Research Science Institute (RSI)

Mathematics Section

Abstracts of Final Research Papers

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Leigh Marie Braswell

Angles of the Cookie Monster Problem

under the direction of Mr. Benjamin Iriarte

Abstract

The Cookie Monster Problem supposes that the Cookie Monster wants to empty a set S of jars filled with various numbers of cookies. On each of his moves, he may choose any subset of jars and take the same number of cookies from each of those jars. The *Cookie Monster number* of S, $CM(S)$, is the minimum number of moves the Cookie Monster must use to empty all of the jars. We find that for a set S of m jars containing $S = \{s_1, s_2, \ldots, s_m\}$ cookies to have $CM(S) < m$, the set S must satisfy some equation of the form $\sum_{i=1}^{m} a_i s_i = 0$ where $a_i \in \mathbb{Z}$. By modeling the problem with matrices, we recursively compute the equations which describe $CM(S) = n$ where S is any cookie sequence of length $n + 1$. We bound the number of these equations and describe some of their coefficients using hyperplanes. We find that using established techniques to determine whether a set S of m jars has $CM(S) < m$ is NP-hard. We also analyze a generating function and an algebra that models the Cookie Monster Problem.

Evan Chen

Diagrammatic Computation of Morphisms Between Bott-Samelson Bimodules via Libedinsky's Light Leaves

under the direction of Mr. Francisco Unda

Abstract

This paper considers left/right action-preserving morphisms between Bott-Samelson bimodules from a combinatorial perspective; specifically, using a diagrammatic representation. We examine the maps formed by a basis known as Libedinsky's light leaves, which constructs a map based on a string $\underline{r}$, composed of the letters s and t, and a binary string $\underline{b}$. The eventual goal is to find a formula for the composition of two such maps based on the two original binary strings. The paper provides a complete formula for the case $\underline{r} = sss\ldots$, and shows that the possible structures of maps when $\underline{r} = ststs\ldots$ are, in a certain sense, limited. We additionally provide a formula for the number of maps between two arbitrary Bott-Samelson bimodules.

Rumen Dangovski

On the Lower Central Series of PI-Algebras

under the direction of Mr. Nathan Harman

Abstract

In this paper we study the *lower central series* $\{L_i\}_{i\geq 1}$ of *algebras with polynomial identities*. More specifically, we investigate the properties of the quotients $N_i = M_i/M_{i+1}$ of successive ideals generated by the elements L_i. We give a complete description of the structure of these quotients for the free metabelian associative algebra $A/(A[A,A][A,A])$. With methods from *Polynomial Identities* theory, linear algebra and representation theory we also manage to explain some of the properties of larger classes of algebras satisfying polynomial identities.

Annie Hu

On the Number of Linear Extensions of Graphs

under the direction of Mr. Benjamin Iriarte

Abstract

Given a bipartite graph, let us pick an acyclic orientation of its edges. Then, if we consider the partially ordered set (poset) induced by this orientation, the number of linear extensions of such a poset is maximal whenever the orientation is bipartite, or such that no directed path of length two exists. We define a sequence of automorphisms that injectively but non-bijectively map the set of linear extensions of a nonbipartite orientation to the set of linear extensions of a bipartite orientation. Additionally, we discuss extending mappings to apply to simple odd cycle graphs and general nonbipartite graphs.

Rohil Prasad

Blocks and Formations: Bounding Sequence Extremal Functions

under the direction of Mr. Jesse Geneson

Abstract

This paper studies a particular type of sequence called an (r, s)-formation. An (r, s)-formation is a concatenation of s permutations on r distinct letters. We define $Form(u)$ for some pattern of letters u to be the smallest s such that every (r, s)-formation contains u for some r. In this paper, we expand on a previous technique for bounding $Form(u)$. We calculate $Form(u)$ explicitly for the $(c, k+1)$-formation consisting of k of the same permutation concatenated to its reverse. We also bound $Form(u)$ for k of the same permutation concatenated to k of its reverse, and for k of the same permutation with its reverse concatenated to it on both sides. In addition, we find a lower bound for the $Form(u)$ value for every (r, s)-formation in which each pair of adjacent permutations are in reverse order.

Raj Raina

Minimum Degrees of Minimal Ramsey Graphs

under the direction of Mr. Rik Sengupta

Abstract

For graphs F and H, we say F is Ramsey for H if every two-coloring of the edges of F contains a monochromatic copy of H. The graph F is H-minimal if the deletion of any edge or vertex of F results in a new graph F' not Ramsey for H. Burr, Erdős, and Lovász defined $s(H)$ to be the minimum degree of F over all Ramsey H-minimal graphs. Define $H_{t,d}$ to be a graph on $t+1$ vertices that contains the complete graph on t vertices and one additional vertex of degree d. The value of $s(H_{t,d})$ was known only for $d = t$ and very recently found for $d = 1$. We determine $s(H_{t,d})$ for all values $1 < d < t$. Next, we generalize results of Burr et. al. and Fox et. al. by determining sharp bounds on $s(H)$ where H is the complete equipartite graph. Finally, Szabó et. al. asked what graphs G have $s(G) = 2\delta(G) - 1$, where $\delta(G)$ is the minimum degree of G. We answer this question for a very large and general class of graphs. Throughout the paper, we use both classical and new combinatorial arguments. We note that these graphs are the first time $s(H)$ has been determined for several well-connected classes of graphs which are not vertex-transitive.

Sarah Shader

Weighted Catalan Numbers and Divisibility Properties

under the direction of Mr. Gaku Liu

Abstract

The weighted Catalan numbers, like the Catalan numbers, enumerate mathematical objects. For example, the number of Morse links with n critical points is the n^{th} weighted Catalan number with weights $1^2, 3^2, 5^2, \ldots, (2n+1)^2, \ldots$, denoted L_n. However, they typically do not have a closed form for the n^{th} term, so we must prove their divisibility properties combinatorially. In this paper, we produce recurrences for the number of orbits of size 1 and size 3 under a group action on binary trees as well as the weight of all orbits of size 1 with particular weights under this group action. This allows us to investigate the weighted Catalan numbers modulo 3. We show that L_n is periodic modulo 3^r with period $2^p \cdot 3^q$ for some positive integer q and $p \in \{0, 1\}$. We also provide conditions for b so that $C_n^{(b)}$ is periodic modulo p^k for a prime p with a period of $(p-1)^x \cdot p^y$ for a positive integer y and $x \in \{0, 1\}$.

Jessica Shi

The Speeds of Families of Intersection Graphs

under the direction of Mr. Francisco Unda

Abstract

We explore the bounds on the number of intersection graphs on n vertices of various families, including systems of parabolas, conic sections, polynomials, and rational functions. For each system we establish a set of polynomials whose sign patterns give an intersection graph. We use Warren's Theorem to obtain an upper bound on the number of sign patterns of this set of polynomials and, as such, on the number of intersection graphs of the system. Next, we use a constructive approach to calculate the lower bounds of the number of intersection graphs. In general, the bounds on the intersection graphs of these systems is $n^{fn(1+o(1))}$, where f is the degree of freedom. The lower bounds confirm the coefficient of the exponent, f.

Albert Soh

Crossing Numbers on the Disk with Multiple Holes

under the direction of Mr. Gaku Liu

Abstract

In this paper we look at the crossing number, pair crossing number, and odd crossing number of graphs on a disk with multiple holes. We provide an upper bound $\left\lfloor \frac{(n-1)^2}{4} \right\rfloor$ for the odd crossing number of graphs on the annulus with n edges. For graphs on a disk with multiple holes, we develop polynomial time algorithms to find the number of crossings between two edges, given their homotopy classes. We describe each edge as a word referring to its homotopy class. The algorithms also proved the crossing number, pair crossing number, and odd crossing number for graphs on a disk with multiple holes.

Bertrand Stone

Characterization of the Line Complexity of Cellular Automata Generated by Polynomial Transition Rules

under the direction of Mr. Chiheon Kim

Abstract

Cellular automata are discrete dynamical systems which consist of changing patterns of symbols on a grid. The changes are specified in such a way that the symbol in a given position is determined by the symbols surrounding that position in the previous state. Despite the simplicity of their definition, cellular automata have been applied in the simulation of complex phenomena as disparate as biological systems and universal computers. In this paper, we investigate the line complexity $a_T(k)$, or number of accessible coefficient blocks of length k, for cellular automata arising from a polynomial transition rule T. We first derive recursion formulas for the sequence $a_T(k)$ associated to polynomials of the form $1 + x + x^n$ where $n \geq 3$ and the coefficients are taken modulo 2. We then derive functional relations for the generating functions associated to these polynomials. Extending to a more general setting, we investigate the asymptotics of $a_T(k)$ by considering a generating function ϕ defined by a power series whose coefficients are translates of $a_T(k)$. We assume that ϕ satisfies a certain general functional equation relating $\phi(z)$ and $\phi(z^p)$ for some prime p. We show that for certain subsequences s_k which are dependent upon a real number $x \in [1/p, 1]$, the ratio $\alpha(s_k)/s_k^2$ tends to a piecewise quadratic function of x.

Nickolay Stoyanov

On Lower Central Series of the r,q-polynomial algebra

under the direction of Mr. Nathan Harman

Abstract

We study the lower central series of an associative algebra, defined as follows: $L_1 = A$, $L_{i+1} = [L_i, A]$, where $[\ ,\ ]$ is the bilinear Lie bracket operation. We look at the successive quotients $B_i = L_i/L_{i+1}$ and $N_i = M_i/M_{i+1}$, where M_i is the two-sided ideal generated by L_i. We aim to study the decomposition of N_i and B_i into free and torsion components using the structure theorem of finitely generated abelian groups. Using the computational algebra system *Magma* we gather lots of data and observe and prove various interesting patterns about these ranks and torsion. We mainly concentrate on the algebra $\mathbb{Z}\langle x, y\rangle/(qxy - ryx)$ where $(q, r) = 1$, also known as the $q, r-$ polynomial algebra. We completely describe and prove the pattern of N_i and B_i for this algebra. We give some conjectures for algebra $\mathbb{Z}\langle x, y\rangle/((f_1), (f_2))$ where f_1 and f_2 are two homogeneous polynomials of degree two and three.