On Binary Formations and Sequence Extremal Functions

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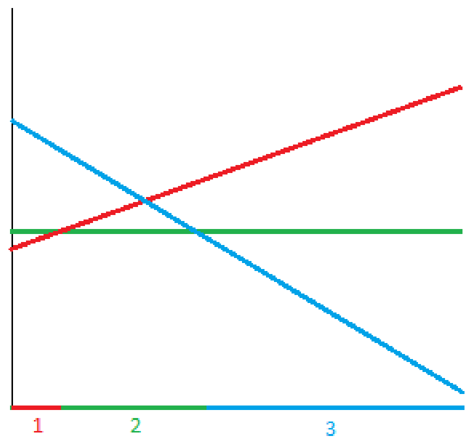
Abstract

We calculate the formation width for a large class of sequences. In some cases, we use the formation width to improve bounds on $Ex(u, n)$ for these sequences. We introduce a new algorithm for calculating the formation width, determine its approximate runtime and use it to find classes of sequences with tight bounds on $Ex(u, n)$. We also prove a Ramsey-type result on sparse sequences.

1 Introduction

In 1965, Harold Davenport and Andrzej Schinzel [3] introduced a type of sequence known as a *Davenport-Schinzel sequence*. The definition of these sequences is motivated by examining the lower envelope of a set of polynomial functions.

In general, the *lower envelope* of a set of real-valued functions $\{f_1, f_2 \dots f_n\}$ is simply the function $L(x) = \min_i(f_i(x))$, which assigns to a real x the index i of the function $f_i(x)$ taking the minimum value out of all the functions in the set at that point.



These lower envelopes can be better understood pictorially. A simple example for a set of three linear functions shown above, with the values of $L(x)$ at each point shown on the x-axis. The lower envelope can be viewed as a partition of the real line into discrete intervals, each with a natural number associated to it. In addition, the boundaries between each of these intervals corresponds to an intersection of two of the underlying functions. Therefore, the values of these intervals form a sequence of natural numbers that can give us information about the underlying set of functions (intersections) without any knowledge of the actual set.

It is easy to analyse the lower envelope of a set of polynomial functions that all have the same degree s. Any two of these polynomials can intersect at most s times. As was noted earlier, the sequence that can be derived from the lower envelope encodes an intersection between function f_i and f_j as the subsequence ij or ji. Therefore, if we are considering the lower envelope of polynomial functions, the associated sequence cannot contain ij or ji repeated $s+1$ times as a subsequence.

Thus, the geometric problem of studying lower envelopes can be approached combinatorially by analysis of these associated *Davenport-Schinzel* sequences.

Formally, a *Davenport-Schinzel sequence* is a sequence S on a finite alphabet of letters satisfying the following two conditions:

1. S does not contain the alternation $abab\ldots$ of length $s+2$ as a subsequence, where a and b are any two distinct letters of its alphabet.

2. No two adjacent letters in S are the same.

Astonishingly, results on Davenport-Schinzel sequences have a variety of applications in computational geometry besides lower envelope analysis. Several are detailed in a book written by Agarwal and Sharir [2].

Then, in 1992, Adamec, Klazar, and Valtr [11] considered a generalized version of Davenport-Schinzel sequences, which have turned out to be an even deeper and more interesting object of study with numerous applications in other areas. Instead of looking at ordinary Davenport-Schinzel sequences which avoided alternations, they considered sequences that avoided an arbitrary fixed subsequence u.

Before we define these generalized sequences rigorously, we need a concrete definition of what 'avoid' means. Recall that for ordinary Davenport-Schinzel sequences, the sequence is required to not have a sequence of the form $abab\ldots$ as a subsequence, where a and b are *any* two distinct letters in the alphabet of the sequence. Thus, in addition to not containing $abab\ldots$, our sequence also cannot contain subsequences such as $cdcd\ldots$ or $xyxy\ldots$. However, these sequences all have the same basic structure. For example, we can get from $abab\ldots$ by replacing all the a's with c's and all the b's with d'. In general, we say two sequences are *isomorphic* when one can be transformed into the other via a renaming of its alphabet. With this notion of sequence isomorphism, we can now rigorously define generalized Davenport-Schiznel sequences.

Given some fixed sequence of letters u over an alphabet of size r, a *generalized Davenport-Schinzel sequence* is a sequence $S(u)$ satisfying the following two conditions:

1. S does not contain any sequence isomorphic to u as a subsequence.

2. r-sparsity: No two letters in $S(u)$ within r of each other are the same.

Both of these conditions are a natural generalization of the two conditions used to define regular Davenport-Schinzel sequences.

The main problem in this field is to determine the maximum lengths of generalized Davenport-Schinzel sequences. The function $Ex(u, n)$ denotes the maximum length of a generalized Davenport-Schinzel sequence on an alphabet of n letters avoiding the sequence u. Bounds on the value of $Ex(u, n)$ have been used in a variety of situations, including bounding the complexity of faces in arc arrangements [2] and tightening bounds on the complexity of double-ended queue operations on splay trees [7].

Another application of generalized Davenport-Schinzel sequences is in the study of simple k-quasiplanar topological graphs. Fox. et al. [4] found that the bound on the number of edges in a simple k-quasiplanar graph with n vertices depends on the value of $Ex((a_1a_2\ldots a_c)^t, n)$, where $(a_1a_2\ldots a_c)^t$ denotes $a_1a_2\ldots a_c$ repeated t times.

In Section 2 we introduce notation and basic results used in the paper. In Section 3, we find the formation width of a sequence and show its application in bounds on quasiplanar graphs. In Section 4, we find the formation width of any sequence on two distinct letters. In Section 5, we partially characterize how inserting or concatenating a single letter onto a sequence affects its formation width. We also make a conjecture that, if assumed true, gives a better upper bound on the formation width of any sequence. In Section 6, we discuss a new method for finding a lower bound on the formation width of a sequence. In Section 7, we calculate and bound the formation width for a variety of sequences. In Section 8, we construct an algorithm for calculating the formation width and determine its approximate runtime. We also characterize some large classes of sequences with tight bounds on $Ex(u, n)$. In Section 9, we prove a Ramsey-like result for sparse sequences. In Section 10, we propose some open problems.

2 Preliminaries

The following are concepts and notation used in the Results section.

Recall our standard definitions. A sequence is *r-sparse* if no set of r consecutive letters contain two equal letters. A sequence is said to be *isomorphic* to u if it can be transformed to u by some renaming of its alphabet. A sequence *contains* u when it has some subsequence isomorphic to u, and *avoids* u if it has no such subsequence. The function $Ex(u, n)$ denotes the maximum length of an r-sparse sequence on an alphabet of n distinct letters avoiding a pattern u.

For the purpose of brevity, let I_c be the increasing sequence on c letters, denoted by either $a_1a_2\dots a_c$ or $12\dots c$. Likewise, D_c is the decreasing sequence on c letters, denoted by either $a_ca_{c-1}\dots a_1$ or $c(c-1)\dots 1$. We refer to $alt(l, t)$ as a concatenation of t permutations, alternating between I_l and D_l. For example, $up(3, 3) = a_1a_2a_3a_1a_2a_3a_1a_2a_3$ and $alt(3, 3) = a_1a_2a_3a_3a_2a_1a_1a_2a_3$. For any sequence S, S^k denotes S repeated k times.

In this paper, permutations on sequences permute the *alphabet* of the sequence, not the sequence itself. For example, $123321 \to 312231$ is achieved by permuting 1 to 3, 2 to 1, and 3 to 2. Given a permutation π, denote the sequences $\pi(1)\pi(2)\dots\pi(c)$ and $\pi(c)\pi(c-1)\dots\pi(1)$ as I_π and D_π respectively.

Definition 2.1. *An (r, s)-formation is a concatenation of s permutations on r letters.*

For example, the sequence $(abc)(acb)$ is a $(3, 2)$-formation, while

$$(adbc)(acbd)(abcd)(dcba)(dcab)$$

is a $(4, 5)$-formation.

Definition 2.2. *For a pattern u, $fw(u)$ is defined as the smallest s for which there exists r such that every (r, s)-formation contains u.*

As a simple example, consider $u = abab$. It is clear that for any c, the $(c, 2)$-formation $(12\dots c)(c(c-1)\dots 1)$ avoids $abab$. Therefore, $fw(abab) \geq 3$. We can then check every $(2, 3)$-formation and find that they all contain $abab$ under some permutation, therefore $fw(abab) = 3$.

A *binary formation* is a formation with every permutation being either I_c or D_c. The following lemma by Geneson et al. relates binary formations to general formations.

Lemma 2.3. *[1] There exists a function $\gamma(r, s)$ such that every $(\gamma(r, s), s)$-formation contains a binary (r, s)-formation.*

By this lemma, it suffices to show that only every *binary* (r, s)-formation contains u and that there exists some binary $(r, s-1)$-formation that avoids u in order to prove $fw(u) = s$.

2.1 Bounding $Ex(u, n)$ using (r, s)-formations

We denote by $F_{r,s}(n)$ the maximal length of a sequence on n distinct letters that avoids every (r, s)-formation. Given a pattern u on r letters, we define $Ex_c(u, n)$ to be the length of the longest c-sparse sequence on n letters that avoids a pattern u for some fixed $c \geq r$.

Lemma 2.4. *[5] If u is a pattern with at most r distinct letters, then $Ex_d(u, n) \leq Ex_c(u, n) \leq (1 + Ex_c(u, d-1))Ex_d(u, n)$ for all $n \geq 1$ and $d \geq c \geq r$.*

Lemma 2.4 directly implies Lemma 2.5.

Lemma 2.5. *[1] For any pattern u with r letters and fixed $c \geq r$, $Ex_c(u, n) = O(F_{r,fw(u)}(n))$.*

Lemma 2.6. *[6] For $s \geq 4$ and $t = \lfloor (s-3)/2 \rfloor$,*

$$F_{r,s}(n) = \begin{cases} n \cdot 2^{(1/t!)\alpha(n)^t + O(\alpha(n)^{t-1})} & : s \text{ is even} \\ n \cdot 2^{(1/t!)\alpha(n)^t \log_2 \alpha(n) + O(\alpha(n)^t)} & : s \text{ is odd.} \end{cases}$$

Using Lemma 2.5, set $c = r$ and get $Ex(u, n) = O(F_{r,fw(u)}(n))$. Using the bounds in Lemma 2.6 then gives an upper bound on $Ex(u, n)$ that depends on the value of $fw(u)$.

2.2 The $\ell(u)$ Function

Let $\ell(u)$ be the smallest k such that I_c^k contains u, where u has c distinct letters. Geneson et al. [1] used $\ell(u)$ to find a lower bound for $fw(u)$. Since I_c^k is a binary (c, k)-formation, it follows immediately that $fw(u) \geq \ell(u)$. In addition, for a permutation π, define $\ell_\pi(u)$ to be the smallest k such that I_c^k has u under the permutation π as a subsequence.

Lemma 2.7. *[1] For any permutation π, $\ell_\pi(I_c) + \ell_\pi(D_c) = c + 1$.*

Lemma 2.7 also implies $\ell(I_c D_c) = c + 1$. We use this lemma in several of the subsequent results.

3 $fw(I_c^k)$ and k-Quasiplanar Graphs

We prove $fw(I_c^k) = 2k-1$. We show its use in tightening the bounds on the number of edges in simple k-quasiplanar topological graphs.

Theorem 3.1. *$fw(I_c^k) = 2k - 1$.*

Proof. By Theorem 6.2, $fw(I_c^k) \geq 2k - 1$.

Every binary $(c, 2k-1)$-formation contains exactly two types of permutations. Thus, it contains $\geq k$ copies of a single permutation, so I_c^k or D_c^k is a subsequence of every binary $(c, 2k-1)$-formation and $fw(I_c^k) \leq 2k-1$. We combine these two bounds to get $fw(I_c^k) = 2k - 1$ as desired. □

Geneson et al. [1] used this result along with Lemma 2.5 and 2.6 to show $Ex(I_c^k, n) = n2^{\frac{1}{(k-2)!}\alpha(n)^{t-2} \pm O(\alpha(n)^{t-3})}$. The previous bound on $Ex(I_c^k, n)$ used in [4] was $nc2^{ck-3}(10c)^{10\alpha(n)^{ck}}$, proven in [5].

Lemma 3.2. *[4] If $Ex(I_c^{2^{k^2+k}}, n) = O(nf_k(n))$ for some function $f_k(n)$, then there are $O((n \log n)f_k(n))$ edges in a simple k-quasiplanar graph on n vertices.*

Lemma 3.3. *[1] There are $O((n \log n)2^{\frac{1}{(2^k-2)!}\alpha(n)^{2^k-2} - O(\alpha(n)^{2^k-3})}$ edges in a simple k-quasiplanar topological graph on n vertices.*

Lemma 3.3 follows immediately from substituting the improved bound on $Ex(I_c^k, n)$ into Lemma 3.2.

4 Two letter patterns

We find $fw(u)$ for any pattern u on an alphabet of 2 letters.

Theorem 4.1. *If u is a pattern of length s composed of two distinct letters, then $fw(u) = s - 1$.*

Proof. It suffices to prove this lemma for sequences with different first and second letters. The upper bound follows since every $(2, t-1)$-formation contains u. For the lower bound we construct a $(2, t-1)$-formation $f(u)$ which only contains copies of u for which the last letter of the copy of u is the last letter of $f(u)$. Therefore the $(2, t-2)$-formation in the first $t-2$ permutations of $f(u)$ avoids u.

For each sequence u with two distinct letters and different first and second letters, the first permutation of $f(u)$ is ab. If the first i permutations of $f(u)$ are defined for $i < t-1$, then permutation $i+1$ of $f(u)$ is the same as permutation i if and only if letters $i+1$ and $i+2$ of u are the same. Let u' denote the sequence obtained by deleting the last letter of u and suppose u has letters x and y. We prove that $f(u)$ contains only copies of u for which the last letter of the copy of u is the last letter of $f(u)$ by induction on the length of u.

Since $f(xy) = ab$, then $f(xy)$ contains exactly one copy of the sequence xy and the last letter of the copy of xy is the last letter of $f(xy)$. Suppose by inductive hypothesis that $f(u')$ contains only copies of u' for which the last letter of the copy of u' is the last letter of $f(u')$. If the last two letters of u are the same, then the first letter of the last permutation of $f(u)$ is different from the last letter of $f(u')$, so the last letter of $f(u)$ will be the last letter of any copy of u in $f(u)$. If the last two letters of u are different, then the first letter of the last permutation of $f(u)$ is the same as the last letter of $f(u')$, so the last letter of $f(u)$ will be the last letter of any copy of u in $f(u)$.

If u has the same first and second letters, then we can use Lemma 5.1 and find $fw(u) = fw(u') + 1$, where u' is the pattern created by removing the first letter of u. □

We note that any binary (c, k)-formation must contain a $(2, k)$-formation which has a formation width of $2k-1$ and find the following corollary.

Corollary 4.2. *For any binary (c, k)-formation u, $fw(u) \geq 2k-1$.*

5 Letter insertion

We examine the change in $fw(u)$ upon insertion or concatenation of a single letter a.

Lemma 5.1. *If u is a sequence beginning with the letter a, then $fw(au) = fw(u)+1$.*

Proof. Assume u has c distinct letters. Let $f(u)$ be any $(c, fw(u))$ formation such that the first $fw(u)-1$ permutations of $f(u)$ do not contain u. We will show that $f(u)$ avoids au. Assume to the contrary that it contains au. Since the first and second letters of au are the same, they are in different permutations. Therefore, u is contained in the last $fw(u)-1$ permutations of $f(u)$, so we arrive at a contradiction and $fw(au) \geq fw(u)+1$. In addition, any binary $(c, fw(u)+1)$-formation contains au since a can be found in the first permutation and u under some permutation can be found as a subsequence of the remaining $fw(u)$ permutations, so $fw(au) = fw(u)+1$. □

Lemma 5.2. *If u' is the sequence created by inserting the letter a into a subsequence u with r distinct letters, then $fw(u') \leq fw(u) + \lfloor \frac{r}{2} \rfloor + 1$.*

Proof. Any binary $(c, fw(u) + \lfloor \frac{r}{2} \rfloor + 1)$- formation can be constructed by inserting an arbitrary binary $(c, \lfloor \frac{r}{2} \rfloor + 1)$-formation into a binary $(c, fw(u))$-formation $f(u)$. Assume u is a subsequence of $f(u)$ under some permutation π. Consider the sequence u'. Let the letter a be inserted between the letters x and y.

If x and y occur in different permutations in $f(u)$, then the formation created by inserting any permutation between their occurrences in $f(u)$ contains u'.

Assume that the k_1 letters immediately to the left of a and the k_2 letters immediately to the right of a all occur in the same permutation of $f(u)$. If $k_1 \geq k_2$, then the formation created by inserting any arbitrary binary $(c, \lfloor \frac{r}{2} \rfloor + 1)$-formation to the right of this permutation will contain u' since the inserted formation must contain a and the $k_2 \leq \lfloor \frac{r}{2} \rfloor$ letters to its right. If $k_1 \leq k_2$, we insert the formation to the left of the permutation instead.

Therefore, $fw(u') \leq fw(u) + \lfloor \frac{r}{2} \rfloor + 1$. □

6 The $r(u)$ function

Recall that I_c is the sequence $a_1 a_2 \cdots a_c$ and D_c is the sequence $a_c a_{c-1} \cdots a_1$. Let $r(u)$ be the smallest k such that $alt(c, k)$ contains u, where u is a pattern with c distinct letters.

Lemma 6.1. $fw(u) \geq r(u)$.

Proof. Assume for the sake of contradiction that $fw(u) < r(u)$ and that, without loss of generality, u has c distinct letters. Since $alt(c, fw(u))$ is a binary formation, $alt(c, fw(u))$ contains u. However, $fw(u) < r(u)$ and $r(u)$ is defined as the *minimum* k such that $alt(c, k)$ contains u, so we arrive at a contradiction and $fw(u) \geq r(u)$. □

By Lemma 6.1 $r(u)$ gives a lower bound on $fw(u)$. We find $r(u)$ for a binary formation u.

Lemma 6.2. $r(I_c^{e_1} D_c^{e_2} I_c^{e_3} \cdots \mathcal{L}^{e_n}) = 2\sum_{i=1}^{n} e_i - n$.

Proof. First we show that $r(I_c^{e_1}) = 2e_1 - 1$. We also show the *last letter condition*, namely that $alt(c, r(I_c^{e_1}))$ contains $I_\pi^{e_1}$ as a subsequence only if $\pi(c) = c$. We define $\pi_r(alt(c,k))$ to be the (c,k)-formation $D_c I_c D_c \ldots$.

We proceed by induction on e_1. The base case $r(I_c) = 1$ is evident by the definition of $r(u)$. In addition, I_π is only a subsequence of I_c if π is the identity permutation, therefore $\pi(c) = c$. Assume that $r(I_c^{e_1}) = 2e_1 - 1$ and that the last letter condition holds for this k. We claim that $r(I_c^{e_1+1}) = 2e_1 + 1$ and that the last letter condition also holds for $e_1 + 1$.

Let π be an arbitrary permutation. We will first show $I_\pi^{e_1+1}$ is not a subsequence of $alt(c, 2e_1)$. If $\pi(c) = c$, then the last letter of $alt(c, 2e_1 - 1)$ corresponds to to the last letter of $I_\pi^{e_1}$. In order for $I_\pi^{e_1+1}$ to be a subsequence of $alt(c, 2e_1 + 1)$ under these conditions, D_c must contain a copy of I_π as a subsequence. However, c is the first letter of D_c and c is the last letter of I_π, so I_π is not a subsequence of D_c and consequently $I_\pi^{e_1+1}$ is not a subsequence of $alt(c, 2e_1 + 1)$.

Assume $\pi(c) = i$ for some $1 \leq i < c$, and assume for the sake of contradiction that $I_\pi^{e_1+1}$ is a subsequence of $alt(c, 2e_1 + 1)$. The last letter condition does not hold, so the last letter of $I_\pi^{e_1}$ has a leftmost occurence in the last permutation of $alt(c, 2e_1)$. The sequence I_π must be a subsequence of the remaining letters. The last letter of $I_\pi^{e_1}$ is i. Since the last letter of the remaining I_π is also i, then this letter occurs in the last permutation of $alt(c, 2e_1 + 1)$. There are exactly $i - 1$ distinct letters between these two i's. The remaining $c - 1$ letters of the final I_π cannot be a subsequence of this, so $I_\pi^{e_1+1}$ is not a subsequence of $alt(c, 2e_1 + 1)$. Therefore, $r(I_c^{e_1}) = 2e_1 - 1$, and $I_\pi^{e_1}$ satisfies the last letter condition.

By an identical argument, we find that $D_\pi^{e_1}$ is contained in $\pi_r(alt(c, 2e_1 - 1))$ and satisfies the last letter condition.

We prove the claim that $r(I_c^{e_1} D_c^{e_2} I_c^{e_3} \cdots \mathcal{L}^{e_n}) \leq 2\sum_{i=1}^{n} e_i - n$. We proceed by induction. The base case for $n = 1$ has been proven above. Assume $r(I_c^{e_1} D_c^{e_2} I_c^{e_3} \cdots \mathcal{L}^{e_k}) = 2\sum_{i=1}^{k} e_i - k$ for some k. We get $r(I_c^{e_1} D_c^{e_2} I_c^{e_3} \cdots \mathcal{L}^{e_{k+1}}$ by adding $I_c^{e_{k+1}}$ if k is even and $D_c^{e_{k+1}}$ if k is odd.

In either case, the first letter of the $I_c^{e_{k+1}}$ or $D_c^{e_{k+1}}$ is equal to the last letter of the

$D_c^{e_k}$ or $I_c^{e_k}$, respectively, so they occur in separate permutations. If $\mathcal{L} = I_c$, then since $r(I_c^{e_{k+1}}) = 2e_{k+1} - 1$, we have $r(I_c^{e_1} D_c^{e_2} I_c^{e_3} \cdots \mathcal{L}^{e_{k+1}}) \leq r(I_c^{e_1} D_c^{e_2} I_c^{e_3} \cdots \mathcal{L}^{e_k}) + 2e_{k+1} - 1$, which simplifies to $2\sum_{i=1}^{k+1} e_i - k - 1$ as desired. The case for $\mathcal{L} = D_c$ is similar.

We now prove the lower bound. Let π be an arbitrary permutation. For all permutations π, $I_\pi^{e_1}$ is a subsequence of $alt(c, k)$ only for some $k \geq 2e_1 - 1$. Similarly, $D_\pi^{e_2}$ is a subsequence of $\pi_r(alt(c, k))$ for some $k \geq 2e_2 - 1$. We can iterate this from $i = 1$ to $i = k$, adding all the expressions to get $r(I_c^{e_1} D_c^{e_2} I_c^{e_3} \cdots \mathcal{L}^{e_k}) \geq 2\sum_{i=1}^{k} e_i - k$. The upper and lower bounds are equal, so $r(I_c^{e_1} D_c^{e_2} I_c^{e_3} \cdots \mathcal{L}^{e_k}) = 2\sum_{i=1}^{k} e_i - k$.

□

We compare when $r(u)$ gives a better lower bound that $\ell(u)$ on $fw(u)$. Given some binary formation, let A and B be the number of permutations that are equal to I_c and D_c respectively in the binary formation. Geneson et al. [1] found that $\ell(I_c^{e_1} D_c^{e_2} I_c^{e_3} \cdots \mathcal{L}^{e_n}) = (c-1)m + M + \lfloor \frac{n}{2} \rfloor$, where $m = \min(A, B)$ and $M = \max(A, B)$. Qualitatively, we find that $r(u)$ gives a better lower bound on binary formations composed mostly of copies of one permutation.

Lemma 6.3. *For a binary formation u, $r(u) \geq l(u)$ iff $M - (c-3)m \geq n + \lfloor \frac{n}{2} \rfloor$.*

Proof. $r(u) \geq l(u)$ whenever $2\sum_{i=1}^{n} e_i - n \geq (c-1)m + M + \lfloor \frac{n}{2} \rfloor$. Since $\sum_{i=1}^{n} e_i = m + M$, we can simplify the expression to obtain the condition that $M - (c-3)m \geq n + \lfloor \frac{n}{2} \rfloor$. □

We extend Lemma 2.7 to a large class of sequences u. We call a *greedy monotonic partition* of a pattern u on c distinct letters with length s under a permutation π to be a partitioning of $\pi(u) = a_1 a_2 \ldots a_s$ (where $a_i \in \{1, 2 \ldots c\} \forall i$) into sets of the form $\{a_i, a_{i+1} \ldots a_j\}$. The letters in each set are in monotonic order under some well-defined ordering of the alphabet of u. In addition, each of these intervals is greedy, so if $[x_i, x_j]$ is monotonically increasing, then $x_{j+1} \leq x_j$, with a similar definition for monotonically decreasing.

Lemma 6.4. *Let u be a sequence on the alphabet $\{1, 2, \ldots c\}$ with length s under some permutation π. If u_r is the pattern obtained by reversing u and the order on the alphabet is $1 \leq 2 \leq \ldots \leq c$, and the monotonic greedy partition of $\pi(u)$ consists solely of increasing sets, then $\ell_\pi(u) + \ell_\pi(u_r) = s + 1$.*

Proof. Assume $\ell_\pi(u) = k$, so there are k increasing sets in the greedy partition. This is equivalent to there being k decreasing sets in the greedy partition of u_r. The last letter of set i and the first letter of set $i+1$, for $1 \leq i \leq k-1$ form exactly $k-1$ disjoint pairs where the letter x is greater than the letter y to its left. By the structure of the partition, we can see that there are no other such pairs. Each letter not in one of these pairs will occur in its own I_c and each pair will occur in its own I_c, so $\ell_\pi(u_r) = (k-1) + (s - 2(k-1)) = s - k + 1$ and $\ell_\pi(u) + \ell_\pi(u_r) = s + 1$. □

7 Formation width of binary formations

This section contains results on the value of $fw(u)$ for some binary formations u. We first prove a general lower bound for $fw(alt(c,k))$ and compute exact values for small k.

Lemma 7.1. $fw(I_cD_cI_c) = c + 3$

Proof. We claim that the binary $(c, c+2)$-formation $I_c^cD_c^2$ avoids $I_cD_cI_c$. Assume for the sake of contradiction that $I_c^cD_c^2$ contains $I_\pi D_\pi I_\pi$ as a subsequence under some permutation π. From [1], $\ell(I_cD_c) = c + 1$. Therefore, the last letter of the D_π in $alt(c,3)$ must occur in the first D_c in $I_c^cD_c^2$. However, the letter after D_π is the same, so it must occur in a different permutation, namely the last D_c of $I_c^cD_c^2$. There are c letters in D_c to fit the c letters in the last I_c of the $I_cD_cI_c$, so π must rename $I_cD_cI_c$ to $D_cI_cD_c$. We can see instantly that $I_c^cD_c^2$ does not contain $D_cI_cD_c$ as a subsequence.

We prove that every binary $(c, c+3)$-formation contains $I_cD_cI_c$. For any formation that is not a string of I_c's followed by a string of D_c's, it obviously contains $I_cD_cI_c$. Thus we are only concerned with the binary formation $I_c^aD_c^b$ where $a + b = c + 3$. We find that using the permutation π mapping $12\ldots c$ to $1\ldots(b-1)c\ldots b$, $I_c^aD_c^b$ contains $I_\pi D_\pi I_\pi$ as a subsequence. □

Lemma 7.2 and Corollary 7.3 are central to the proofs of Lemma 7.4 and Theorem 7.5.

Lemma 7.2. *$I_\pi D_\pi$ is a subsequence of $I_c^cD_c$ iff $\pi(1) < \pi(2)$.*

Proof. Let π be a permutation.

By Lemma 2.7, the last letter of $I_\pi D_\pi$, namely $\pi(1)$, occurs in the last D_c of $I_c^c D_c$. If it not the only letter of $I_\pi D_\pi$ occurring in that last D_c, then $\pi(2)\pi(1)$ is a subsequence of this D_c. This is possible iff $\pi(1) < \pi(2)$.

Assume that the final D_c only contains $\pi(1)$. If $\pi(1) > \pi(2)$, this is impossible since its adjacent $\pi(2)$ occurs in some I_c, and $\pi(1)$ can then fit in the same I_c, implying that $I_\pi D_\pi$ is a subsequence of I_c^c, which is impossible. However, this is possible if $\pi(1) < \pi(2)$. Therefore, $I_\pi D_\pi$ is a subsequence of $I_c^c D_c$ iff $\pi(1) < \pi(2)$. □

Corollary 7.3. *$I_\pi D_\pi$ is a subsequence of $D_c I_c^c$ iff $\pi(2) < \pi(1)$.*

Proof. From Lemma 7.2, we can reverse the sequences to get that $D_\pi I_\pi$ is a subsequence of $I_c D_c^c$ iff $\pi(2) < \pi(1)$. Therefore, $\pi_r(D_\pi I_\pi)$ is a subsequence of $\pi_r(I_c D_c^c)$ iff $\pi(2) < \pi(1)$. Simplifying, $I_\pi D_\pi$ is a subsequence of $D_c I_c^c$ iff $\pi(2) < \pi(1)$. □

Using Lemma 7.2 and Corollary 7.3 we calculate $fw(alt(c, 4))$.

Lemma 7.4. *$fw(I_c D_c I_c D_c) = 2c + 3$.*

Proof. We have $c + fw(I_c D_c I_c) \geq fw(I_c D_c I_c D_c)$ so $2c + 3 \geq fw(I_c D_c I_c D_c)$. In addition, the $(c, 2c+2)$ formation $F = I_c^c D_c^2 I_c^c$ avoids $I_\pi D_\pi I_\pi D_\pi$ for all permutations π.

First assume that $\pi(1) < \pi(2)$. The first $I_c^c D_c$ of F avoids $I_\pi D_\pi$, therefore the last $I_\pi D_\pi$ be a subsequence of the remaining I_c^c, which is impossible since $\ell(I_c D_c) = c+1$.

The proof is similar for $\pi(1) > \pi(2)$. □

We extend the technique used in the proof of Lemma 7.4 to bound $alt(c, k)$ below for general c and k.

Theorem 7.5. *$fw(alt(c, 2k)) \geq k(c + 2) - 1$ and $fw(alt(c, 2k + 1)) \geq k(c + 2) + 1$.*

Proof. We claim that the $(c, k(c + 2) - 2)$-formation $S_{2k} = I_c^c D_c^2 I_c^c \dots I_c^c$ avoids $alt(c, 2k)$. We proceed by induction. The base cases have already been proven in Lemma 7.4 for $k = 2$.

Assume S_{2j} avoids $alt(c, 2j)$ for all $j \leq \lfloor \frac{k}{2} \rfloor$. Also assume for the sake of contradiction that S_{2k+2} contains $alt(c, 2k+2)$ under some permutation π as a subsequence. Let G be $alt(c, 2k+2)$ without the rightmost two permutations. Then $G = alt(c, 2k)$. The leftmost $(c, k(c+2)-2)$-formation $I_c^c D_c^2 I_c^c \dots I_c^c$, or S_{2k+2} without the rightmost

$D_c^2 I_c^c$, avoids $alt(c, 2k)$. Therefore, the last letter of G must occur somewhere in the rightmost $D_c^2 I_c^c$ of S_{2k+2}. In addition, the letter directly after G in $alt(c, 2k+2)$ is the same as the last letter of G, so it must be found at least one permutation to the right of where the last letter of G occurs. Thus, if the last letter of G occurs in anywhere but the first D_c of $D_c^2 I_c^c$, this means we must have $I_\pi D_\pi$ as a subsequence of some subsequence of I_c^c, which is impossible since $\ell(I_c D_c) = c+1$ by [1]. Thus, the last letter of G occurs in the first D_c of $D_c^2 I_c^c$, so $I_\pi D_\pi$ must be a subsequence of $D_c I_c^c$. Using Lemma 7.3, we can see that $\pi(2) < \pi(1)$.

Consider the $I_\pi D_\pi$ in $alt(c, 2k+2)$ directly to the left of the rightmost $I_\pi D_\pi$ and the rightmost $c+4$ permutations of S_{2k+2}, namely $D_c^2 I_c^c D_c^2 I_c^c$. This new $I_\pi D_\pi$ occurs at its rightmost in the leftmost $D_c I_c^c$ of these $c+4$ permutations since $\pi(2) < \pi(1)$ and otherwise it would be contained in $I_c^c D_c$. We can iterate this for every block of $I_\pi D_\pi$ in $alt(c, 2k+2)$ to show that at their rightmost, every block is contained in $D_c I_c^c$ and the D_c directly to the left of that is unused, so each block uses at least $c+2$ permutations. A minimum of $k(c+2)$ permutations are used, but since there are only $k(c+2)-1$ permutations in $alt(c, 2k+2)$, we arrive at a contradiction and S_{2k+2} avoids $alt(c, 2k+2)$.

For the odd case, we claim the $(c, k(c+2)+1)$-formation $S_{2k+1} = I_c^c D_c^2 I_c^c \dots D_c^2$ avoids $alt(c, 2k+1)$. The base case for $k = 1$ has been proven in Lemma 7. We proceed by induction. Assume for some k, S_{2k} avoids $alt(c, 2k)$. Also assume for the sake of contradiction that S_{2k+1} contains $alt(c, 2k+1)$ under some permutation π. Let G be $alt(c, 2k+1)$ without the rightmost permutation, so $G = alt(c, 2k)$. Since the leftmost S_{2k} of S_{2k+1} avoids G, the last letter of G must occur in the last D_c^2 of S_{2k+1}. The last letter of G is equal to the preceding letter, so they must be in different permutations and the last letter of G must occur in the first D_c of the last D_c^2 of S_{2k+1}. The remaining I_π of $alt(c, 2k+1)$ must be contained as a subsequence of the final D_c of S_{2k+1}. Therefore, π can only be the permutation mapping I_c to D_c and vice versa. It is easy to check that, however, S_{2k+1} does *not* contain $\pi_r(alt(c, 2k+1))$, so we arrive at a contradiction.

Therefore, $fw(alt(c, 2k)) \geq k(c+2)-1$ and $fw(alt(c, 2k+1)) \geq k(c+2)+1$. □

Lemma 7.6. $fw(I_c^k D_c) = c + 2k - 1$.

Proof. From Theorem 3.1, $fw(I_c^k) = 2k-1$. The upper bound for $fw(I_c^k D_c)$ follows immediately from this result, since $fw(I_c^k D_c) \leq c + fw(I_c^k) \leq c + 2k - 1$.

We prove the lower bound by constructing a binary formation avoiding $I_c^k D_c$. Let S_k be the $(c, c+2k-2)$ binary formation created by concatenating $alt(c, 2k-2)$ and I_c^c. We claim S_k avoids $I_c^k D_c$.

Let π be some permutation. Assume for the sake of contradiction that S_k *does* contain $I_\pi^k D_\pi$ as a subsequence.

We claim that the last $I_\pi D_\pi$ of $I_\pi^k D_\pi$ must be found as a subsequence of the final $D_c I_c^c$ block of S_k.

Assume for the sake of contradiction that it is not, so that the first letter of this $I_\pi D_\pi$ lies at its rightmost in the last I_c of the remaining $alt(c, 2k-3)$ block. Then the remaining I_π^{k-1} must be a subsequence of the remaining $alt(c, 2k-3)$ block. Note that since the first letter of the last $I_\pi D_\pi$ lies at its rightmost in the last I_c of the remaining $alt(c, 2k-3)$ block, the last letter of $alt(c, 2k-3)$ cannot be part of the I_π^{k-1}. Since this does not satisfy the last letter condition from the proof of Lemma 6.2, I_π^{k-1} cannot be a subsequence of $alt(c, 2k-3)$. Therefore we arrive at a contradiction and the last $I_\pi D_\pi$ of $I_\pi^k D_\pi$ must be found as a subsequence of the final $D_c I_c^c$ block.

As a consequence of $D_c I_c^c$ containing $I_\pi D_\pi$, $\pi(1) > \pi(2)$.

We proceed by induction. The base case $k = 1$ is immediate, $I_c D_c$ is avoided by I_c^c since $\ell(I_c D_c) = c + 1$ [1].

Assume $I_c^k D_c$ is avoided by S_k. Also assume for the sake of contradiction that S_{k+1} contains $I_c^{k+1} D_c$. Since S_k avoids $I_\pi^k D_\pi$, we must have the first letter $\pi(1)$ of $I_\pi^k D_\pi$ must occur in the initial $I_c D_c$ of S_{k+1}. In addition, the first I_π of $I_\pi^{k+1} D_\pi$ has $\pi(1)$ and $\pi(2)$ in that order, so $\pi(1)\pi(2)\pi(1)$ must be a subsequence of $I_c D_c$. First consider the case of $\pi(2)$ occurring in the D_c. This produces a contradiction since $\pi(1) > \pi(2)$, so it cannot occur after $\pi(2)$ in the D_c. If $\pi(2)$ occurs in the I_c, we get another contradiction since $\pi(1) > \pi(2)$ so it cannot occur before $\pi(2)$ in the I_c. Therefore, $\pi(1)\pi(2)\pi(1)$ cannot be a subsequence of $I_c D_c$, we arrive at a contradiction and S_{k+1} avoids $I_c^{k+1} D_c$.

Therefore, S_k avoids $I_c^k D_c$ for all k and $fw(I_c^k D_c) \geq c + 2k - 1$. The upper and lower bounds are equal, so $fw(I_c^k D_c) = c + 2k - 1$.

□

8 Computer algorithms and pattern classification

We determine an upper bound on the time complexity of calculating $fw(u)$ using a new algorithm.

Theorem 8.1. *Let u be a pattern of length s on r distinct letters. An algorithm that calculates $fw(u)$ requires $O((r^2+s)s^{2s+r(u)-1})$ time.*

Proof. The time to check if one subsequence of length s is equivalent to u upon permutation of its alphabet takes $O(r^2+s)$ time. Calculating $\ell(u)$ requires checking subsequences of a formation of at most $s-r$ permutations. There are $\sum_{i=1}^{s-r}\binom{ri}{s} = O(\binom{r(s-r)}{s})$ subsequences to check, so calculating $\ell(u)$ requires $O((r^2+s)\binom{r(s-r+1)}{s})$ time. Calculating $r(u)$ takes the same amount of time.

Let $l(u)=x$, $r(u)=y$. Given $r(u)=y$, u is contained in all binary formations of the fw $I_c^{e_1}D_c^{e_2}\ldots\mathcal{L}^{e_n}$ where $n \geq y$ by definition. We also have $\min(2x-1, s-r+1) \geq fw(u) \geq \max(x,y)$.

We search every possible binary formation in order to find $fw(u)$. We only need to check all binary formations of the fw $I_c^{e_1}D_c^{e_2}\ldots\mathcal{L}^{e_n}$ where $n<y$ and $\sum e_i$ ranges from $\max(x,y)$ to $s-r+1$. Therefore we search at most $\sum_{i=\max(x,y)}^{s-r+1}\binom{i-1}{y-1}$ binary formations. Therefore, the total time to check all of these is

$$
\begin{aligned}
O((r^2+s)\sum_{i=\max(x,y)}^{s-r+1}\binom{ri}{s}\binom{i-1}{y-1}) &= O((r^2+s)\binom{r(s-r+1)}{s}\binom{s-r}{y-1}) \\
&= O((r^2+s)\binom{r(s-r+1)}{s}\binom{s-r}{y-1}) \\
&= O((r^2+s)r^s(s-r)^{s+y-1}) \\
&= O((r^2+s)s^{2s+y-1})
\end{aligned}
$$

. ∎

Nivasch [6] proved that $Ex(u,n) = O(n\alpha(n))$ for any pattern u with $fw(u)=4$, where $\alpha(n)$ is the inverse Ackermann function.

We implemented this algorithm in the Java programming language and used it to partially classify all patterns u for which $fw(u)=4$, and as a consequence determine a class of sequences with quasilinear bounds on $Ex(u,n)$.

A lemma by Geneson et al. [1] states that $fw(ua) = fw(u)$ given a is some letter not in the pattern u. Therefore, we need only regard patterns with at least two occurrences of each letter as distinct, since a pattern with exactly 1 occurence of a letter has the same fw as a pattern with no occurences of that letter.

Corollary 8.2. *The patterns u on 3 letters with $fw(u) = 4$ described up to isomorphism are: abccba, abcbca, aabccb, aabcbc, aabbcc, abaccb, abacbc, ababcc, baaccb, baacbc, baabcc, aabcabc, abacabc, abcaabc, abcabac, abcabca, aabcacb, abcaacb, abcacba, and aabcbac.*

Corollary 8.3. *The patterns u on 4 letters with $fw(u) = 4$ described up to isomorphism are: abcdabdc, abcdadbc, abcdadcb, abcdacbd, abcdacdb, abcdbacd, abcdbcad, abcdbcda, abcddabc, abcddacb, abcddbac, abcddbca, abcddcab, abcdbdac, abcdbdca, abcdcabd, abcdcadb, abcdcbad, abcdcbda, abcdcdab, abcdcdba, aabcddbc, aabcdbdc, aabcdbcd, aabcdcbd, baacddbc, baacdbdc, baacdbcd, baacdcbd, bcaaddbc, bcaabddc, bcaabcdd, bcaadbdc, bcaadcdb, bcaabdcd, bcaadbcd, bcaadcbd, abacddbc, abacdbdc, abacdbcd, abacdcbd, bacaddbc, bacaddcb, bacabddc, bacabcdd, bacadbdc, bacadcdb, bacadbcd, bacadcbd, abcaddbc, abcaddcb, abcabddc, abcacddb, abcabcdd, abcacbdd, abcadbdc, abcadcdb, abcabdcd, abcacdbd, abcadbcd, and abcadcbd.*

We classify a large number of patterns with tight bounds on $Ex(u, n)$ below.

Lemma 8.4. *Any sequence $u = 0v0v'0$, with v being a sequence of distinct letters not including 0 and v' being the sequence obtained by either moving the first letter of v or shifting any of the other letters to the end has $fw(u) = 4$.*

Proof. Assume without loss of generality that $v = 123\dots(n-1)$. First consider $v' = 123\dots c1(c+1)\dots(n-1)$ for some $2 \leq c \leq n-1$. It is apparent that $0v0v'0$ is a subsequence of any binary $(n, 4)$-formation with containing at least 3 of I_n or D_n. Thus, it suffices to show that it is contained by $I_n^2 D_n^2$, $I_n D_n^2 I_n$, and $I_n D_n I_n D_n$. The first contains a copy of u such that all of the letters in the last permutation are used. The second contains a copy of u such that all of the letters in the third permutation are used. The final contains a copy of u where $0v$ is mapped to $1(r-c+1)\dots r(r-c)\dots 2$.

The case where v' is obtained by shifting a letter in v to the end can be obtained by reversing $123\dots c1(c+1)\dots(n-1)$ and renaming the alphabet. □

The corollary below follows from [6].

Corollary 8.5. *Let* $u = 0v0v'0$ *as described in the previous lemma. Then* $Ex(u, n) = O(n\alpha(n))$.

Lemma 8.6. $fw((abc)^s(acb)^t) = 2(s + t) - 1$.

Proof. We show that any binary $(3, 2(s+t)-1)$-formation contains either $(cba)^s(cab)^t$, $(acb)^s(abc)^t$, or $(bac)^s(bca)^t$ as a subsequence. We assume without loss of generality that the last $2(s + t) - 3$ permutations contain at least as many increasing as decreasing permutations. The base case for $s = 1$ is easy to check.

Now consider assume the inductive hypothesis holds for $s = k-1$. Consider some binary $(3, 2(k+t)-1)$-formation. The last $2(k+t)-3$ permutations of the formation contain either $(cba)^{k-1}(cab)^t$, $(acb)^{k-1}(abc)^t$, or $(bac)^s(bca)^t$ as a subsequence.

Assume that they only contain $(cba)^{k-1}(cab)^t$. If the first two permutations are anything but *abcabc*, the formation contains $(cba)^k(cab)^t$ as desired. However, if they are *abcabc*, then the formation contains $(bac)^k(bca)^t$ as a subsequence.

Assume instead that they contain $(acb)^{k-1}(abc)^t$. If the first two permutations are anything but *cbaabc*, the formation contains $(acb)^k(abc)^t$ as desired. However, if they are *cbaabc*, then the formation contains $(cba)^k(cab)^t$ as a subsequence.

Finally assume that they contain $(bac)^{k-1}(bca)^t$. If the first two permutations are anything but *abccba*, the formation contains $(bac)^k(bca)^t$ as desired. However, if they are *abccba*, then the formation has $(acb)^k(abc)^t$ as a subsequence.

Therefore, $fw((abc)^s(acb)^t = 2(s + t) - 1$. □

The corollary follows from [1].

Corollary 8.7. $Ex((abc)^s(acb)^t) = n2^{\frac{1}{(k-2)!}\alpha(n)^{k-2} \pm O(\alpha(n)^{k-3})}$ *where* $k = s + t \geq 3$.

Lemma 8.8. *Any sequence* $u = ax_1ax_2 \dots ax_{t-1}ax_t$*, where each* x_i *is a rearrangement of bcd, has* $fw(u) = 2t - 1$ *if and only if all of the* x_i *are equal.*

Proof. Since u contains $(ab)^t$ as a subsequence, $fw(u) \geq 2t - 1$.

If any of the x_i equal *dbc* or *cbd* when x_t is mapped to *abcd*, it suffices to show that $alt(4, 2t-1)$ avoids any sequence that does not satisfy the condition. Assume for the sake of contradiction that $alt(c, 2t-1)$ contains $u = ax_1ax_2 \dots x_t$. If ax_i occurs solely in the kth permutation of $alt(4, 2t-1)$, then since ax_{i+1} has the same initial letter, we can say without loss of generality that it does not occur in the $(k+1)$th permutation. Similarly, if ax_i occurs in the kth and $(k+1)$th permutations, ax_{i+1} occurs only in the $(k+2)$th permutation onwards. Therefore, since ax_1 has its leftmost occurrence in the 1st permutation, in general ax_i has its leftmost occurrence in the $(2i-1)$th permutation. Therefore, ax_t must occur solely in the final permutation of $alt(4, 2t-1)$, so ax_t must be renamed to $abcd$. Therefore, there exists exactly one renaming for which $alt(4, 2t-1)$ contains u, and all the other ax_i must be subsequences of $abcddcba$ under this renaming. This is only true when none of the ax_i are equal to $adbc$ or $acbd$.

Given any $ax_i \neq ax_t$, we can show the formation $alt(4, 2i-4)(dcba)(abcd)alt(4, 2t-2i+1)$ avoids $ax_1ax_2 \dots ax_t$. Assume for the sake of contradiction that this formation contains $ax_1ax_2 \dots ax_t$. Similarly to $alt(4, 2t-1)$, the leftmost possible occurrence of a_j is in the $(2j-1)$th permutation for $1 \leq j \leq t$. Therefore, ax_t occurs in the $(2t-1)$th permutation and the formation can only contain $ax_1ax_2 \dots ax_t$ under the renaming mapping ax_t to $abcd$. However, since $ax_i \neq ax_t$, we find ax_i under this mapping is avoided by $dcbaabcd$ since $dcbaabcd$ does not contain $abdc$, $acdb$, or $adcb$. Therefore, its leftmost occurrence spans the $(2i-1)$th, $(2i)$th, and $(2i+1)$th permutations. Since the $(2i+1)$th permutation is equal to $abcd$, we find that ax_{i+1} must have its first letter occur at its leftmost in the $(2i+2)$th permutation. Using the same argument as before, we find that ax_{t-1} must have its leftmost occurrence in the $(2t-2)$th permutation. However, since the $(2t-2)$th permutation is equal to $dcba$, the last letter of ax_{t-1} must occur in the $(2t-1)$th permutation because ax_{t-1} can only equal $abdc$, $acdb$, or $adcb$. Therefore, $ax_t = abcd$ must occur in the remaining ≤ 3 letters, which is impossible. We conclude our formation avoids $ax_1ax_2 \dots ax_t$. Therefore, $fw(u) = 2t-1$ only if all x_i are equal to x_t.

Assuming all the x_i are equal, u is isomorphic to $(abcd)^t$ and $fw(u) = 2t-1$ by Theorem 3.1. We conclude that $fw(u) = 2t-1$ if and only if all x_i are equal to x_t. □

9 Monotonic subsequences of k-sparse sequences

It is easy to show by Ramsey's Theorem that any sequence of integers will contain a monotonically increasing or decreasing subsequence of some fixed size, but not necessarily both. We show that sparsity is a strong enough condition on the sequence to ensure that it contains both an increasing and a decreasing subsequence of a certain size.

Lemma 9.1. *Any k-sparse sequence on the alphabet $\{1, 2, \ldots n\}$ of length $\geq nk$ contains a monotonically decreasing and a monotonically increasing subsequence, each of length $\frac{k}{2}$.*

Proof. It suffices to prove this for sequences of length nk. We split our sequence into n consecutive blocks $B_1, B_2, \ldots B_n$ of size k. From our sequence, we construct the $n \times n$ matrix $P = [a_{ij}]$, setting $a_{ij} = 1$ if the number i is in block B_j and 0 otherwise.

Note that if P has the $m \times m$ identity matrix I_m (not to be confused with I_c) as a submatrix, then our sequence contains a monotonically increasing subsequence of length m. The matrix P has k nonzero entries in each column, it has a total of kn entries equal to 1. By a result from [10], any $0-1$ square matrix of dimension n with $\geq 2kn$ nonzero entries contains I_k as a submatrix. Therefore, we see that P contains $I_{\frac{k}{2}}$ as a submatrix, so our sequence contains a monotonically increasing subsequence of length $\frac{k}{2}$.

The proof is identical for finding the decreasing subsequence. □

10 Open problems

Several problems remain open.

The algorithm described in Section 9 runs in exponential time and is unfeasible for calculating $fw(u)$ for longer patterns. Improving this algorithm would allow easy calculation of $fw(u)$ for many more patterns u, which could yield many new insights.

Define an *n-shaped sequence* to be a sequence of the form

$$a_1 a_2 \ldots a_k a_{k-1} \ldots a_2 a_1 a_2 \ldots a_{k-1} a_k.$$

It is proven in [9] that $Ex(u, n) = O(n)$. This result is used in [4] to bound the number of edges in simple x-monotone topological graphs. Improving the constant term in the bound would lead to tighter edge number bounds on x-monotone graphs.

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