximal at v* if the function formed by increasing the value of $\vec{a}_i^*$ at v by 1 is not a partial $\widehat{G}$-parking function. We say a partial $\widehat{G}$-parking function $\vec{a}_i^*$ is *maximal* if it is maximal at the second vertex in G, i.e. the first vertex to which it assigns a value. A partial $\widehat{G_i}$-parking function is maximal if it is maximal at v_i. Let $\operatorname{cont}_G(\vec{a}^*)$ be the total *contribution* of $\vec{a}^*$ to $I_{\widehat{G}}(-1)$, specifically

$\sum_{\vec{a} \text{ ending in } \vec{a}^*}(-1)^{\sum \vec{a}-|G|}$. For a partial $\widehat{H}$-parking function $\vec{a}^*$, let notmax($\vec{a}^*$) be the set of natural numbers i such that $\vec{a}^*$ is not maximal at v_i. If $U \subseteq \{1,\ldots,N\}$, let cont($U$)= $\sum_{\mathrm{notmax}(\vec{a}^*)=U} \mathrm{cont}_H(\vec{a}^*)$.

We construct the bijection $\mathcal{F}$ from $\mathcal{P}^*_{\widehat{G}_1} \times \cdots \times \mathcal{P}^*_{\widehat{G}_N}$ to $\mathcal{P}^*_{\widehat{H}}$ by concatenation.

Lemma 2.5. *Let U be a subset of $\{1,\ldots,N\}$. Then*

$$\mathrm{cont}(U) = \begin{cases} 0 & |U| \text{ odd} \\ (-1)^{N-1}\prod_{i\notin U} I_{\widehat{G}_i}(-1)\prod_{i\in U} I_{\widehat{G}'_i}(-1) & |U| \text{ even.} \end{cases}$$

Proof. We examine $|U|$ odd and $|U|$ even separately.

1. Suppose first that $|U| = 2k-1$ is odd. Consider a partial $\widehat{H}$-parking function $\vec{a}^*$ with notmax($\vec{a}^*$) $= U$. Then $(i)\oplus\vec{a}^*$ is an $\widehat{H}$-parking function for any $0 \le i \le 2k-1$. Of these $2k$ possible completions of $\vec{a}^*$, k have even sum and k have odd sum, so $\mathrm{cont}_H(\vec{a}^*) = 0$. Summing over all such $\vec{a}^*$, we find that $\mathrm{cont}(U) = 0$.

2. Now suppose $|U| = 2k$ is even. Consider a partial $\widehat{H}$-parking function $\vec{a}^*$ with notmax($\vec{a}^*$) $=$ U. Let $\mathcal{F}(\vec{a_1}^*,\ldots,\vec{a_N}^*) = \vec{a}^*$. Then $(i)\oplus\vec{a}^*$ is an $\widehat{H}$-parking function for any $0 \le i \le 2k$. As in the odd case, most of these contributions cancel, but this time we find that

$$\mathrm{cont}_H(\vec{a}^*) = (-1)^{\sum \vec{a}^* - n}.$$

Because $n = n_1 + \cdots + n_N - N + 1$,

$$\begin{aligned}\mathrm{cont}_H(\vec{a}^*) &= (-1)^{\sum \vec{a_1}^* - n_1}\ldots(-1)^{\sum \vec{a_N}^* - n_N}(-1)^{N-1} \\ &= (-1)^{N-1}\mathrm{cont}_{G_1}(\vec{a_1}^*)\ldots\mathrm{cont}_{G_N}(\vec{a_N}^*).\end{aligned}$$

But $\vec{a_i}^*$ is maximal if and only if $i \in U$. Hence

$$\begin{aligned}\operatorname{cont}(U) &= (-1)^{n-1} \sum_{\operatorname{notmax}(\vec{a}^*)=U} \operatorname{cont}_H(\vec{a}^*) \\ &= (-1)^{N-1} \prod_{i \in U} \sum_{\vec{a_i}^* \text{ non-maximal}} \operatorname{cont}_{G_i}(\vec{a_i}^*) \prod_{i \notin U} \sum_{\vec{a_i}^* \text{ maximal}} \operatorname{cont}_{G_i}(\vec{a_i}^*).\end{aligned}$$

The non-maximal partial $\widehat{G}_i$-parking functions are exactly the $\widehat{G}'_i$-parking functions, so the summation in the first product is equal to $-I_{\widehat{G}'_i}(-1)$. Since $|U|$ is even, the extra minus signs cancel. The contribution of each non-maximal partial $\widehat{G}_i$-parking function is 0 since vertex l_i can take on the values 0 and 1, so the summation in the second product is equal to $I_{\widehat{G}_i}(-1)$. Therefore

$$\operatorname{cont}(U) = (-1)^{N-1} \prod_{i \in U} I_{\widehat{G}'_i}(-1) \prod_{i \notin U} I_{\widehat{G}_i}(-1),$$

proving Lemma 2.5. □

To finish the proof of Theorem 2.3, we notice that by Lemma 2.5,

$$\begin{aligned}I_{\widehat{H}}(-1) &= \sum_{\substack{U \subseteq \{1,\ldots,N\} \\ |U| \text{ even}}} \operatorname{cont} U \\ &= (-1)^{N-1} \sum_{\substack{U \subseteq \{1,\ldots,N\} \\ |U| \text{ even}}} \prod_{i \notin U} I_{\widehat{G}_i}(-1) \prod_{i \in U} I_{\widehat{G}'_i}(-1).\end{aligned}$$ □

Figure 2 illustrates Theorem 2.3 for $N = 5$. Let H be the first graph. We can decompose $I_{\widehat{H}}(-1)$ into a sum of products, one of which is represented by the graphs below the line. Because G'_1 and G'_3 are used instead of G_1 and G_3, this product corresponds to $U = \{1, 3\}$. Summing all such products for any U with $|U|$ even yields $I_{\widehat{H}}(-1)$.

Because any vertex in a tree can be used to decompose the tree by Theorem 2.3, $I_{\widehat{T}}$ where

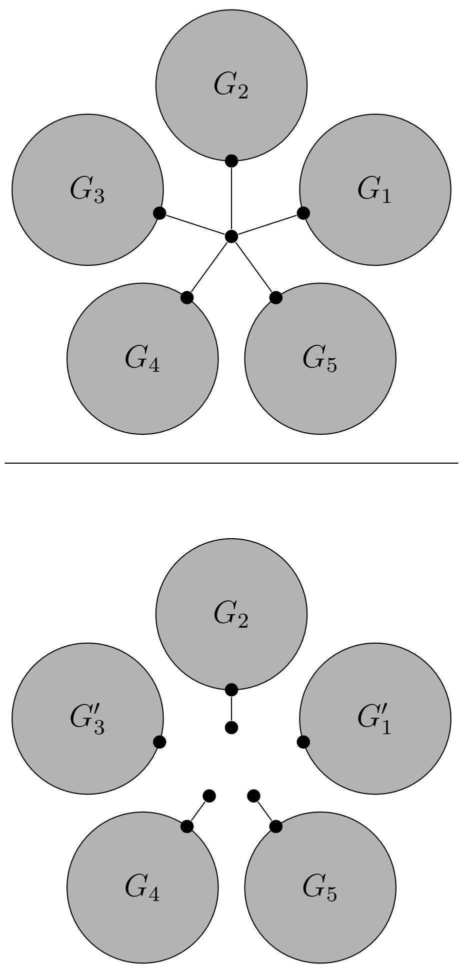

Figure 2: Example application of Theorem 2.3. If H is the top graph, then $I_{\widehat{H}}(-1)$ is a sum of products, one of which is shown below the line.

T is a tree can be expressed in terms of the same expression for smaller trees. We notice that $I_{\widehat{A}}(-1) = I_{\widehat{B}}(-1) = -1$, where A is the graph with a single vertex and B is the graph with two vertices and an edge between them. Using only Theorem 2.3, we can recursively find $I_{\widehat{T}}(-1)$ for any tree T from these two base graphs. Note that $I_{\widehat{T}}(-1)$ is always negative.

We consider a third graph decomposition of a different class of graphs. Theorem 2.7 can reduce any graph with a leaf l, such that removing l and its neighboring vertex leaves multiple connected components.

Definition 2.6. Suppose $G_1, \ldots, G_N$ are graphs, and that l_i is a leaf of G_i connected to v_i for each i. Let $\uparrow_{i\in\{1,\ldots,N\}} G_i$ be the graph formed by merging all l_i and all v_i in $\bigcup_{i\in\{1,\ldots,N\}} G_i$ into vertices l and v, respectively. See Figure 3 for an example.

At first glance, this may seem like a special case of Theorem 2.3. However, we are now allowing multiple edges from v to the same subgraph, whereas Theorem 2.3 only allowed a single edge from the center vertex to each subgraph. Like Lemma 2.1, this theorem can be proved for $N = 2$ and generalized by induction. However, we instead present a direct proof of the general version.

Theorem 2.7. *Let $H = \uparrow_{i\in\{1,\ldots,N\}} G_i$. Then*

$$I_{\widehat{H}}(-1) = (-1)^{N-1} \prod_{i\in\{1,\ldots,N\}} I_{\widehat{G}_i}(-1).$$

Proof. We assume that l and v are the first and second vertices of H, and l_i and v_i are the first and second vertices of G_i, respectively. We also assume that H has n vertices and G_i has n_i vertices. As in the proof of Theorem 2.3, we need to define *partial $\widehat{G}$-parking function.* Note that these definitions are slightly different from those in the proof of Theorem 2.3; we now assign values to all but *two* vertices.

Definition 2.8. A *partial $\widehat{G}$-parking function* is a restriction of a $\widehat{G}$-parking function to the vertics of G except for the first *two*. Let $\mathcal{P}^*_{\widehat{G}}$ be the set of partial $\widehat{G}$-parking functions. Let $\max_G(\vec{a}^*)$ be the maximum natural number k such that $(0,k) \oplus \vec{a}^*$ is a $\widehat{G}$-parking function. Let $\mathrm{cont}_G(\vec{a}^*)$ be the total contribution of $\widehat{G}$-parking functions ending in $\vec{a}^*$ to $I_{\widehat{G}}(-1)$.

For any $i < \max_H(\vec{a}^*)$, both $(0,i) \oplus \vec{a}^*$ and $(1,i) \oplus \vec{a}^*$ are $\widehat{H}$-parking functions. Since the sums of these sequences differ by 1, their contributions to $I_{\widehat{H}}(-1)$ cancel. However, $(1, \max_H(\vec{a}^*)) \oplus \vec{a}^*$ is not an $\widehat{H}$-parking function, so $\mathrm{cont}_H(\vec{a}^*) = (-1)^{\sum \vec{a}^* + \max_H(\vec{a}^*) - n}$. Similarly, $\mathrm{cont}_{G_i}(\vec{a_i}^*) = (-1)^{\sum \vec{a_i}^* + \max_{G_i}(\vec{a_i}^*) - n_i}$. Each partial $\widehat{H}$-parking function $\vec{a}^*$ is a concatenation of partial $\widehat{G}_i$-parking functions. In particular, this provides a bijection $\mathcal{F}$ from $\mathcal{P}^*_{\widehat{G}_1} \times \cdots \times \mathcal{P}^*_{\widehat{G}_N}$ to $\mathcal{P}^*_{\widehat{H}}$. Also, $\max_H(\mathcal{F}(\vec{a_i}^*, \ldots, \vec{a_N}^*)) = \max_{G_i}(\vec{a_i}^*) + \cdots + \max_{G_N}(\vec{a_N}^*) - N + 1$. Hence

$$\mathrm{cont}_H(\mathcal{F}(\vec{a_1}^*, \ldots, \vec{a_N}^*)) = (-1)^{N-1} \prod_{i \in \{1,\ldots,N\}} \mathrm{cont}_{G_i}(\vec{a_i}^*).$$

Summing over all partial $\widehat{H}$-parking functions,

$$\begin{aligned} I_{\widehat{H}}(-1) &= \sum_{\vec{a}^* \in \mathcal{P}^*_{\widehat{H}}} \mathrm{cont}_H(\vec{a}^*) \\ &= (-1)^{N-1} \prod_{i \in \{1,\ldots,N\}} \sum_{\vec{a_i}^* \in \mathcal{P}^*_{\widehat{G}_i}} \mathrm{cont}_{G_i}(\vec{a_i}^*) \\ &= (-1)^{N-1} \prod_{i \in \{1,\ldots,N\}} I_{\widehat{G}_i}(-1). \end{aligned}$$ □

Figure 3 illustrates Theorem 2.7 when $N = 2$. Let the graph above the line be H. Then $I_{\widehat{H}}(-1) = I_{G_1}(-1) I_{\widehat{G}_2}(-1)$. Figures 2 and 3 illustrate graphically why we use the symbols $*$ and $\uparrow$ for the graphs in question; The symbols look like the graphs they represent.

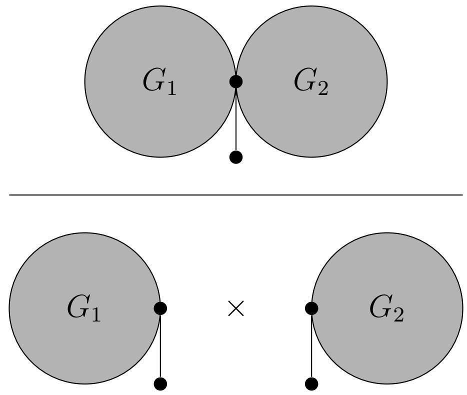

Figure 3: Example application of Theorem 2.7. The first graph can be split into the product of the two graphs underneath.

3 Partial Orientations

We now explore the connection between partial orientations of G and $I_{\widehat{G}}(-1)$. First, we define partial orientation.

Definition 3.1. Let G be an undirected graph. A *partial orientation* of G is an assignment of directions to some subset of the edges of G. Given a partial orientation of G, the *in-degree* of a vertex v is the number of edges oriented to point towards v.

Backman and Hopkins [5] studied the $\widehat{G}$-parking functions and their relation to partial orientations, proving for example that the number of $\widehat{G}$-parking functions of a graph is the number of acyclic partial orientations of G.

For reasons that will become apparent in Section 4, we are interested in counting partial orientations for which a specific set of vertices has even in-degree, and all others have odd in-degree.

Definition 3.2. Let U be a subset of the vertices of G. A partial orientation of G is *U-even* if the vertices in U have even in-degree and the vertices in $G \backslash U$ have odd in-degree. Let $\text{even}_G(U)$ denote the number of U-even partial orientations of G.

In this section, we are interested in G-even partial orientations, which give every vertex even in-degree. We use the shorthand $\text{even}(G)$ for $\text{even}_G(G)$ In Section 4, we generalize some of these results using U-even partial orientations.

We show that $\text{even}(T) = -I_{\widehat{T}}(-1)$ for any tree T. It suffices to show that the $\text{even}(G)$ obeys the same recurrences and base cases as $I_{\widehat{G}}(-1)$. In fact, an analogue of Theorem 2.3 alone is sufficient, but we will also prove an analogue of Theorem 2.7.

Lemma 3.3. *Let* $H = \ast_{i\in\{1,\ldots,N\}} G_i$. *Then*

$$\text{even}(H) = \sum_{\substack{U\subseteq\{1,\ldots,N\}\\ |U|\ \textit{even}}} \prod_{i\notin U} \text{even}(G_i) \prod_{i\in U} \text{even}(G_i').$$

Proof. We count $\text{even}(H)$. Because l, the center vertex, must have even in-degree, let U be the set of $v \in \{v_1, \ldots, v_N\}$ such that the edge from l to v is oriented to point to l. We sum over all such U with $|U|$ even.

Consider the following cases:

1. $v_i \in U$. The number of ways to partially orient the rest of G_i is $\text{even}(G_i')$.

2. $v_i \notin U$. The number of ways to partially orient G_i is $\text{even}(G_i)$.

For a fixed U, the number of ways to finish our partial orientation is the product of the number of ways to partially orient each G_i, i.e.

$$\prod_{i\notin U} \text{even}(G_i) \prod_{i\in U} \text{even}(G_i').$$

Summing over all U with $|U|$ even, we find

$$\text{even}(H) = \sum_{\substack{U\subseteq\{1,\ldots,N\}\\ |U|\ \text{even}}} \prod_{i\notin U} \text{even}(G_i) \prod_{i\in U} \text{even}(G_i').$$

□

Lemma 3.3 is the equivalent of Theorem 2.3 for partial orientations. Using Lemma 3.3 and Theorem 2.3, we prove Theorem 3.4, describing $I_{\widehat{T}}(-1)$ for any tree T.

Theorem 3.4. *Let T be a tree. Then* $\text{even}(T) = -I_{\widehat{T}}(-1)$.

Proof. Let A and B be the graph with a single vertex and the graph with two vertices connected by an edge, respectively. Then $\text{even}(A) = \text{even}(B) = -I_{\widehat{A}}(-1) = -I_{\widehat{B}}(-1) = 1$. It is straightforward to check that combining graphs with $*$ preserves the equality between $\text{even}(G)$ and $-I_{\widehat{G}}(-1)$. Since any tree can be built out of A and B using the $*$ operation, by induction $\text{even}(T) = -I_{\widehat{T}}(-1)$. □

Note that Theorem 3.4 does not hold in general for non-trees. We also prove an equivalent of Theorem 2.7 for partial orientations.

Lemma 3.5. *Let $H = \uparrow_{i\in\{1,\dots,N\}} G_i$. Then*

$$\text{even}(H) = \prod_{i\in\{1,\dots,N\}} \text{even}(G_i).$$

Proof. Consider a partial orientation of each G_i. We can combine these partial orientations into one of H by straightforward union, except we leave the edge between v and l unoriented for now. If each vertex in each G_i had even in-degree before, they still do, except for vertex v. To deal with v, notice that there is exactly one way to orient the edge between v and l so that both v and l have even in-degree. Orienting the edge this way creates a partial orientation of H with even in-degrees. This describes a bijection between the partial orientations of H with even in-degrees and the partial orientations of each G_i with even in-degrees. Therefore $\text{even}(H) = \prod_{i\in\{1,\dots,N\}} \text{even}(G_i)$. □

4 More General Graphs

In this section we consider graphs that do not always have exactly one edge from any vertex to 0. With identical proofs, the results of Section 2 hold for general graphs as long as vertices discussed in the proofs have edges to 0. Plautz and Calderer [6] proved that

$$T_G(1, y) = \sum_{(a_1,\dots,a_n)\in\mathcal{P}_G} y^{|E|-|V|+1-a_1-\dots-a_n},$$

where T_G is the Tutte Polynomial of G and $|E|$ and $|V|$ are the numbers of edges and vertices in G, respectively, so $|V| = n+1$. This is already remarkably similar to the sum enumerator. At $y = -1$, we find

$$\begin{aligned} T_G(1, -1) &= \sum_{(a_1,\dots,a_n)\in\mathcal{P}_G} (-1)^{|E|+a_1+\dots+a_n-n} \\ &= (-1)^{|E|} I_G(-1). \end{aligned}$$

Equivalently, $I_G(-1) = (-1)^{|E|} T_G(1, -1)$

Notice that $(-1)^{|E|}$ and $T_G(1, -1)$ are invariant to relabellings of the vertices of G. In particular, we can designate a different vertex to be 0, and these expressions remain the same. Therefore $I_G(-1)$ is invariant to our choice of vertex 0.

Another implication of this connection to the Tutte Polynomial is that $I_G(-1)$ obeys the *deletion-contraction recurrence*. For an edge e of G, let $G\backslash e$ denote G with e deleted, and let G/e denote G with e contracted, merging the vertices on e into a single vertex. Then $T_G = T_{G/e} + T_{G\backslash e}$ [5]. The $(-1)^{|E|}$ factor in $I_G(-1)$ gives

$$-I_G(-1) = I_{G/e}(-1) + I_{G\backslash e}(-1).$$

Figure 4 is an illustration of this, with relevant vertices labeled by the number of edges to 0.

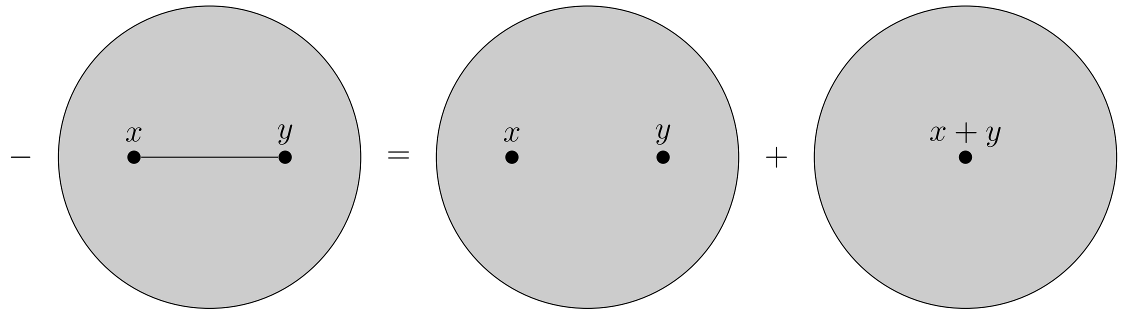

Figure 4: Example application of the deletion-contraction recurrence for $I_G(-1)$. If the first graph is G, the second and third graphs are $G\backslash e$ and G/e, respectively. Vertex labels indicate the number of edges to vertex 0.

We show that some features in G can be removed without affecting $I_G(-1)$.

Lemma 4.1. *1. Suppose G is a graph with a loop, i.e. an edge from v to v. Let G' be G with the loop removed. Then $I_G(-1) = I_{G'}(-1)$.*

2. Suppose G is a graph with two edges between u and v. Let G' be G with both of these edges removed. Then $I_G(-1) = I_{G'}(-1)$.

Proof. 1. The presence or absence of a loop does not change the set of parking functions of G, so it does not change $I_G(-1)$. Therefore $I_G(-1) = I_{G'}(-1)$.

2. Assign u to be vertex 0 and v to be vertex 1. Let $\vec{a}^*$ be a partial G-parking function, as defined in the proof of Theorem 2.3. Then $(k) \oplus \vec{a}^*$ is a G'-parking function if and only if $(k+2) \oplus \vec{a}^*$ is a G-parking function. Exactly two G-parking functions ending in $\vec{a}^*$ are not G'-parking functions, and these two sequences have sums of different parity, so they cancel in $I_G(-1)$. Hence $I_G(-1) = I_{G'}(-1)$. □

If there are multiple edges between two vertices in G, we can remove them in pairs until only 0 or 1 edges remain. While using deletion-contraction, we often end up with graphs with double edges and loops, which we can ignore.

Definition 4.2. For a graph G, the *partial cone* over G at $U \subseteq G$ is the graph formed by adding a vertex (usually 0) to G and connecting the vertices in U to the new vertex. The partial cone over G at G is the ordinary cone over G.

In the previous sections, we only dealt with ordinary cones, and now we are interested in partial cones at arbitrary sets of vertices. We prove a generalization of Theorem 3.4, where only some vertices have edges to 0. We use G_0 to denote the set of vertices of G with an edge to 0.

Theorem 4.3. *Let G be a partial cone over a tree T on $\{1, \ldots, n\}$. Then*

$$\text{even}_T(G_0) = (-1)^{|G \setminus U|} I_G(-1).$$

It is possible to prove Theorem 4.3 by generalizing Lemma 3.3. It is easier to use the deletion-contraction recurrence, which is what we do here.

Proof. We show that $\text{even}_T(G_0)$ obeys the deletion-contraction recurrence. Pick an edge e, and partition the G_0-even partial orientations of T into two sets: those that orient e and those that do not.

Consider first the G_0-even partial orientations that do not orient e. These partial orientations are in bijection with the G_0-even partial orientations of $T \setminus e$, because the edges other than e have to satisfy G_0-evenness. There are $\text{even}_{T/e}(G_0)$ of such partial orientations, accounting for the deletion part.

Now consider G_0-even partial orientations of T that orient e. We show that these are in bijection with $G \setminus e_0$-even partial orientations of T/e. Here the merged vertex is in $G \setminus e_0$ if it has exactly one edge to 0. Contracting e in a G_0-even partial orientation of T that orients e creates a $G \setminus e_0$-even partial orientation of T/e. Each $G \setminus e_0$-even partial orientation of T/e is created from exactly one G_0-even partial orientation of T, since there is exactly one way to orient e such that the in-degrees of its endpoints have the correct parity. Therefore there are $\text{even}_{T/e}(G\backslash e_0)$ G_0-even partial orientations of T, accounting for the contraction part.

Hence $\text{even}_T(G_0)$ obeys the same recurrence as $I_G(-1)$, at least up to sign. To account for sign, notice that $\text{even}_T(G_0)$ is always nonnegative, and the sign of $I_G(-1)$ is $(-1)^{|E|}$. Since T is a tree, it has $n-1$ edges, and there are $|U| = n+1-|G\backslash U|$ edges to 0. Thus

$$\begin{aligned}\text{even}_T(G_0) &= (-1)^{|E|} I_G(-1) \\ &= (-1)^{n-1+n+1-|G\backslash U|} I_G(-1) \\ &= (-1)^{|G\backslash U|} I_G(-1).\end{aligned}$$

□

Theorem 4.3 does not hold for general graphs, although $I_G(-1)$ and $\text{even}_T(G_0)$ obey the same recurrence. This is because loops and double edges increase the number of partial orientations, and thus $\text{even}_T(G_0)$, but not $I_G(-1)$. For example, let G be the graph with two loops at vertex 1, and an edge between 0 and 1. The only G-parking function is (0), but there are five partial orientations such that 1 has even in-degree.

5 Conclusion

We found ways to calculate $I_{\widehat{G}}(-1)$ from subgraphs of G whenever G is disconnected, centered around a star with separated components, or has a leaf that yields a disconnected graph when removed along with the vertex to which it has an edge. The recurrence for graphs centered around stars provided a method to find $I_{\widehat{T}}(-1)$ when T is a tree by repeatedly decomposing T into its subgraphs. We found that, when T is a tree, $I_{\widehat{T}}(-1)$ is the number of partial orientations of T with all in-degrees even. We generalized this fact to partial cones over a tree, counting partial orientations such that exactly the vertices connected to 0 have even in-degrees. Because $I_G(-1)$ is closely related to $T_G(1,-1)$, we found a deletion-contraction recurrence for $I_G(-1)$, and also connected this to the number of partial orientations, especially those of trees with specific vertices having even in-degree.

6 Acknowledgments

I would like to thank Dr. Alex Postnikov for providing this problem, and Mr. Pavel Galashin for direction throughout my research. I would like to thank Dr. John Rickert, Dr. Tanya Khovanova, and Noah Golowich for their assistance in writing this paper, and the MIT Mathematics Department coordinators, Prof. David Jerison and Prof. Ankur Moitra, for organizing the math section of RSI. Additionally, I would like to thank Jared Tramontano for a helpful insight into the problem that ultimately led to Theorem 2.7.

I would also like to thank the Center for Excellence in Education and the Massachusetts Institute of Technology for supporting this research and providing me with the necessary resources. I would like to thank Dr. Diane Tang, Ms. Elizabeth Williams, Ms. Christie Nelson, and Dr. and Mrs. William Cordwell for sponsoring my research and RSI.

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