Laplacians Associated with Regular Bipartite Graphs and Application to Finite Projective Planes

Jesse T. Zhang

under the direction of
David B Rush
Massachusetts Institute of Technology

Research Science Institute
July 30, 2014

Abstract

Internet search technology is a pervasively used utility that relies on techniques from the field of spectral graph theory. We present a novel spectral approach to investigate an existing problem: the critical group of the line graph has been characterized for regular nonbipartite graphs, but the general regular bipartite case remains open. Our approach aims to obtain the relationship between the spectra of the Laplacians of the graph G and its line graph $\widehat{G}$. We obtain a theorem for the spectra of all regular bipartite graphs and demonstrate its effectiveness by completely characterizing the previously unknown critical group for a particular class of regular bipartite graphs, the incidence graphs of finite projective planes with square order. This critical group is found to be $\mathbb{Z}_2 \oplus (\mathbb{Z}_{q+1})^{q^3-1} \oplus (\mathbb{Z}_{q^2+q+1})^{q^2+q-1}$, where q is the order of the finite projective plane.

Summary

Many processes and relations in physical, biological, social, and information systems can be modeled by a network of objects with some pairs of objects connected by links. This project studies a structure that describes neighboring links in the network. We focus on the algebraic representations of the network in a branch of mathematics that is essential to internet search technology. So far, methods have proved unsuccessful in dealing with a particular class of networks. This project draws ideas from widely distinct areas of math, from number theory, to combinatorics of graphs, to algebra of matrices, to geometry of the projective plane. With an innovative approach to the problem, we obtain the desired results for a previously unstudied class of networks.

1 Introduction

Algebraic graph theory is a branch of mathematics with a long history. It is concerned with studying the structure of graphs using the algebraic properties of associated matrices. In particular, the *graph Laplacian* and the *incidence matrix* are two of the most fundamental matrices associated with a graph that encapsulate important information about the graph. Spectral graph theory is the branch of algebraic graph theory that studies the spectrum, or set of eigenvalues, of the graph Laplacian. The field has significant applications. Google's search technology, an innovation that made Google's founders billionaires, is based on computing the Perron–Frobenius eigenvector of the web graph. Historically, algebraic approaches have proven especially effective in dealing with regular graphs.

The *critical group* of a graph is a finite abelian group whose order is the number of spanning trees in the graph. It is an invariant of the graph and is closely related to the *chip-firing game* played on the vertices of a graph. Other names for the critical group include the *abelian sandpile model* in physics, the *Jacobian group*, and the *Picard group.* In this paper, we investigate the relationship between the structure of a graph and the critical group of the graph's *line graph*, which describes adjacencies between the edges of the graph. We assume that no graph has self-loops, but multiple edges between two vertices are permitted. The critical group of the line graph has been studied in detail and completely classified for regular nonbipartite graphs by Berget, Manion, Maxwell, Potechin, and Reiner [1], but results for the general regular bipartite case are still missing. This desired relationship was further investigated by Machacek [2] using the same approach, but the critical group of the line graph was only obtained for some classes of regular bipartite graphs.

Hence, we investigate the overarching relationship between the regular bipartite graph and its associated line graph. To accomplish this, we introduce a novel approach that directly examines their Laplacians. We present results on the respective spectra and ultimately establish that excluding a set of eigenvalues all equal to twice the vertex degree, the eigenvalues of the graph Laplacian are precisely equal to those of the line graph Laplacian. Finally, we demonstrate the effectiveness of the theorem by completely characterizing the critical groups of incidence graphs of projective planes over finite fields of square order, a class of regular bipartite graphs.

2 Preliminaries

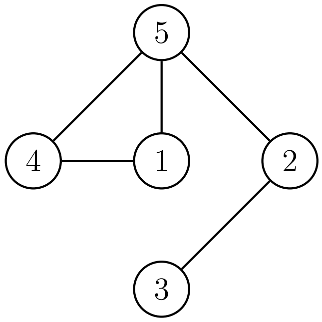
We proceed by introducing the concepts and definitions fundamental to the project. We also describe the existing results upon which our work is built.

2.1 The graph Laplacian and incidence matrix

Suppose we have a graph $G = (V, E)$ where V is the set of vertices and E is the set of edges. Let $A(G)$ be its adjacency matrix, a zero-one symmetric matrix with $A(G)_{x,y} = 1$ if and only if vertices x and y are connected. Let d_i be the degree of vertex v_i, and define the graph's degree matrix as the diagonal matrix $D(G) := \text{diag}(d_1, d_2, ..., d_n)$. The corresponding *graph Laplacian* $L(G)$ of the graph is a $|V| \times |V|$ matrix defined by the difference between the degree matrix and adjacency matrix, $L(G) = D(G) - A(G)$. The graph Laplacian and its spectrum characterize many important aspects of the graph, such as the number of spanning trees or forests. Alternatively, let $m_{x,y}$ denote the multiplicity of the edges between vertices x and y. Then the graph Laplacian can also be defined by its individual entries:

$$L(G)_{x,y} = \begin{cases} \deg(x) & \text{if } x = y \\ -m_{x,y} & \text{otherwise.} \end{cases}$$

Figure 1 shows an example of a graph G and its corresponding Laplacian $L(G)$.



$$L(G) = \begin{bmatrix} 2 & 0 & 0 & -1 & -1 \\ 0 & 2 & -1 & 0 & -1 \\ 0 & -1 & 1 & 0 & 0 \\ -1 & 0 & 0 & 2 & -1 \\ -1 & -1 & 0 & -1 & 3 \end{bmatrix}$$

Figure 1: G (left) and $L(G)$ (above)

We can view the graph Laplacian as an abelian group homomorphism $L(G) : \mathbb{Z}^{|V|} \to \mathbb{Z}^{|V|}$. Consequently, from the cokernel, we define the unique finite abelian group called the critical group, $\Phi(G)$:

$$\mathbb{Z}^{|V|}/\text{im } L(G) \cong \mathbb{Z}^c \oplus \Phi(G),$$

where c is the number of connected components of the graph and im $L(G)$ denotes the image of the mapping. We note that the rank of $L(G)$ is $|V| - c$.

Moreover, we observe that the critical group of a graph is the product of the critical groups of its connected components, so it suffices to study only connected graphs. When G is connected, we have $c = 1$, so im $L(G)$ is completely determined by any $(|V|-1) \times (|V|-1)$ submatrix of $L(G)$. Thus, assuming G is connected, we can write the critical group as

$$\Phi(G) \cong \mathbb{Z}^{|V|-1}/\text{im } \overline{L(G)}^{x,y},$$

where $\overline{L(G)}^{x,y}$ is the reduced graph Laplacian obtained by deleting any row x and any column y from $L(G)$.

We now present Kirchhoff's Matrix Tree Theorem for the number of spanning trees, $\kappa(G)$, in a graph. It follows from the theorem that the order of $\Phi(G)$ is precisely $\kappa(G)$.

Theorem 2.1. *(Kirchhoff's Matrix Tree Theorem [3]). Let $G = (V, E)$ be a connected graph* *with* $|V| = n$ *and let* $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_{n-1} > \lambda_n = 0$ *be the eigenvalues of* $L(G)$*. Then,*

$$\kappa(G) = \frac{1}{n} \prod_{i=1}^{n-1} \lambda_i.$$

That is, $\kappa(G)$ *is equal to any 1,1-cofactor of* $L(G)$*.*

Another structure closely related to the graph and its graph Laplacian is the *incidence matrix*. The incidence matrix, $B(G)$, is a $|V| \times |E|$ matrix that results from assigning an orientation to the edges of G. Its components are defined by

$$B(G)_{x,y} = \begin{cases} 1 & \text{if } e_y \text{ enters } v_x \\ -1 & \text{if } e_y \text{ leaves } v_x \\ 0 & \text{otherwise.} \end{cases}$$

The *unoriented* or *unsigned incidence matrix*, which we denote by $C(G)$, is also of interest in this paper. Its terms are the absolute values of those in $B(G)$. It is well known that the incidence matrix is related to the Laplacian by the equation $L(G) = B(G)B(G)^T$.

2.2 The line graph and edge subdivision graph

In addition to the bipartite graph, we investigate its associated *line graph* and the critical group of the line graph. In doing so, we also consider the *edge subdivision graph* and its critical group. The line graph and edge subdivision graph are denoted by $\widehat{G}$ and $\text{sd } G$ respectively. Given a graph $G = (V, E)$, define $\text{sd } G$ to be the graph obtained by placing a vertex at the midpoint of every edge in G. The line graph $\widehat{G} = (V_{\widehat{G}}, E_{\widehat{G}})$ is slightly more difficult to define. We set $V_{\widehat{G}} = E$ so that each edge of G corresponds to a vertex in $\widehat{G}$. Two vertices of the line graph are connected by an edge if the corresponding edges in G are incident on the same vertex. Figure 2 depicts a graph and its corresponding line graph.

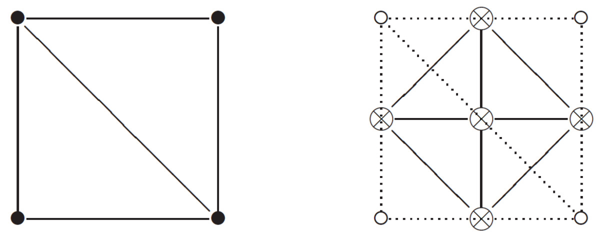


Figure 2: A graph G and its line graph $\widehat{G}$ with G underlayed [1]

We provide a few more definitions as follows. Let the abelian group $\mathbb{Z}/d\mathbb{Z}$ be denoted by $\mathbb{Z}_d$. Assuming G is connected, let $\beta(G)$ be the number of independent cycles of G. It is well-known that $\beta(G) = |E| - |V| + 1$. In fact, it has been shown that $\beta(G)$ is an upper bound on the number of generators required for $\Phi(G)$. Thus, we can formulate the following simple relationship between $\Phi(G)$ and $\Phi(\text{sd } G)$.

Theorem 2.2. *(Lorenzini [4]). Given a connected graph G, we can write*

$$\Phi(G) = \bigoplus_{i=1}^{\beta(G)} \mathbb{Z}_{k_i},$$

where the k_i are all positive integers. Furthermore,

$$\Phi(sd\ G) = \bigoplus_{i=1}^{\beta(G)} \mathbb{Z}_{2k_i}.$$

In addition, the relationship between the number of spanning trees of G and of $\widehat{G}$ has been characterized for regular graphs.

Theorem 2.3. *(Cvetković, Dobb, Sax [5]). If graph G is d-regular, then*

$$\begin{aligned}\kappa(\widehat{G}) &= (2d)^{\beta(G)-2}\kappa(G) \\ &= d^{\beta(G)-2}\kappa(sd\ G).\end{aligned}$$

Theorem 2.3 motivates the search for a nice relationship between $\Phi(G)$ and $\Phi(\widehat{G})$. Several important theorems proven by Berget, Manion, Maxwell, Potechin, and Reiner [1] relate $\Phi(G)$ and $\Phi(\widehat{G})$. In particular, these authors are able to entirely characterize the desired relationship for simple, connected, d-regular graphs that are nonbipartite. A central idea is that because the critical group is a finite abelian group, it is completely determined if the *p-Sylow subgroups* are know for each prime p. Furthermore, we also have the exact sequence relating $\Phi(G)$ and $\Phi(\widehat{G})$ through the edge subdivision graph for any d-regular graph.

Theorem 2.4. *(Berget, et al. [1]) For any connected d-regular simple graph G with $d \geq 3$, there is a natural group homomorphism $f : \Phi(\widehat{G}) \to \Phi(sd\ G)$ whose kernel-cokernel exact sequence takes the form*

$$0 \to \mathbb{Z}_d^{\beta(G)-2} \oplus C \to \Phi(\widehat{G}) \to \Phi(sd\ G) \to C \to 0,$$

where C is the cokernel and is described by the following cyclic d-torsion group:

$$C = \begin{cases} 0 & \text{if } G \text{ non-bipartite and } d \equiv 1 \pmod 2 \\ \mathbb{Z}_2 & \text{if } G \text{ non-bipartite and } d \equiv 0 \pmod 2 \\ \mathbb{Z}_d & \text{if } G \text{ bipartite.} \end{cases}$$

When G is nonbipartite, the aforementioned results (Theorems 2.2 and 2.4) use $\Phi(G)$ to completely determine $\Phi(\widehat{G})$. The relationship is expressed in Corollary 2.5. Note, however, that the same approach has not been successfully utilized to address and generalize the case where G is bipartite.

Corollary 2.5. *(Berget, et al. [1]). For a simple, connected d-regular graph G with $d \geq 3$ which is nonbipartite,*

$$\Phi(\widehat{G}) \cong \bigoplus_{i=1}^{\beta(G)-2} \mathbb{Z}_{2dd_i} \oplus \begin{cases} \mathbb{Z}_{2d_{\beta(G)-1}} \oplus \mathbb{Z}_{2d_{\beta(G)}} & \text{if } |V| \text{ is even} \\ \mathbb{Z}_{4d_{\beta(G)-1}} \oplus \mathbb{Z}_{d_{\beta(G)}} & \text{if } |V| \text{ is odd.} \end{cases}$$

2.3 Smith normal form and Laplacian eigenvalues

The *Smith normal form* of an integer matrix is a diagonal matrix obtained by multiplying the original matrix on the left and right by invertible square matrices. Interestingly, given a graph $G = (V, E)$, the Smith normal form of the graph Laplacian also determines the critical group $\Phi(G)$. When computing the Smith normal form, acceptable operations are those that can be performed by left/right multiplication by an integer matrix in the *general linear group* of order n, or $GL_n(\mathbb{Z})$. Equivalently, the following operations on the original Laplacian are allowed:

- permute rows or columns
- scale rows or columns by ± 1
- add an integer multiple of one row/column to another row/column.

The Smith normal form is the primary tool was use to examine the structure of the critical group. There exist several relevant results on the general form of the Smith normal matrix, and thus the critical group, from the eigenvalues of the Laplacian. To start, we present a result for a symmetric $n \times n$ integer matrix M with rank $n-1$. Denote this matrix by M. Let R be the $n \times 1$ column vector that generates the kernel of M and let $r = R \cdot R$. For $n \in \mathbb{Z}$, let $\text{ord}_p(n)$ be the largest power of p that divides n, where p is a prime. Then, $\Phi(M)$ contains subgroups characterized by the eigenvalues of M.

Lemma 2.6. *(Lorenzini [6]) Let M be any $n \times n$ integer matrix of rank $n-1$. Let $\lambda \neq \pm 1$ be an integer eigenvalue of M and let $m(\lambda)$ be its multiplicity. Then*

1. *If there exists a prime p such that $p|\lambda$ but $p \nmid r$, then $\Phi(M)$ contains a subgroup isomorphic to $(\mathbb{Z}/p^{ord_p(\lambda)}\mathbb{Z})^{m(\lambda)}$.*

2. *If the vector R has one entry equal to ± 1, then $\Phi(M)$ contains a subgroup isomorphic to $(\mathbb{Z}/\lambda\mathbb{Z})^{m(\lambda)-1}$.*

Proof. See [6], Proposition 2.3. □

Furthermore, let $f(x) \in \mathbb{Z}[x]$ be the minimal polynomial of λ in M. Define $L(\lambda)$ as the least common multiple of the roots of $f(x)$ when computed in the ring of algebraic integers.

Lemma 2.7. *Let M be an $n \times n$ integer matrix of rank $n-1$. Let λ be an eigenvalue of M. Then $\Phi(M)$ contains an element of order*

$$\frac{L(\lambda)}{\gcd(L(\lambda), r)}.$$

Proof. See [6], Proposition 2.1. □

Note that in our case where M is a graph Laplacian, we have $R = (1, 1, ..., 1)^T$ and $r = n$, where n is the number of vertices in the graph.

2.4 Projective planes over finite fields

A *projective plane* is a geometric structure that extends the concept of a plane as a 2-dimensional projective space. It consists of a set of points, a set of lines, and a relation between points and lines called *incidence*. Every projective plane has the following three properties. First, given any two distinct points, there is exactly one line incident with both of them. Second, given any two distinct lines, there is exactly one point incident with both of them. Finally, there are four points such that no line is incident with more than two of them. Our focus lies with projective planes over finite fields, also known as field planes. Consider $\mathbb{Z}_q$, the finite field of order q, where q is a perfect power of a prime. We denote the corresponding field plane by $P^2(\mathbb{Z}_q)$. We also say its order is q. This field plane has an associated regular bipartite graph, known as the *incidence graph,* that has two disjoint sets of vertices, one corresponding to the points in the field plane and one corresponding to the lines. Edges of the incidence graph are drawn between the two sets if the respective point and line of the field plane are incident. Each vertex of the incidence graph has degree $q+1$. To determine the total number of vertices, $2v$, we count the number of pairs of adjacent edges in the incidence graph. There are $q+1$ edges meeting at each vertex, so the desired number is $2v\binom{q+1}{2}$. From the properties of the projective plane, there is a path of length 2 between any two vertices in the same set, so the desired number is also equal to $2\binom{v}{2}$. Consequently, we have

$$\frac{v(q)(q+1)}{2} = \frac{v(v-1)}{2}.$$

Solving for v yields $v = q^2 + q + 1$. Therefore, each set of the incidence graph has $q^2 + q + 1$ vertices for a total of $2q^2+2q+2$. The spectra of these incidence graphs have been completely characterized.

Lemma 2.8. *(Shirrell [7]) The eigenvalues of the Laplacian of the incidence graph of a finite projective plane of order q are $0,\ 2(q+1),\ q+1+\sqrt{q},\ q+1-\sqrt{q}$ with multiplicities $1,\ 1,\ q^2+q,\ q^2+q$, respectively.*

Proof. See [7], Proposition 1. □

3 Results on Spectra

We present our results on the spectra of $L(G)$ and $L(\widehat{G})$ that ultimately lead to a strong relationship between the two spectra. We start with a lemma that utilizes a new insight of assigning a particular orientation to bipartite graphs to give a result for regular bipartite graphs.

Lemma 3.1. *Let $\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \cdots \geq \lambda_{n-1} > \lambda_n = 0$ be the eigenvalues of $L(G)$. Then the eigenvalues of $L(\widehat{G})$ are*

$$2d - \lambda_1,\ 2d - \lambda_2,\ 2d - \lambda_3,\ ...,\ 2d - \lambda_{n-1},\ 2d,\ \underbrace{2d,\ 2d,\ ...,\ 2d}_{|E|-|V|}.$$

Proof. Recall that $L(G) = BB^T$. Another useful observation is that $A(\widehat{G}) = C^TC - 2I_n$, where I_n is the identity matrix with dimensions $n = |V|$. Moreover, since we are considering d-regular graphs, each edge is incident to $2d-2$ other edges, so each vertex of $\widehat{G}$ has a degree of $2d-2$. Hence, the degree matrix is given by $D(\widehat{G}) = (2d-2)I_n$. It follows that the graph Laplacian of $\widehat{G}$ can be expressed in terms of $C := C(G)$, the unsigned incidence matrix of G :

$$\begin{aligned} L(\widehat{G}) &= D(\widehat{G}) - A(\widehat{G}) \\ &= (2d)I_n - C^TC. \end{aligned}$$

Since G is bipartite, we can partition its vertices into two disjoint sets where edges only connect vertices in different sets. Assign a canonical orientation to G in which the edges are all directed from one set to the other. The result of this is seen in the signed incidence matrix B, where all the nonzero terms in a row now have the same sign. Consequently, there exists a diagonal matrix I' that consists of only 1's and -1's along its diagonal such that

$B = I'C$. Equivalently, $C = I'B$, because I' is clearly its self-inverse. We now have a explicit relationship between B and C. Substituting,

$$
\begin{aligned}
L(\widehat{G}) &= (2d)I_n - (I'B)^T I'B \\
&= (2d)I_n - B^T I'I'B \\
&= (2d)I_n - B^T B.
\end{aligned}
$$

It is well-known that B^TB and BB^T, which are $|E| \times |E|$ and $|V| \times |V|$ matrices respectively, have the same nonzero eigenvalues. Thus, if $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_{n-1} > \lambda_n = 0$ are the eigenvalues of $L(G) = BB^T$, then the eigenvalues of B^TB are $\lambda_1,\ \lambda_2,\ ...,\ \lambda_{n-1},\ 0,\ 0,\ ...,\ 0$. Let matrix $M = B^TB$ and let v_i be the eigenvector corresponding to eigenvalue λ_i. That is, $Mv_i = \lambda_i v_i$. Hence, we have $((2d)I_n - M)v_i = (2d - \lambda_i)v_i$, implying that $2d - \lambda_i$ is an eigenvalue of $L(\widehat{G})$. So by linearity of matrices, the eigenvalues of $L(\widehat{G})$ are then $2d-\lambda_1,\ 2d-\lambda_2,\ 2d-\lambda_3,\ ...,\ 2d-\lambda_{n-1},\ 2d,\ \underbrace{2d,\ 2d,\ ...,\ 2d}_{|E|-|V|}$. □

Corollary 3.2. *The spectrum of $L(G)$ contains $2d$.*

Proof. Since the rows and columns of any Laplacian sum to 0, the rows are not linearly independent. Hence, 0 is an eigenvalue of all Laplacians. By the spectrum of $L(\widehat{G})$ determined in Lemma 3.1, $2d$ is an eigenvalue of $L(G)$, as desired. □

Now we proceed with a result on the eigenvalues of $L(G)$ and their relationship with each other. Again, we have an approach that is well-suited for the case of bipartite graphs that we are investigating.

Lemma 3.3. *The n eigenvalues of $L(G)$ can be paired up into $n/2$ pairs such that each pair sums to $2d$.*

Proof. We want to show that $\{2d - \lambda_1, 2d - \lambda_2, ..., 2d - \lambda_n\}$ is exactly the set of values as $\{\lambda_1, \lambda_2, ..., \lambda_n\}$. This is true if and only if the two corresponding characteristic polynomials are identical, that is, $L(G)-\lambda I_n$ and $L(G)-(2d-\lambda)I_n$ have the same determinant. Because G is regular bipartite, the matrix $L(G)-\lambda I_n$ takes the form:

$$\left[\begin{array}{cccccc} d-\lambda & 0 & 0 & & & \\ 0 & \ddots & 0 & & X & \\ 0 & 0 & d-\lambda & & & \\ & & & d-\lambda & 0 & 0 \\ & X^T & & 0 & \ddots & 0 \\ & & & 0 & 0 & d-\lambda \end{array}\right]. \tag{1}$$

Submatrix X consists entirely of elements equal to 0 and -1, with d elements equal to -1 in each row and column. Similarly, $L(G)-(2d-\lambda)I_n$ takes the form:

$$\left[\begin{array}{ccc|ccc} \lambda-d & 0 & 0 & & & \\ 0 & \ddots & 0 & & X & \\ 0 & 0 & \lambda-d & & & \\ \hline & & & \lambda-d & 0 & 0 \\ & X^T & & 0 & \ddots & 0 \\ & & & 0 & 0 & \lambda-d \end{array}\right]. \tag{2}$$

Note that the matrices are identical except that the respective diagonals have opposite signs. The key idea lies in the two $(n/2)\times(n/2)$ submatrices in the top left and bottom right corners, separated by the vertical and horizontal axes. Upon expansion of each of the determinants of Matrices (1) and (2), we obtain a sum of terms such that each term consists of a product of n elements, no two of which lie in the same row or column. Suppose that for one of the terms, m of the n elements lie in the top left submatrix. Then $n/2-m$ of the top $n/2$ rows have yet to be selected, so there must be $n/2 - m$ elements chosen from the top right submatrix. We have an identical result for the bottom left submatrix. It follows that m of the elements lie in the bottom right submatrix. Now, if any of the elements in the top left and bottom right submatrices are 0, the product of the n elements is 0, which makes the term containing these elements irrelevant in the expansion of the determinant. Because the submatrices in top left and bottom right are diagonal matrices, all nonzero terms of the determinant contain an equal number of elements along the diagonals of each submatrix. Thus, each term in the determinant has an even number of elements along the diagonal of the overall matrix. Consequently, $L(G) - \lambda I_n$ and $L(G) - (2d - \lambda)I_n$ have the same determinant. □

Finally, Lemmas 3.1 and 3.3 can be combined to give a strong result on the spectra of the regular bipartite graph and its associated line graph.

Theorem 3.4. *Excluding $|E| - |V|$ additional eigenvalues equal to $2d$, the eigenvalues of $L(\widehat{G})$ are precisely the same as the eigenvalues of $L(G)$, including multiplicity.*

Proof. By Lemma 3.3, consider the $n/2$ pairs of eigenvalues of $L(G)$. By Lemma 3.1, there exists a bijective mapping between each pair of eigenvalues in the spectrum of $L(G)$ and an identical pair of eigenvalues in the spectrum of $L(\widehat{G})$. Lemma 3.1 also accounts for the $|E| - |V|$ additional eigenvalues in the spectrum of $L(\widehat{G})$ equal to $2d$, so we are done. □

4 Critical Group of Projective Planes

Theorem 3.4 allows for a new perspective in relating $\Phi(\widehat{G})$ to $\Phi(G)$ for regular bipartite graphs G. We have obtained an exact relationship between the spectra of the Laplacian that can aid us in characterizing their respective critical groups. Now, we utilize the result to completely determine the critical group of the line graph of a particular class of regular bipartite graphs, incidence graphs of projective planes with square order.

Theorem 4.1. *Let $G = (V, E)$ be the incidence graph of a nondegenerate projective plane with order $q = p^{2y}$, where y is an integer and p is prime. Then the critical group of the line graph of G is isomorphic to $\mathbb{Z}_2 \oplus (\mathbb{Z}_{q+1})^{q^3-1} \oplus (\mathbb{Z}_{q^2+q+1})^{q^2+q-1}$.*

Proof. Recall from Lemma 2.8 that the eigenvalues of $L(G)$ are exactly known to be $0,\ 2q+2,\ q+1+\sqrt{q},\ q+1-\sqrt{q}$ with multiplicities $1,\ 1,\ q^2+q,\ q^2+q$, respectively. Note that there are $2(q^2+q+1)$ total vertices and $(q+1)(q^2+q+1)$ total edges in G, so $|E| - |V| = (q-1)(q^2+q+1) = q^3 - 1$.

By Theorem 3.4, the eigenvalues of $L(\widehat{G})$ are $0,\ 2q+2,\ q+1+\sqrt{q},\ q+1-\sqrt{q}$ with multiplicities $1,\ q^3,\ q^2+q,\ q^2+q$, respectively. First, we consider $\lambda = 2q+2$. Applying part 2 of Lemma 2.6, we have that $\Phi(\widehat{G})$ contains a subgroup isomorphic to $(\mathbb{Z}_{2q+2})^{q^3-1}$.

Since q is a square, all the eigenvalues are integers. We now consider $g = \gcd(q+1+\sqrt{q}, q+1-\sqrt{q})$. Rewriting $q+1+\sqrt{q}$ as $\sqrt{q}(\sqrt{q}+1)+1$, we see that this expression must be odd, so g is odd as well. Moreover, $g|2\sqrt{q}$, so $g|q$. Since q and q^2+q+1 are relatively prime, it follows that $g = 1$, so the two eigenvalues $q+1+\sqrt{q}$ and $q+1-\sqrt{q}$ are also relatively prime. Again applying part 2 of Lemma 2.6, we have that $\Phi(\widehat{G})$ contains subgroups isomorphic to $(\mathbb{Z}_{q+1+\sqrt{q}})^{q^2+q-1}$ and $(\mathbb{Z}_{q+1-\sqrt{q}})^{q^2+q-1}$. Thus, $\Phi(\widehat{G})$ contains a subgroup isomorphic to $(\mathbb{Z}_{q^2+q+1})^{q^2+q-1}$.

Now, we consider the eigenvalue $\lambda = 2q+2$ again. Because it is an integer, its minimal polynomial is $f(x) = x - (2q+2)$. In $\widehat{G}$, the number of vertices is $(q+1)(q^2+q+1)$, so we also have $r = (q+1)(q^2+q+1)$. Then, by Lemma 2.7, we have that $\Phi(\widehat{G})$ contains an element of order

$$\frac{2q+2}{\gcd(2q+2, (q+1)(q^2+q+1))} = \frac{2q+2}{q+1} = 2.$$

We now know several components of the desired critical group $\Phi(\widehat{G})$.

Recall from Kirchhoff's Matrix Tree Theorem that the order of $\Phi(\widehat{G})$ is the number of spanning trees in $\widehat{G}$ and can be calculated by the product of the nonzero eigenvalues divided by the number of vertices in $\widehat{G}$. That is,

$$\kappa(\widehat{G}) = \frac{(2q+2)^{q^3}(q+1+\sqrt{q})^{q^2+q}(q+1-\sqrt{q})^{q^2+q}}{(q+1)(q^2+q+1)} = 2(2q+2)^{q^3-1}(q^2+q+1)^{q^2+q-1}.$$

With the subgroups we have already determined, the order of the determined portion of $\Phi(\widehat{G})$ is precisely

$$2(2q+2)^{q^3-1}(q^2+q+1)^{q^2+q-1}.$$

Hence, the critical group is $\Phi(\widehat{G}) \cong \mathbb{Z}_2 \oplus (\mathbb{Z}_{q+1})^{q^3-1} \oplus (\mathbb{Z}_{q^2+q+1})^{q^2+q-1}$. □

5 Conclusion

The project primarily consisted of two aspects. The first was the development of an innovative approach to characterize the critical groups of the line graphs of regular bipartite graphs. This was done by determining the respective spectra of the Laplacians of the graph and its line graph. This approach took advantage of the bipartite nature of the graph and ultimately established a strong explicit relationship between between the two spectra. The second aspect of the project was the application of the spectral results to entirely classify previous unknown critical groups of line graphs. We demonstrated our approach's effectiveness by characterizing the line graphs of incidence graphs of finite projective planes of square order, a specific class of regular bipartite graphs. However, despite the progress in this project, we found that the eigenvalues alone may not be enough for us to expand our result to more general graphs. In particular, the next step in this research is to completely determine the critical group of the line graph for incidence graphs of finite planes with order equal to the odd power of a prime. This will generalize the result for all finite projective planes. We have a conjecture for this open problem.

Conjecture 5.1. *Let G be the incidence graph of a nondegenerate projective plane with* *order* $q = p^{2y+1}$*, where* y *is an integer and* p *is prime. Then the critical group of the line graph of* G *is isomorphic to* $\mathbb{Z}_2 \oplus (\mathbb{Z}_{q+1})^{q^3-1} \oplus (\mathbb{Z}_{q^2+q+1})^{q^2+q-1}$.

That is, we conjecture that the critical group takes the same form regardless of the order of the finite plane. With more work on the relationship between a graph's spectrum and its critical group, we hope our method will be extended to characterize the critical group of the line graph for all regular bipartite graphs.

6 Acknowledgments

Special thanks to Mr. David B Rush for introducing me to and mentoring me throughout the project, to Dr. Tanya Khovanova for her guidance and suggestions, to Dr. Richard Stanley for his valuable ideas and insight on the direction of the project, and to my tutor Dr. John Rickert for his consistent advice on writing and presenting. Thanks also to Dr. Pavel Etingof and Dr. David Jerison for supervising the research which was done at the MIT math department. This paper would not have been possible without the help of Mr. Adam Sealfon, Ms. Leigh Marie Braswell, Mr. Raj Raina, and Dr. Jenny Sendova for their thorough and thoughtful review of my paper. Finally, I express my gratitude to the Center for Excellence in Education, the Research Science Institute and its staff, the Massachusetts Institute of Technology, and my sponsors, the United States Department of Defense, Building Engineering and Science Talent (BEST), and the Kenneth and Myra Monfort Charitable Foundation Inc. for providing me the resources and opportunity to conduct this research.

References

[1] A. Berget, A.Manion, M. Maxwell, A. Potechin, and V. Reiner. The critical group of a line graph. *arXiv*, 0904.1246, 1991.

[2] J. Machacek. The critical group of a line graph: The bipartite case. *www-users.math.umn.edu/ reiner-/HonorsTheses/Machacek_thesis.pdf*, 2011.

[3] N. Biggs. Algebraic graph theory. *Cambridge mathematical library*, 1993.

[4] D. J. Lorenzini. A finite group atached to the laplacian of a graph. *Discrete Math*, 91:277–282, 1991.

[5] D. Cvetkovic, M. Dobb, and H. Sachs. Spectra of graphs: Theory and application. *Pure and Applied Mathematics*, 2009.

[6] D. J. Lorenzini. Smith normal form and laplacians. *www.math.uga.edu/ lorenz-/PaperAccepted.pdf*, 2007.

[7] S. Shirrell. On graphs arising from projective planes. *www.math.uga.edu/ lenny/-REU%20Summer%202009/StudentWriteUps/ShirrellWriteup.pdf*, 2009.