NP-Hard Triangle Packing Problems

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Abstract

In computational geometry, packing problems ask whether a set of rigid pieces can be placed inside a target region such that no two pieces overlap. The triangle packing problem is a packing problem that involves triangular pieces, and it is crucial for algorithm design in many areas, including origami design, cutting industries, and warehousing. Previous works in packing algorithms have conjectured that triangle packing is NP-hard. In this paper, we mathematically prove this conjecture. We prove the NP-hardness of three distinct triangle packing problems: (i) packing right triangles into a rectangle, (ii) packing right triangles into a right triangle, and (iii) packing equilateral triangles into an equilateral triangle. We construct novel reductions from the known NP-complete problems 3-partition and 4-partition. Furthermore, we generalize that packing arbitrary triangles into an arbitrary target region is strongly NP-hard. Because triangle packing is NP-hard, triangle packing must be determined by approximation or heuristic algorithms rather than exact algorithms.

1 Introduction

In a packing problem, we wish to determine whether a set of objects can be placed into a container such that no two objects overlap. These problems have been studied extensively and are motivated by a number of applications, including warehousing, origami design, newspaper paging, and cutting industries. In wood, glass, or steel industries, for example, packing is involved when determining how to cut pieces from large sheets of material.

Complexity theory involves the classification of decision problems — problems where the answer is either yes or no — by computational hardness. Tractable problems are problems that can be solved with efficient algorithms, or algorithms that require a number of resources that grows as a polynomial function of the problem instance; intractable problems are ones that cannot be solved with efficient algorithms. **P** is the class of decision problems with efficient algorithms for finding solutions. **NP** is the class of problems with efficient algorithms for verifying solutions. Problems that are at least as hard as the hardest problem in NP are **NP-hard**. Unless all problems that are efficiently verifiable are also efficiently solvable, all NP-hard problems are intractable. Hence, if a problem can be proven NP-hard, programmers would know to look for approximate or case-by-case solutions rather than attempt to find an efficient exact algorithm.

Many packing problems are NP-hard. Common one-dimensional versions, such as the Knapsack problem, are NP-hard. Several two-dimensional geometric packing problems have been proven NP-hard, including packing squares into squares [7], packing circles into equilateral triangles or squares [4], and packing identical simple polygons into a larger polygon [1]. Nevertheless, there do exist two-dimensional packing problems for which a polynomial time exact algorithm has been found. The guillotine pallet loading problem is closely related to rectangle packing but involves cutting patterns. Tarnowski proposed a polynomial time algorithm for the guillotine pallet loading problem [8].

We study the computational complexity of triangle packing. Triangle packing problems in general ask this: Can a set of given triangular pieces be placed inside a given target region such that no two pieces overlap? Chen and He [2] were the first to conjecture that "triangle packing problem is a special case of polygon packing problem and also NP-hard." Wang et. al [9] and Chen et. al [3] later made similar statements. The previous literature does not provide proof for the NP-hardness of triangle packing other the mention that triangle packing is a special case of polygon packing, an NP-hard problem. However, the reasoning that triangle packing is NP-hard because polygon packing is NP-hard would be conceptually flawed. Because triangle packing is a special case of polygon packing, it follows that polygon packing is at least as hard as triangle packing; the fact that polygon packing is NP-hard is not sufficient to conclude that triangle packing is NP-hard. Thus, the claim that triangle packing is NP-hard requires non-trivial proof.

We prove the NP-hardness of three important triangle packing problems: (i) packing right triangles into a rectangle, (ii) packing right triangles into a right triangle, and (iii) packing equilateral triangles into an equilateral triangle. Right triangle packing is utilized in cutting industries and origami engineering, and equilateral triangle packing has been a topic of interest since Friedman's and Morandi's work [5]. These problems are special cases of the general triangle packing problem, which asks whether arbitrary triangular pieces can be placed into an arbitrary triangular region. Hence, we may generalize from our NP-hardness results that triangle packing is NP-hard. We conclude that approximation or heuristic algorithms are necessary to determine triangle packing.

Most noteworthy of our triangle packing problems is equilateral triangle packing. This problem has been previously studied in the form of an optimization problem. Friedman and Morandi [5] studied the unit equilateral triangle packing problem, which asks: Given a set of k unit equilateral triangles, what is the minimum side length s of an enclosing equilateral triangle? Non-trivial cases include $k = 5$, for which which s has been proven by Friedman in 1997 to be $1 + \sqrt{3} \approx 2.732$, and $k = 6$, for which s has been found but not proven to be $13/8 + 3\sqrt{13}/8 \approx 2.977$. Though reasonable bounds have been found for cases up to $k = 30$, most non-trivial cases have not been proven or are difficult to find. We prove that the problem of packing equilateral triangles into equilateral triangles is NP-hard. Because equilateral triangles scale in two dimensions, it is particularly difficult to construct formations that constrain equilateral packing. Thus, our NP-hardness proof of equilateral triangle packing utilizes novel gadgets and requires techniques different from previous proofs in packing other objects.

In Section 2, we discuss the methods and concepts with which we prove NP-hardness. In Section 3, we prove that packing right triangles into a rectangle is NP-hard. In Section 4, we prove that packing right triangles into a right triangle is NP-hard. In Section 5, we show that packing equilateral triangles into an equilateral triangle is NP-hard. Finally, in Section 6, we summarize our work and present directions for further research.

2 Definitions and Concepts

Reductions allow us to classify the hardness of problems relative to other problems. We prove the NP-hardness of triangle packing problems using reductions.

Definition 2.1. A polynomial reduction from problem Π_A to problem Π_B is a function $f(x)$ that transforms in polynomial time an instance x_A of problem Π_A to an instance $x_B = f(x_A)$ of problem Π_B with polynomial size such that x_A is a "yes" instance of Π_A if and only if x_B is a "yes" instance of Π_B. We say that Π_B is *as hard as* Π_A, and we write $\Pi_A \leq_p \Pi_B$.

Suppose we wish to show that a problem Π_B is NP-hard. We do this in three steps: (1) Select a known NP-hard problem Π_A; (2) Construct a reduction from Π_A to Π_B. Prove that the reduction is a polynomial transformation; (3) Show that x_A is a "yes" instance of Π_A if and only if x_B is a "yes" instance of Π_B. Likewise, to show that a problem Π_A is easy, we reduce from a problem Π_A to a known tractable problem Π_B. We then know that problem Π_A can be solved in polynomial time by first converting to problem Π_B and then solving with the algorithm for Π_B.

In our packing proofs, we reduce from the NP-hard problems 3-partition and 4-partition.

Definition 2.2. 3-Partition. Given a set A of $3m$ integers with sum mt and a bound on each elements $a_i \in A$ such that $t/4 < a_i < t/2$, a *3-partition* is a partitioning of A into m disjoint subsets $A_1, A_2, \ldots, A_m$ such that the sum of the elements in each A_i is equal to t.

Garey and Johnson [6] proved that 3-partition is *strongly* NP-hard. A strongly NP-hard problem remains hard even when its parameters are bounded by a polynomial function, rather than exponential function, of the input size. To show that a problem is strongly NP-hard, we reduce from a known strongly NP-hard problem.

We also reduce from 4-partition, which is analogous to 3-partition but forms m quadruples of the same sum from a set of $4m$ integers, and each element $a_i \in A$ is bounded by $t/5 < a_i < t/3$. The 4-partition problem is known to be strongly NP-hard by a fairly standard reduction from 3-partition.

3 Packing Right Triangles into a Rectangle

We prove that packing right triangles into a rectangle is strongly NP-hard.

Theorem 3.1. *It is strongly NP-hard to decide whether n specified right triangular pieces can be packed into a rectangular region.*

Proof. The NP-hardness proof is a reduction from 3-partition. Suppose we are given a set $A = \{a_1, a_2, \ldots, a_{3m}\}$. Let $\sigma = \sum_{i=1}^{3m} a_i$ be the sum of all elements in A and $t = \frac{\sigma}{m}$ be the target sum. We construct the set of triangular pieces and a target rectangular region as follows: for each integer a_i in A, create two congruent right triangles with legs of lengths 1 and $a_i + 24m + \sigma$, and call these $6m$ pieces the *slim pieces*. Let the target region be an $m \times (t + 72m + 3\sigma)$ rectangle. This transformation from an instance of 3-partition to right triangle packing is shown in Figure 1. Because all quantities are polynomial functions of m and t, the reduction is a polynomial transformation.

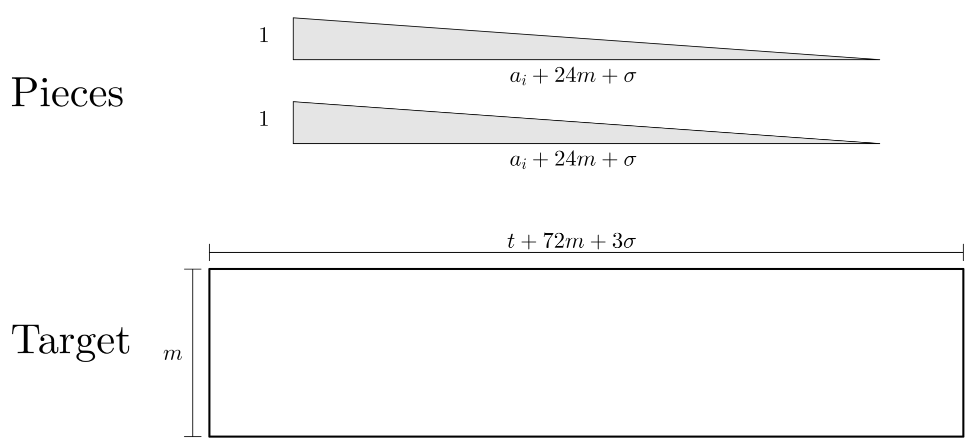


Figure 1: Reduction from 3-partition to packing right triangles into a rectangle.

We show that if there exists a 3-partition of set A, then there exists a way to pack the triangles. For each pair of congruent slim triangles, we place them together to a $1 \times (a_i + 24m + \sigma)$ rectangular unit. For every triple (a_j, a_k, a_l) of the 3-partition that sums to t, we fill a row of the target rectangle with the corresponding $1 \times (a_j + 24m + \sigma)$, $1 \times (a_k + 24m + \sigma)$, and $1 \times (a_l + 24m + \sigma)$ rectangular units. Each row fills length $(a_j + 24m + \sigma) + (a_k + 24m + \sigma) + (a_l + 24m + \sigma) = t + 72m + 3\sigma$, which is exactly the length of the target rectangle. There are m triples, so we can fill all m rows of the box given a 3-partition assignment, as shown in Figure 2. Note that the packing is an exact packing because the total area of the triangular pieces equals the area of the target rectangle.

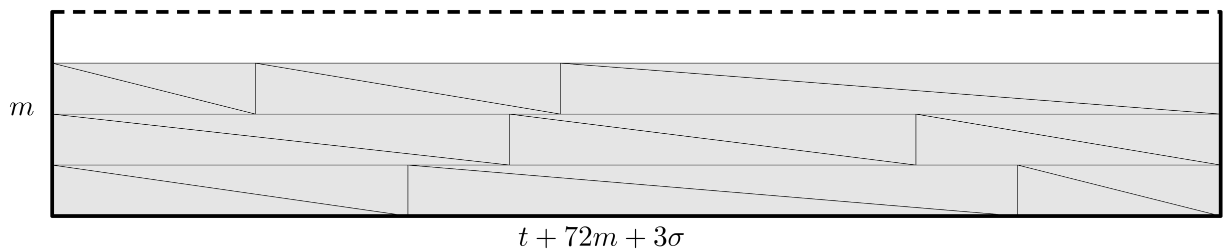


Figure 2: Existence of packing given existence of 3-partition

We now show if there exists a packing, then there exists a 3-partition of set A. Consider the leftmost available $90°$ corner at any step in the packing. We show that this corner must be exactly packed by a rectangular unit formed by two slim triangles. Suppose a slim triangle placed in this right angle corner has dimensions $1 \times (a_i + 24m + \sigma)$. This triangle is placed on one of its three sides. The slim triangle cannot be placed vertically on its side of length 1 because the other leg of the triangle has length $a_i + 24m + \sigma$, which is greater than the height m of the rectangular region. Because the triangle cannot be placed vertically, it must be placed on its longer leg or on its hypotenuse. The only four possible configurations of the placement are shown in Figure 3.

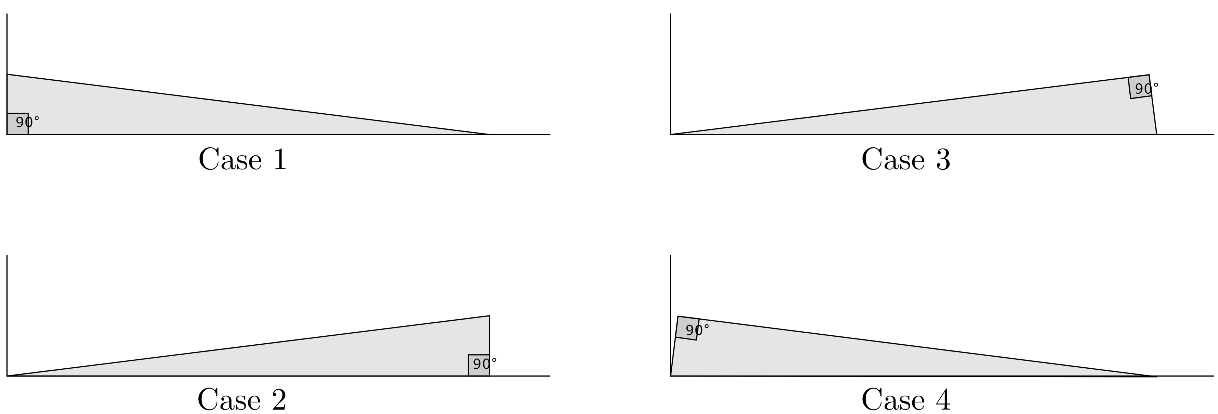


Figure 3: Four ways to place a slim triangle in $90°$ corner

The hypotenuse of this triangle has length $\sqrt{1 + (a_i + 24m + \sigma)^2}$. Let x be the fractional part of the length of the hypotenuse. We have

$$(a_i + 24m + \sigma) + x = \sqrt{1 + (a_i + 24m + \sigma)^2}$$

$$\implies (a_i + 24m + \sigma)^2 + 2(a_i + 24m + \sigma)x + x^2 = 1 + (a_i + 24m + \sigma)^2$$

$$\implies 2(a_i + 24m + \sigma)x + x^2 = 1$$

$$\implies 2(a_i + 24m + \sigma)x < 1$$

$$\implies x < \frac{1}{2(a_i + 24m + \sigma)} < \frac{1}{2(24m + \sigma)}.$$

There are $6m$ triangles, which have a total fractional part of less than $\frac{6m}{2(24m+\sigma)} < 1$. Hence, the lengths of hypotenuses cannot sum to an integer. The length of the target rectangle is an integer, so the length of the rectangle cannot be partitioned by the length of a hypotenuse. Thus, the only two possible configurations are Cases 1 and 2 in Figure 3.

In both cases, there must be at least one triangle that lies along the hypotenuse of the bottom triangle. No triangle may be placed on its side of length 1 along a hypotenuse because the slope of the hypotenuse is close to 0. Suppose two triangles are placed along the hypotenuse of the bottom triangle, as shown in Figure 4. Then their combined length is at least $a_k + a_j + 48m + 2\sigma$ for some a_k and a_j. However, the length of the bottom triangle's hypotenuse is $\sqrt{1 + (a_i + 24m + \sigma)^2} < a_i + 24m + \sigma + 1 < a_k + a_j + 48m + 2\sigma$. Hence, we can place only one triangle on top of the bottom one. This means that the top triangle must fully cover the bottom triangle; the top and bottom triangles must be congruent.

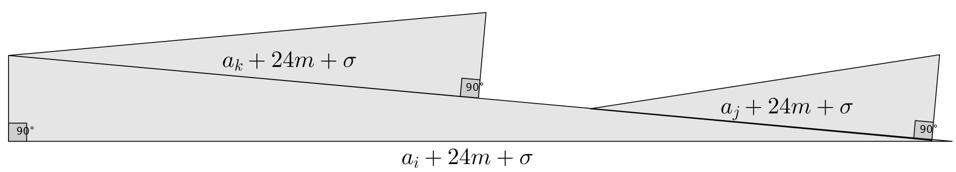


Figure 4: It is not possible in a packing for two triangles to be placed completely on top of one triangle.

There are two possible ways to place a congruent triangle on top, as shown in Figure 5. In Case 1, the triangles are placed to form a rectangle, and in Case 2, the triangles are placed to form a kite.

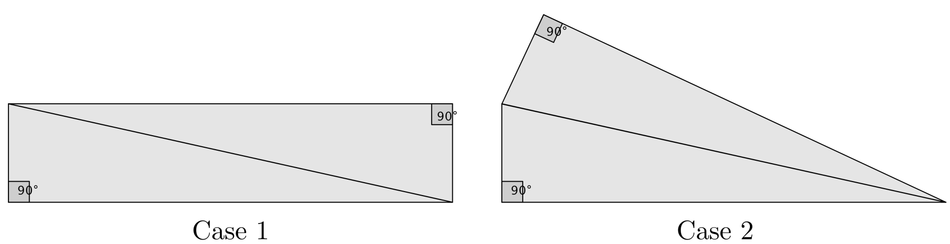


Figure 5: Two ways to place congruent triangles on top of each other

Because we chose the leftmost 90° angle, the kite in Case 2 is either packed against the left wall or in a nub and facing either left or right, as shown in Figure 6. If the kite is packed against the left wall and facing left, then a slim triangle must be placed vertically in the angle formed by the wall and the kite. The vertical placement causes the triangle to protrude out of the boundary, so this case is not possible.

If the kite is packed facing right, either against the wall or in a nub, then the angle in the left corner cannot be packed by slim triangles. Let θ_i denote the smallest angle of a $1 \times (a_i + 24m + \sigma)$ triangle. As we show independently in Section 4, the θ_i angles are too small to exactly pack a $90°$ corner. Furthermore, we can construct the slim triangular pieces such that no $90° - \theta_j$ angle is small enough to pack a $90° - 2\theta_i$ corner. If the pieces are not already scaled such that all $90° - 2\theta_i < 90° - \theta_j$ for all θ_i, θ_j, we can lengthen each long leg of a slim triangle by a large constant B such that all θ are arbitrarily close and adjust the length of the target rectangle by $3B$. Then $2\theta_i > \theta_j \implies 90° - 2\theta_i < 90° - \theta_j$ for all θ_i, θ_j. Hence, the against wall (right) and in nub (left) configurations are not possible. Finally, if the kite is placed in a nub facing left, then its top piece would overlap with a rectangle from the top. Thus, Case 2 is also not possible.

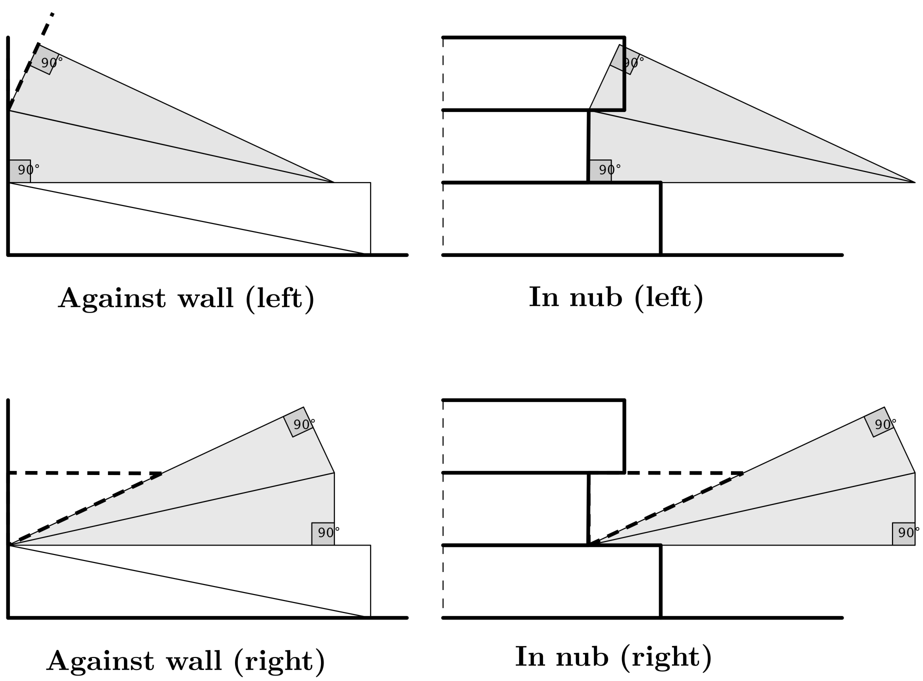


Figure 6: Four ways to place congruent triangles according to Case 2.

Because each $a_i \in A$ is bounded by $t/4 < a_i < t/2$, any filled row must contain exactly three rectangular units. The packing is exact, so the lengths of the rectangular units in each row determine a 3-partition of A.

Hence, there exists a polynomial time conversion of the set A into a specific instance of the packing problem such that the packing problem is solvable precisely if the set A has a 3-partition. This concludes our NP-hardness proof of packing right triangles into rectangles. $\square$

4 Packing Right Triangles into a Right Triangle

We extend Theorem 3.1 for packing right triangles into a right triangle.

Theorem 4.1. *It is strongly NP-hard to decide whether n specified right triangular pieces can be packed into a right triangular region.*

Proof. We reduce from 3-partition. Given an instance of 3-partition, we construct $6m + 2$ pieces pieces for packing: one isosceles right triangle with legs of length $t + 72m + 3\sigma$, one isosceles right triangle with legs of length m, and $6m$ slim pieces corresponding to the 3-partition set integers as constructed in the proof of Theorem 3.1. Let the target right triangle be an isosceles right triangle with legs of length $t + 73m + 3\sigma$. Any packing must be an exact packing because the area of the target triangle is equal to the sum of areas of the $6m + 2$ packing pieces. Similar to before, this transformation is a polynomial transformation.

In any packing, the two isosceles right triangle pieces must each be in one of the $45°$ corners of the target triangle (Figure 7). We prove this by showing the slim pieces cannot pack a $45°$ corner. Let θ_i be the smaller acute angle in a slim triangle with dimensions $1 \times (a_i + 24m + \sigma)$. We have $\theta_i = \tan^{-1}\left(\frac{1}{24m+a_i+\sigma}\right) < \tan^{-1}\left(\frac{1}{24m}\right)$. Because $\tan^{-1}(n) < n$ for all $n > 0$, where n is in radians, $\tan^{-1}\left(\frac{1}{24m}\right) < \frac{1}{24m} < \frac{\pi}{24m}$. Then $\theta_i < \frac{\pi}{24m}$ and the sum of the smallest angles for all slim triangles is less than $6m(\frac{\pi}{24m}) = \frac{\pi}{4}$ radians.

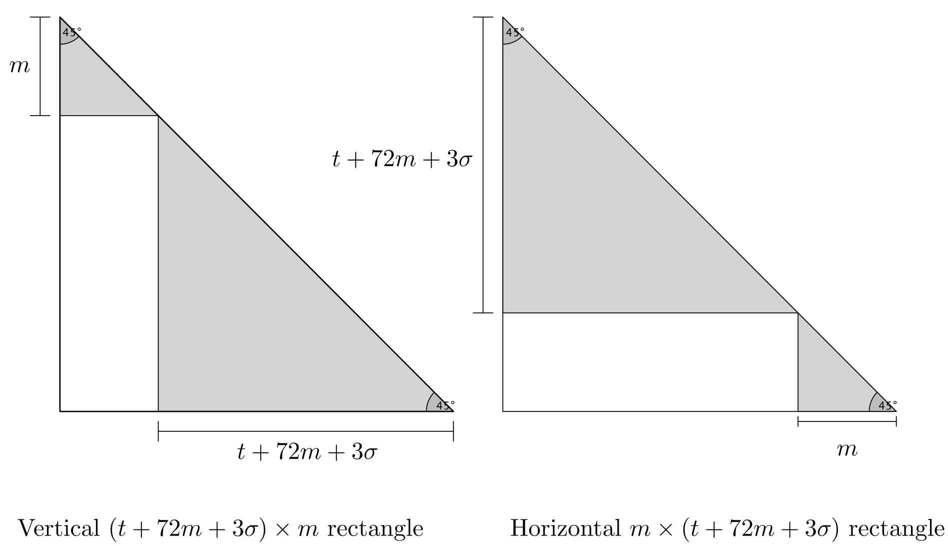


Figure 7: Either placement of the isosceles right triangle packing pieces forms $m \times (t + 72m + 3\sigma)$ rectangle.

Because the sum of all small acute angles is less than $\frac{\pi}{4}$, the small acute angles together cannot exactly pack a $45°$ corner of the isosceles target triangle. Furthermore, neither the large acute angle nor the right angle of a slim piece can be placed at a $45°$ corner because they are greater than $45°$. Thus, the $45°$ corners of the target triangle must be packed with the two isosceles triangle pieces as shown in Figure 7. Once the two isosceles pieces are placed inside the target triangle, there is a leftover $m \times (t + 72m + 3\sigma)$ rectangular region. The remaining $6m$ slim pieces must be packed inside this rectangular box. We have already shown in Theorem 3.1 that these slim pieces can be packed into the rectangle if and only if there exists a 3-partition. Hence, the problem of packing right triangles into a right triangle is also NP-hard. $\square$

5 Packing Equilateral Triangles into an Equilateral Triangle

In this section we prove the NP-hardness of packing equilateral triangular pieces into an equilateral triangular target region. This reduction requires different techniques than those used in the reductions for packing right triangles.

In our previous reductions, we scaled each right triangular piece in exactly one dimension; we constructed the length of each right triangle to match an integer in a three-partition set, while the heights remained constant. We were able to scale in exactly one dimension because right triangles are not constrained to be similar. Hence, the constructed right triangular pieces behaved like one-dimensional objects.

However, because equilateral triangles are constrained to be similar, they must scale in two dimensions. As a result, equilateral pieces behave as two-dimensional objects. Similar to square packing [7], it is much more difficult in equilateral packing than in right triangle packing to construct an exact and determined packing from partition pieces alone. Thus we present a novel reduction in which we construct helper pieces as gadgets to force a determined packing of the pieces.

Theorem 5.1. *It is strongly NP-hard to decide whether n specified equilateral triangles can be packed into a target equilateral triangle.*

Proof. We reduce from 4-partition. Let $A = \{a_1, a_2, \ldots, a_{4m}\}$ be a set of $4m$ positive integers and $t = \frac{\sum a_i}{m}$ be the target sum. Let $B = 4mt^2$. Let the target region be an equilateral triangle with side length $(3B + 2t)2m + (4B + 3t)$.

We define the following equilateral triangular pieces for packing as shown in Table 1. The names of the pieces describe their individual functions. Of the outer triangles, the blocking triangle blocks off most of the target equilateral triangle, the wedge triangles force the blocking triangle into a corner, the container triangles pack such that they create disjoint containers for inner triangles, and the filler triangles fill in these disjoint containers to make them parallelograms. Of the inner triangles, the partition triangles correspond to the integers in a 4-partition, the support triangles fill up the space in between partition triangles and force them against the sides, and the unit triangles are equilateral triangles of side length 1 that fill up the remaining space.

The total number of pieces for packing is bounded by the area of the target region, which is on the order of $B^2 = 16m^2t^4$. Hence, the reduction is a polynomial transformation.

We show that if there exists a 4-partition of A, then there exists a packing. We arrange the outer pieces such that the remaining space consists of m disjoint $(2B + t) \times (B + t)$ parallelograms (Figure 8-a). In each parallelogram, we pack the partition triangles corresponding to the integers in a 4-partition quadruple with the support triangles sandwiched between pairs of partition triangles

	Side Length	Quantity	Category
Partition	$B + a_i$	$\forall a_i \in A$	Inner
Support	t	$2m\,(2B/t)$	Inner
Wedge	$4B + 3t$	$m + 1$	Outer
Container	$3B + 2t$	$3m$	Outer
Filler	$B + t$	m	Outer
Blocking	$(3B + 2t)2m$	1	Outer
Unit	1	remaining space	Inner

Table 1: This table defines the seven types of equilateral triangle pieces for packing. We separate the pieces into two categories, outer and inner pieces, depending on their functions in the packing.

(Figure 8-b). We fill the remaining space with unit triangles. Note that the packing is an exact packing because the total area of the triangular pieces equals the area of the target rectangle.

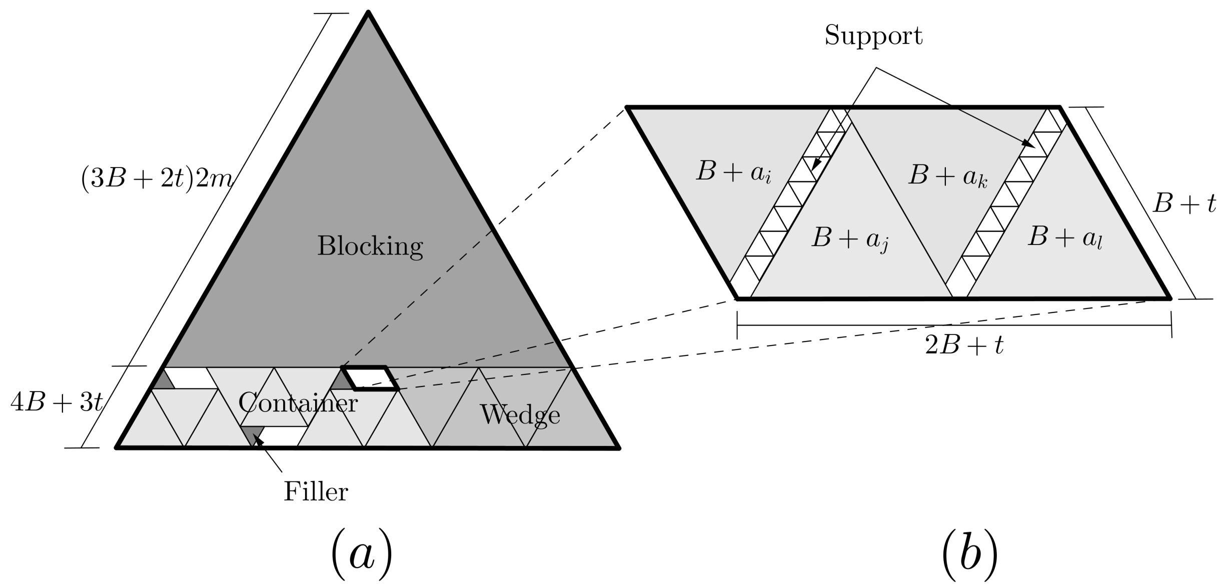


Figure 8: (a) Arrangement of the outer triangles to form m disjoint empty $(2B + t) \times (B + t)$ parallelograms. (b) Arrangement of inner triangles within each parallelogram.

Furthermore, we can show that if there exists a packing of the pieces, then there exists a 4-partition of the set A (Lemma 5.2). The proof of Lemma 5.2 requires many steps and is deferred to Subsection 5.1.

We have constructed a polynomial transformation from 4-partition to equilateral packing. There exists a 4-partition of a set A if and only if there exists a packing of the equilateral triangles. Thus, the problem of packing equilateral triangles is NP-hard. □

5.1 Existence of a 4-Partition

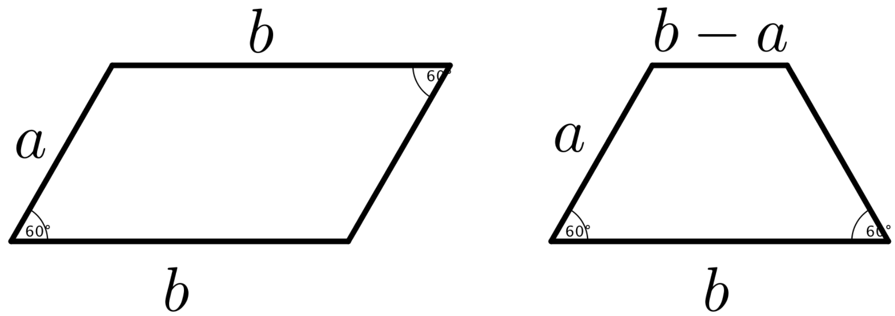
In this subsection, we prove Lemma 5.2, which we used in the proof of Theorem 5.1

Lemma 5.2. *If there exists a packing of the pieces defined by Table 1 into a target equilateral triangle of side length $(3B + 2t)2m + (4B + 3t)$, then there exists a 4-partition of the set A.*

Before we prove this lemma, it is necessary to prove some preliminary results. We first define the terms with which we refer to the packing.

Definition 5.1 ($a \times b$ parallelogram)**.** An $a \times b$ parallelogram is a parallelogram with two adjacent sides of lengths a and b. The sides of the parallelogram intersect at $60°$ angles.

Definition 5.2 ($a \times b$ trapezoid)**.** An $a \times b$ trapezoid is an isosceles trapezoid with non-parallel sides of length a and a longer base of length b. The non-parallel sides intersect the longer base at $60°$ angles. Note that we must have $a < b$.



$a \times b$ parallelogram $\qquad$ $a \times b$ trapezoid

Figure 9: $a \times b$ parallelogram and $a \times b$ trapezoid

We define the length of an $a \times b$ parallelogram to be the length b and the length of an $a \times b$ trapezoid to be the length b.

The *isometric grid* is the grid formed by tiling the plane regularly with unit equilateral triangles. We define area by the number of units in the isometric grid. We place the vertices of the target equilateral triangle at lattice coordinates of the isometric grid. Then all vertices of the packing pieces are at lattice coordinates of the isometric grid. This is because the packing is an exact packing, and at any step of the packing, the remaining space in target region consists of all $60°$ and $120°$ corners. Inductively, the equilateral triangular pieces must exactly fill in these corners and have vertices on lattice points. Hence, all pieces in the packing point either *upward* or *downward*.

Now, we prove Propositions 5.3, 5.4, 5.5, and 5.6 to show how the outer triangles must be packed.

Proposition 5.3. ***Trapezoidal Strip:*** *The blocking triangle must be placed inside the target triangle such that the remaining space is a* $[4B + 3t] \times [(3B + 2t)2m + (4B + 3t)]$ *trapezoid.*

Proof. The blocking triangle has side length $(3B + t)2m$, and each wedge triangle has side length $4B + 3t$, so the sum of their side lengths is exactly the side length of the target triangle. Because

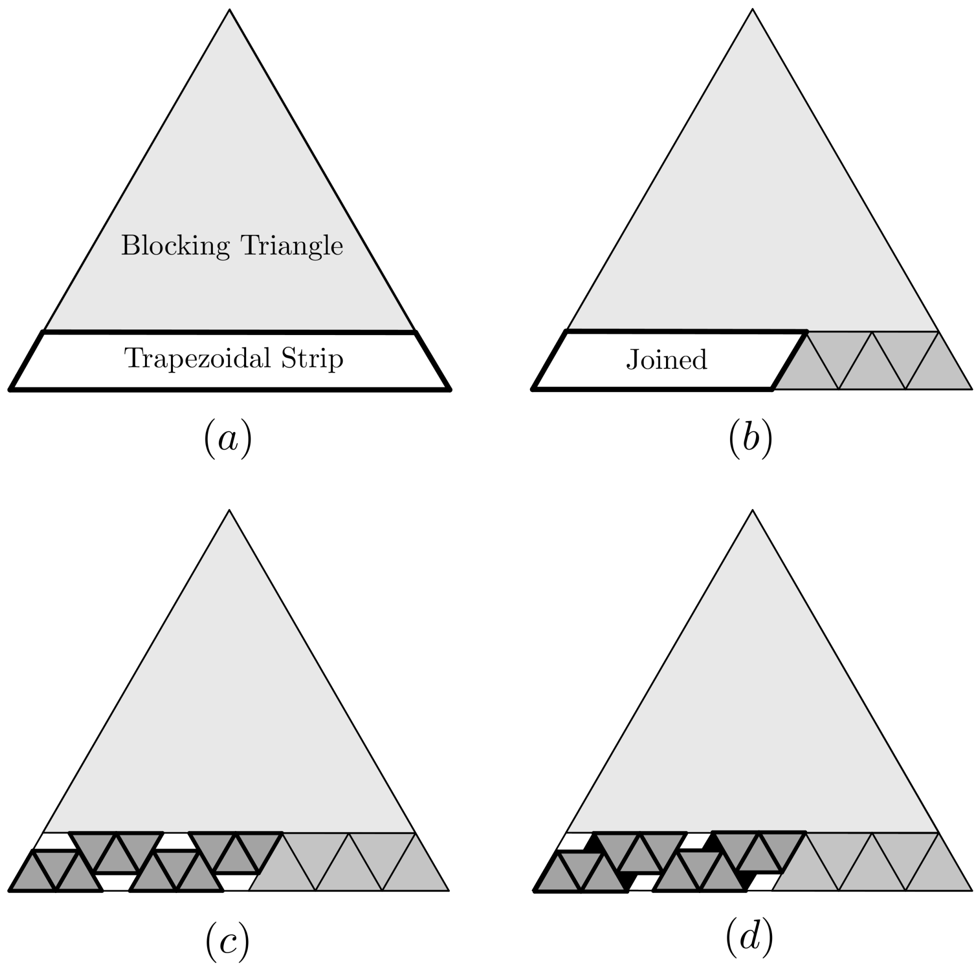


Figure 10: Placing the outer triangles: (a) The blocking triangle must be in a corner of the target triangle, leaving a trapezoidal strip. (b) Without loss of generality, we place the wedge triangles next to each other, leaving the joined region. (c) The container triangles must be placed in the joined region in the blocked pattern, leaving m disjoint smaller trapezoids. (d) Each disjoint trapezoid must contain exactly one filler triangle.

there are $m+1 \geq 2$ wedge triangles, the wedge triangles force the blocking to be rigidly in a corner of the target triangle. The remaining space is a $[4B+3t] \times [(3B+2t)2m+(4B+3t)]$ trapezoid. We call this region the trapezoidal strip (Figure 10-a). □

The placement of the $m+1$ wedge triangles divides the trapezoidal strip into $m+2$ smaller disjoint regions, not necessarily of positive area. If the wedge triangles are placed adjacent to each other, with one wedge triangle adjacent to the side of the trapezoidal strip, then the remaining space is a single continuous region. This region is a trapezoid if m is odd, and the region is a parallelogram if m is even. We call this the joined region (Figure 10-b). Because we can connect the $m+2$ arbitrarily formed disjoint regions to form the joined region, there always exists a packing of the remaining pieces into the joined region if there exists a packing into the $m+2$ disjoint regions. Hence, it suffices to show how the container and filler triangles are packed into the joined region.

Proposition 5.4. *The $3m$ container triangles must pack inside the joined region such that they alternate between pointing up and pointing down.*

Proof. We define the *middle strip* of the joined region to be the strip of height $2B + t$ along the horizontal center line of the joined region (Figure 11. Each container triangle overlaps the entire height of the middle strip, so we may enumerate the container triangles by their order of appearance along the middle strip.

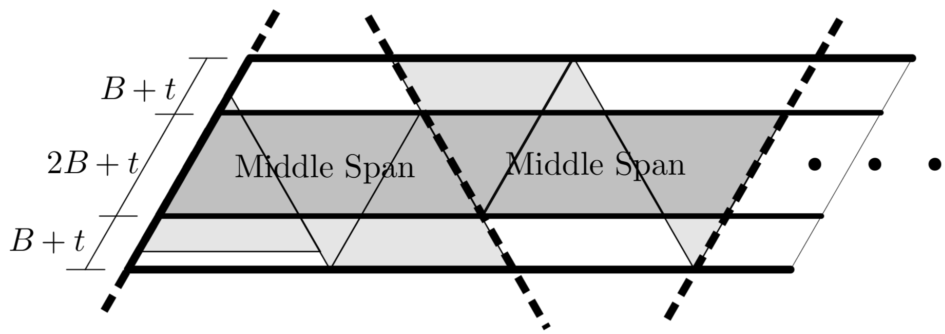


Figure 11: Each container triangle overlaps the entire height of the middle strip. This figure shows the middle strip divided into middle spans.

Let us define the *middle span* of every three consecutive container triangles to be the space within the middle strip bounded by the lines passing through the left edge of the first triangle and the right edge of the third triangle of the triple. For our packing of $3m$ container triangles, there are m middle spans. Because each container triangle fills the full height of the middle strip, none of the middle spans overlap.

By the Pigeonhole Principle, at least two container triangles in each middle span point in the same direction. The left sides of these two triangles form a pair of parallel sides, and the right sides form another pair. At least one of their pairs of parallel sides is at least $3B + 2t$ apart. Thus, the space in the middle strip between the outer sides of these two triangles is at least a $(2B+t)\times(5B+3t)$ trapezoid. It follows that each middle span is at least the area of a $(2B + t) \times (5B + 3t)$ trapezoid.

However, the area of the middle strip is exactly the area of m $(2B + t) \times (5B + 3t)$ trapezoids. Hence, each middle span must be a $(2B+t)\times(5B+3t)$ trapezoid. This can happen if and only if the container triangles within a middle span alternate between pointing up and down. Furthermore, no two consecutive container triangles belonging to different middle spans may point in the same direction because this creates a gap in the middle strip and wastes space. Thus, the $3m$ container triangles alternate between pointing up and pointing down. $\square$

Proposition 5.5. *The $3m$ container triangles must be packed inside the joined region such that the remaining space consists of m disjoint $(B + t) \times (3B + 2t)$ trapezoids.*

Proof. We proceed by induction. When $m = 1$, the joined region is a $(4B+3t)\times(6B+4t)$ trapezoid. The three container triangles must be packed as shown in Figure 12.

When $m = k$, the joined region is a $(4B + 3t) \times \frac{k}{2}(8B + 5t)$ parallelogram, for k even, or a $(4B+3t) \times [\frac{k-1}{2}(8B+5t)+(6B+4t)]$ trapezoid, for k odd. Suppose for $m = 3k$, the $3k$ container triangles must be packed in the blocked pattern into this joined region, as shown in Figure 12, and that they cannot be packed into any region of shorter length. We now show that the container triangles must also be packed in the blocked pattern when $m = k + 1$.

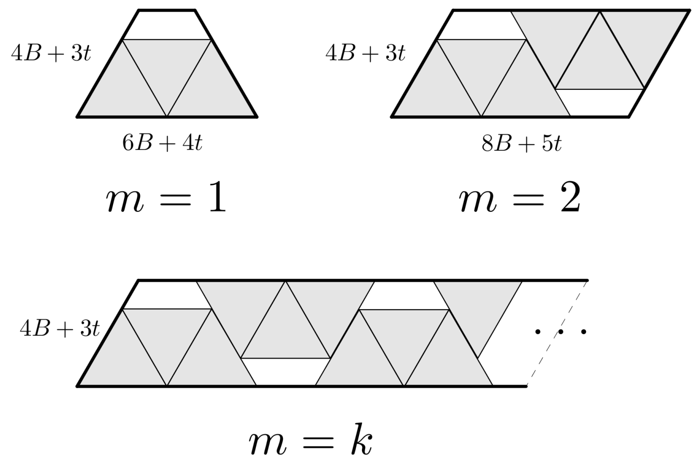


Figure 12: Blocked pattern for placement of container triangles.

Let us define the *span* of three consecutive container triangles to be the space within the joined region bounded by the lines passing through the left edge of the first triangle and the right edge of the third triangle, for every three consecutive container triangles (Figure 13). In a packing of $3(k + 1)$ container triangles, there are $k + 1$ spans. By Proposition 5.4, the container triangles alternate between pointing up and pointing down, so each span is a trapezoid. Because at least one pair of parallel sides of the first and third triangles are at least $3B + 2t$ apart, the length of each span is at least $(3B+2t)+(3B+2t) = 6B+4t$. Because the triangles alternate (Proposition 5.4), the right side of the third triangle in the nth span is always parallel to and to the left of the left side of the first triangle in the $(n+1)$th span. Hence, none of the spans overlap.

But the area of the joined region is exactly the area of m $(4B + 3t) \times (6B + 4t)$ trapezoids, so each span must be a $(4B+3t)\times(6B+4t)$ trapezoid. Moreover, there cannot be any space between the spans or between the leftmost and rightmost spans and the sides of the joined region.

For a span to have a length of exactly $6B+4t$, at least one of the first or third triangle must be adjacent to the base of the span trapezoid. In the first span, the third triangle cannot be above the first triangle; otherwise, the second triangle would not be able to fit between them. Thus, the third triangle of the first span must be adjacent to the base. The remaining $3k$ container triangles must then pack entirely to the right of the dividing line. By the inductive hypothesis, the remaining $3k$ container triangles pack in the blocked pattern, with the fourth container triangle of the joined region in the top left corner of the space to the right of the dividing line. In order for the second and fourth container triangles to not overlap, the sides of the second and third container triangles

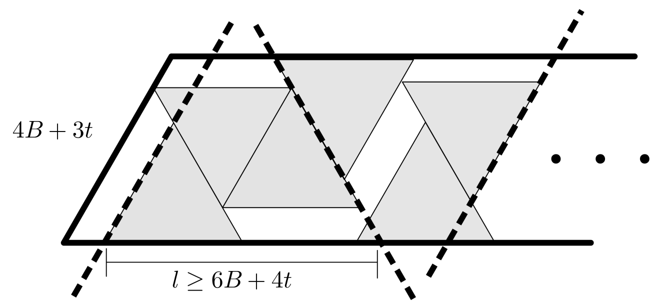


Figure 13: The leftmost six container triangles are shown. They are divided into spans along the dotted lines. The length of each span must be at least $6B + 4t$.

must entirely overlap. This forces the first container triangle to also be adjacent to the base of the span. Hence, the first three triangles are packed as depicted in the $m = 1$ case (Figure 12). This packing yields the blocked pattern for $m = k + 1$. Therefore, the $3m$ container triangles always pack in the blocked pattern, and the remaining space of the joined region comprises m disjoint $(B + t) \times (3B + 2t)$ trapezoids. □

Proposition 5.6. *There must be exactly one filler piece in each of the m disjoint $(B+t)\times(3B+2t)$ trapezoids.*

Proof. Suppose there exists an unpacked $(B + t) \times (3B + 2t)$ trapezoid with two filler pieces. After placing one of the filler pieces, the remaining space can be joined to a $(B+t)\times(2B+t)$ parallelogram. We make the other filler piece act as a partition piece with side length $B + t$. As we show in the proof of Proposition 5.8, the length of a parallelogram containing this partition piece must be at least $2B+t+a_i+a_j+a_k > 2B+t$, for some positive a_i, a_j, a_k, a contradiction. Hence, no unpacked $(B + t) \times (3B + 2t)$ trapezoid may contained more than one filler piece, so each contains exactly one filler piece, and the remaining space can be joined to a $(B + t) \times (2B + t)$ parallelogram. □

The remaining space in each of the m disjoint $(B + t) \times (3B + 2t)$ trapezoids can be connected to each other to form a $(B + t) \times (2B + t)$ parallelogram. Hence, if there exists a packing of the inner pieces into any frame formed by the outer pieces, then there exists a packing of the inner pieces into m disjoint $(B + t) \times (2B + t)$ parallelograms. Thus, it suffices to prove that if there exists a packing of the inner pieces into m disjoint $(B + t) \times (2B + t)$ parallelograms, then there exists a 4-partition of the set A. We prove Propositions 5.7 and Proposition 5.8 to show how the inner triangles must be packed.

Proposition 5.7. *The number of unit equilateral triangles is bounded by $4mt^2$.*

Proof. The area in which to pack the inner pieces is equal to the area of m disjoint $(2B+t)\times(B+t)$ parallelograms. Each of these parallelograms has an area of $2(B + t)(2B + t) = 4B^2 + 6Bt + 2t^2$, so the total area for packing inner pieces is $4mB^2 + 6mBt + 2mt^2$

We now find a lower bound to the total area taken up by the partition and support pieces. An equilateral triangle of side length s has area s^2, so the partition pieces have area $(B+a_i)^2$ for each $a_i \in A$, and each support piece has area t^2. The partition and support pieces fill a total area of

$$
\begin{aligned}
&2mt^2\left(\frac{2B}{t}\right)+\sum_{a_i\in A}(B+a_i)^2 \\
&= 4mBt-2mt^2+\left(4mB^2+2mBt+\sum_{a_i\in A}a_i^2\right) \\
&> 4mB^2+6mBt-2mt^2.
\end{aligned}
$$

Hence, the amount of empty space in which to fill unit equilateral triangles must be less than $(4mB^2+6mBt+2mt^2)-(4mB^2+6mBt-2mt^2)=4mt^2$. □

Proposition 5.8. *If there exists a packing of the inner triangles into m disjoint $(2B+t)\times(B+t)$ parallelograms, then there exists a 4-partition of A.*

Proof. The area of each parallelogram is of order $4B^2$, so no parallelogram can contain five or more partition triangles. Thus, each parallelogram must contain exactly four partition triangles. The four partition triangles in each target parallelogram must be arranged in two pairs with one in each pair pointing upward and one in each pair pointing downward.

Suppose a $(2B+t)\times(B+t)$ parallelogram contains triangles of side lengths $B+a_i$, $B+a_j$, $B+a_k$, and $B+a_l$. Without loss of generality, suppose the $B+a_i$ and $B+a_j$ triangles form the top pair and the $B+a_k$ and $B+a_l$ triangles form the bottom pair. Suppose the $B+a_i$ and $B+a_k$ triangles point upward, and the $B+a_j$ and $B+a_l$ triangles point downward.

Each partition triangle must have an edge adjacent to the left or right side of the parallelogram. We prove this by contradiction. Suppose there is an upward partition triangle of side length $B+x$ that is distance δ away from the left side of the parallelogram. Because all triangles pack at integer coordinates of the isometric grid, we have $\delta \geq 1$, so this gap has area greater than B. By Proposition 5.7, the number of unit triangles is $4mt^2$, which is less than B. Hence, such a gap would be too large to be filled by unit triangles. Furthermore, the maximum value of δ is $t-x$. Because all support triangles have side length t, this gap is too narrow to fit support triangles. Thus, the gap cannot be filled, which cannot happen in an exact packing.

In each pair, the partition triangles must be either adjacent or separated, as shown in Figure 14. Suppose the top pair is adjacent. Then the right edge of the parallelogram has length $(t-a_j)+(B+a_j)+(B+a_l)>2B+t$, a contradiction. The top pair cannot be adjacent. Similarly, the bottom pair cannot be adjacent.

If the triangles are separated, then the space in between them must be packed by support triangles by the same exact packing argument. Let r and s be the number of rows of support triangles between the top pair and bottom pair of partition triangles, respectively. Then the top pair takes up length $B+a_i+a_j+t(r-1)$ along the side of the target parallelogram and the bottom

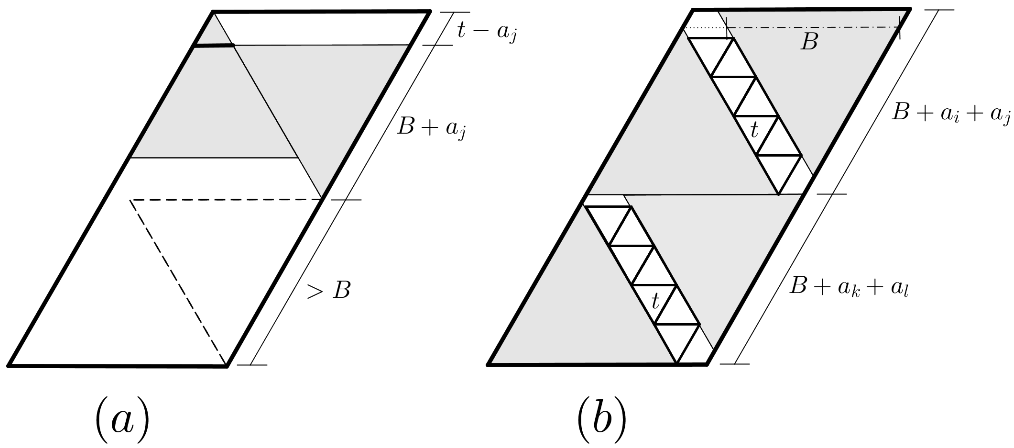


Figure 14: (a) If the partition triangles on the top are adjacent, then they use length $B + t$ along the right edge of the parallelogram. (b) If the partition triangles are separated by one row of support triangles, then they use length $B + a_i + a_j$ along the right edge of the parallelogram.

pair takes up length $B + a_k + a_l + t(s - 1)$ along the side of the parallelogram.

We must have $2B + a_i + a_j + a_k + a_l + t(r + s - 2) = 2B + t$ for the pieces to fit inside the target parallelogram and not leave empty space. Because r and s are integers, this yields $r = s = 1$. Hence, we have $a_i + a_j + a_k + a_l = t$, so the four partition triangles in the packing of the disjoint parallelograms correspond to a 4-partition quadruple. $\square$

We may now prove Lemma 5.2.

Proof of Lemma 5.2. By Propositions 5.3, 5.4, 5.5, and 5.6, any given packing out the outer triangles into the target region leaves remaining space corresponding to m disjoint $(2B + t) \times (B + t)$ parallelograms. By Proposition 5.8, any given packing of the inner triangles into these disjoint parallelograms corresponds to a 4-partition of the set A. Therefore, if there exists a packing of the equilateral triangular pieces, then there exists a 4-partition of set A. $\square$

This is the lemma that is used at the final step of the proof of Theorem 5.1.

6 Conclusions and Directions for Future Research

We studied three triangle packing problems: (i) packing right triangles into a rectangle, (ii) packing right triangles into a right triangle, and (iii) packing equilateral triangles into an equilateral triangle. We showed that each problem is strongly NP-hard. Furthermore, we may generalize that triangle packing with arbitrary triangular pieces is strongly NP-hard by a reduction to any of our three triangle packing problems. Our results indicate that triangle packing in industrial applications presents a difficulty; there does not exist an efficient exact algorithm for triangle packing in general or for each of our three cases of triangle packing. Because triangle packing is NP-hard, approximation or case-by-case algorithms rather than exact algorithms must be used to determine triangle packing.

We present the open problem of whether these NP-hard triangle packing problems are also in NP. If we can show that an NP-hard problem is in NP, then we may classify the problem as an NP-complete. NP-complete problems form the class of problems that are the hardest of all NP problems. If we can efficiently solve one NP-complete problem, then we can efficiently solve all NP problems; in other words, NP-complete problems can be solved efficiently if and only if all problems that are easy to verify are also easy to solve.

To show that a packing problem is in NP, we must show that it is possible to encode and verify a solution using polynomial time and resources. At the encoding step, the tightest packing may require irrational translation and rotation for some input triangles. Because it is not possible to directly encode irrational numbers with polynomial resources, the encoding step presents a difficulty. Note that if we do not allow rotations of the input pieces, then the packing problems are NP-complete. Hence, for each of our NP-hard triangle packing problems, we have a fixed-orientation NP-complete analog.

For future study, it would be fruitful to specify a new way of encoding the packing pieces such that the encoding for arbitrarily rotating pieces is tractable. This may be possible by approximating irrational numbers as terminated decimals in a way that the packing remains rigid, or by creating special structures, such as continued fractions, for encoding irrational numbers.

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