ON THE LOWER CENTRAL SERIES OF PI-ALGEBRAS

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ABSTRACT. In this paper we study the *lower central series* $\{L_i\}_{i\geq 1}$ of *algebras with polynomial identities.* More specifically, we investigate the properties of the quotients $N_i = M_i/M_{i+1}$ of successive ideals generated by the elements L_i. We give a complete description of the structure of these quotients for the free metabelian associative algebra $A/(A[A,A][A,A])$. With methods from *Polynomial Identities* theory, linear algebra and representation theory we also manage to explain some of the properties of larger classes of algebras satisfying polynomial identities.

1. SUMMARY

In this paper we consider algebraic structures which are not commutative, i.e. the order of operations matters. Such structures are derived from quantum physics and have various applications. In contrast, commutative structures appear in classical physics and thus are better studied. We construct a series called the *lower central series* which indicates the level of non-commutativity of an algebraic structure. Our main result is a comprehensive description of important objects related to the lower central series of the so called *algebras with polynomial identities.*

2. INTRODUCTION

The lower central series is a specific type of filtration of groups or algebras which is of fundamental importance in group theory and non-commutative algebra. In this paper we study the lower central series of Lie algebras. More specifically, we consider the lower central series of free associative algebras with the additional structure of Lie algebras induced by means of the commutator. Recently, there has been substantial progress on the study of the properties of these filtrations and their successive quotients ([7], [4], [6], [3]). For an associative algebra A, the lower central series quotients have been seen to be related to both the geometry of $\mathrm{Spec}(A_{ab})$, the spectrum of the abelianization of A, and the representation theory of $\mathrm{Der}(A)$, the Lie algebra of derivations of A. Completely understanding the lower central series of A and the information it encodes remains an elusive open problem.

Let A be an associative algebra over $\mathbb{C}$ with n generators. Set $L_1(A) := A$ and define recursively $L_i(A) = [A, L_{i-1}(A)]$, where the bracket operation is given by the commutator $[a,b] = ab - ba$ for $a, b \in A$. Several series of quotients help us understand how far A is from being a commutative algebra. In this paper we are interested in the quotient series

$$\{B_i(A) = L_i(A)/L_{i+1}(A)\}_{i\geq 1} \text{ and } \{N_i(A) = M_i(A)/M_{i+1}(A)\}_{i\geq 1},$$

where $M_i(A)$ is the two-sided ideal generated by $L_i(A)$. The quotients $B_i(A)$ were first studied by Feigin and Shoikhet [7]. They found an isomorphism between the space $A/M_3(A)$ and the space of closed differential forms of positive even degree on

$\mathbb{C}^n$. Etingof, Kim and Ma [6] gave an explicit description of the quotients $A/M_4(A)$. Dobrovolska, Kim, Ma and Etingof also studied the series B_i [3, 4].

In this paper we are interested in algebras that satisfy polynomial identities or *PI-algebras*. M. Dehn [2] first considered PI-algebras in 1922. His motivation came from projective geometry. Specifically, Dehn observed that if the Desargues theorem holds for a projective plane, we can build this plane from a division ring. Later, in 1936, W. Wagner [16] considered some identities for the quaternion algebras. The actual development of the theory of PI-algebras started with fundamental works of N. Jacobson and I. Kaplansky in 1947–48 ([10], [12]). In PI theory we consider several types of problems. Some of the feasible questions which we ask are about the structure of the identities satisfied by a given algebra, the classes of algebras which satisfy these identities, and the ideals generated by them. We consider PI-algebras as algebraic structures with induced identities on them. For more complete surveys on PI-algebras see the works of Drensky [5], Koshlukov [13] and Jacobson [11].

We study for the first time the lower central series of PI-algebras. More specifically, we are interested in the large classes of PI-algebras — algebras of the form $A/(M_i(A) \cdot M_j(A))$ and $A/(A[L_i(A), L_j(A)])$. To explain the motivation behind studying PI-algebras, consider the case when A is finitely generated by n elements. Feigin and Shoikhet [7] describe an action of W_n, the Lie algebra of polynomial vector fields on $\mathbb{C}^n$, on the lower central series quotients of A. Therefore, the quotients $B_i(A)$ can be considered in terms of the well-understood representation theory of W_n ([7], [6], [3]). The PI-Algebras $A/(M_i(A) \cdot M_j(A))$ are of particular interest in this setting because the structure of the commutator ideals M_i allows the action of W_n to descend to these quotients. In particular, we hope to understand the lower central series quotients in terms of the representation theory of W_n.

In this paper we prove the isomorphism $N_i(A/(M_m(A) \cdot M_l(A))) \cong N_i(A)$ for i large enough. We show that B_i and N_i are isomorphic for some specific algebras. Our main result is the description of the structure of the quotient elements B_i and N_i for the free metabelian associative algebra $R_{2,2}$, which is the associative PI-algebra whose only relations are of the form $[a,b][c,d] = 0$. We describe the basis for the elements $N_i(R_{2,2})$ and consider the growth of the dimensions of the graded components for any number of generators. We use techniques from PI theory, linear algebra and representation theory to prove our results.

The structure of this paper is as follows. In Section 3 we present the basic definitions we need. In Section 4 we explain how we used computer calculations as a basis for our conjectures. In Section 5 we consider the isomorphisms

$$N_i(A/(M_m(A) \cdot M_l(A))) \cong N_i(A) \text{ and } N_i \cong B_i.$$

In Section 6 we prove our main results. Namely, we give a complete description of the lower central series properties of the free metabelian associative algebra $R_{2,2}$. In Section 7 we formulate a conjecture which concerns the universal behavior of the lower central series quotients for the PI-algebras we consider. In Section 8 we present some additional results concerned with the action of the general linear group.

3. Preliminaries

In this section we introduce basic definitions used throughout the paper.

All vector spaces will be over a field $\mathcal{K}$ of characteristic zero. If an algebra A is equipped with an associative multiplication and has a multiplicative identity, then we say that A is a *unitary associative algebra.* We denote the *free associative algebra* on n generators $x_1, \ldots, x_n$ as A_n. The *bracket* or *commutator*

$$[,]: A \times A \to A$$

is $[a, b] = ab - ba$, where $a, b \in A$. We consider the algebra A as a Lie algebra with this bracket multiplication. See Appendix A for more detailed exposition on the basic definitions.

Definition 3.1. The *lower central series* of an algebra A is the series of elements $\{L_i(A)\}_{i\geq 1}$ defined by $L_1(A) = A$ and $L_i(A) = [A, L_{i-1}(A)]$ for $i \geq 1$.

We consider the series $\{M_i(A)\}_{i\geq 1}$ of the two-sided ideals generated by L_i, i.e. $M_i(A) = A \cdot L_i(A) \cdot A$. Due to the identity $[B, cd] + c[d, B] = [B, c]d$ for $B \in L_{i-1}$ and $c, d \in A$, the two-sided ideal M_i is actually a left-sided (right-sided) one.

Definition 3.2. Let the series $\{N_i(A)\}_{i\geq 1}$ be the quotients of successive elements in the series $\{M_i(A)\}_{i\geq 1}$, i.e.

$$N_i(A) = M_i(A)/M_{i+1}(A).$$

A closely related series of quotients are the B-series $\{B_i(A)\}_{i\geq 1}$, which we define as

$$B_i(A) = L_i(A)/L_{i+1}(A).$$

In this paper we consider the lower central series of algebras with polynomial identities.

Definition 3.3. For a polynomial $f = f(x_1, \ldots, x_m)$ in the free associative algebra A, f is a *polynomial identity* for an associative algebra R if $f(r_1, \ldots, r_m) = 0$ for all elements r_j of R.

If an associative algebra R satisfies a nontrivial polynomial identity, we say that R is a *PI-algebra.* We also study PI-algebras with identities of the form $[a_1, \ldots, a_i][b_1, \ldots, b_j]$, where the a-elements and the b-elements are arbitrarily chosen from A.

Definition 3.4. Let $R_{i,j}(A)$ denote the algebra $R_{i,j}(A) = A/(M_i(A) \cdot M_j(A))$.

In PI theory these objects are known as the relatively free algberas in the class $\mathfrak{M}_{i,j}$ of algebras satisfying the identities of the form $[a_1, \ldots, a_i][b_1, \ldots, b_j]$ (see [5]).

The main goal of this paper is to provide a complete description of the lower central series structure of $R_{2,2}(A)$. This algebra is special in PI theory and we call it the *free metabelian associative algebra* defined in the class $\mathfrak{M} = \mathfrak{M}_{2,2}$ of all algebras, satisfying the *metabelian identity* $[a, b][c, d] = 0$. For convenience, we write $R_{i,j}$ instead of $R_{i,j}(A)$ when we work with the free associative algebra $A = A_n$ on n generators.

We are also interested in the relatively free PI-algebras which satisfy

$$[[a_1, \ldots, a_i], [b_1, \ldots, b_j]] = 0.$$

These algebras are of the form $A/(A \cdot [L_i(A), L_j(A)])$ and we write

$$S_{i,j}(A) = A/(A \cdot [L_i(A), L_j(A)]).$$

We may omit the algebra A in the notation and write $S_{i,j}$ only.

4. PATTERNS IN THE LOWER CENTRAL SERIES OF PI-ALGEBRAS

The elements $N_i(R_{m,l})$ are naturally graded by degree, i.e.

$$N_i(R_{m,l}) = \bigoplus_{d\geq 0} N_i(R_{m,l})[d],$$

where $N_i(R_{m,l})[d]$ is the subspace of $N_i(R_{m,l})$ which consists of all elements of degree d. The elements $B_i(R_{m,l})$ exhibit the following analogous grading

$$B_i(R_{m,l}) = \bigoplus_{d\geq 0} B_i(R_{m,l})[d].$$

The results in this project are motivated by computational data, which describes the dimensions of the components of the gradings. We use the software MAGMA. The first part of the research was to state a large number of conjectures about the behavior of the algebras $R_{i,j}$ and $S_{i,j}$. We managed to unify most of the them (see Appendix B as well).

We consider representatives of the classes $R_{i,j}$ and $S_{i,j}$. With $N_i(R)[d]$ we denote the subspace of degree d in the grading of the space $N_i(R)$. In Appendix B we give tables of $\dim N_i(R)[d]$ for different algebras. There we explicitly describe the patterns we have found. In this section we present some of the patterns for the free metabelian associative algebra.

Element in the series:	Degrees of grading:									
$N_i[d]$	0	1	2	3	4	5	6	7	8	9
$N_2[d]$	0	0	1	2	3	4	5	6	7	8
$N_3[d]$	0	0	0	2	4	6	8	10	12	14
$N_4[d]$	0	0	0	0	3	6	9	12	15	18
$N_5[d]$	0	0	0	0	0	4	8	12	16	20
$N_6[d]$	0	0	0	0	0	0	5	10	15	20
$N_7[d]$	0	0	0	0	0	0	0	6	12	18
$N_8[d]$	0	0	0	0	0	0	0	0	7	14
$N_9[d]$	0	0	0	0	0	0	0	0	0	8

TABLE 1. Calculations for $R_{2,2}$.

Table 1 presents the degrees of the components in the gradings of the element $N_i(R_{2,2})$. We observe arithmetic progressions in the rows and we prove them in Section 6.

As one can see, the information in Table 2 is the same as the one in Table 1. For this reason we search for and prove isomorphisms between the elements B_i and N_i for the algebras $R_{m,2}$, where $m \geq 2$.

5. GENERAL BEHAVIOR OF THE LOWER CENTRAL SERIES OF THE ALGEBRAS $R_{i,j}$ AND $S_{i,j}$

In this section we consider some general properties of the classes of algebras $\{R_{i,j}\}$ and $\{S_{i,j}\}$. There is an isomorphism between the first elements of the N-series and the first elements of the respective series for the free associative algebra. We prove results which let us obtain a connection with the known structure of the N-series for the free associative algebra.

Element in the series:	Degrees of grading:									
$B_i[d]$	0	1	2	3	4	5	6	7	8	9
$B_2[d]$	0	0	1	2	3	4	5	6	7	8
$B_3[d]$	0	0	0	2	4	6	8	10	12	14
$B_4[d]$	0	0	0	0	3	6	9	12	15	18
$B_5[d]$	0	0	0	0	0	4	8	12	16	20
$B_6[d]$	0	0	0	0	0	0	5	10	15	20
$B_7[d]$	0	0	0	0	0	0	0	6	12	18
$B_8[d]$	0	0	0	0	0	0	0	0	7	14
$B_9[d]$	0	0	0	0	0	0	0	0	0	8

TABLE 2. Calculations for $R_{2,2}$.

In [9] Gupta and Levin prove the following result

Theorem 5.1 (Gupta and Levin, 1983)**.** *Given $m, l \geq 2$ and an arbitrary algebra A, we have that*

$$M_m(A) \cdot M_l(A) \subset M_{m+l-2}(A).$$

This bound can be improved for a specific type of pairs (m, l). Etingof, Kim and Ma [6] called a pair of natural numbers (m, l) a *null pair* if for every algebra A we have that $M_m(A) \cdot M_l(A) \subset M_{m+l-1}(A)$. They conjectured that a pair (m, l) is null if and only if either m or l is odd. Bapat and Jordan [1] confirmed this conjecture and established the following result

Theorem 5.2 (Bapat and Jordan)**.** *A pair (m, l) is a null pair if and only if m or l is an odd number.*

Here we describe the main results which establish a connection with the free algebra.

Theorem 5.3. *Consider the algebra $R_{m,l}$. The space $N_i(R_{m,l})$ is isomorphic to $N_i(A)$ for $i \leq m+l-2$. If the tuple (m, l) is a null one, then $N_i(R_{m,l})$ is isomorphic to $N_i(A)$ for $i \leq m+l-1$.*

Proof. From Theorem 5.1 we get $M_m(A) \cdot M_l(A) \subset M_{m+l-2}(A)$. Theorem 5.2 gives us $M_m(A) \cdot M_l(A) \subset M_{m+l-1}(A)$ for the null pair (m, l). Thus, consider the filtration

$$M_1(A) \supset \cdots \supset M_{m+l-r-1} \supset M_{m+l-r} \supset M_m(A) \cdot M_l(A) \tag{1}$$

where r is two in the general case and one in the case of (m, l) being null.

For convenience, let us denote with $I_{m,l}(A) = I_{m,l}$ the ideal $M_m(A) \cdot M_l(A)$. Now, let us consider the elements N_i. On the one hand, by definition

$$N_i(A) = M_i(A)/M_{i+1}(A).$$

On the other hand, for the PI-algebra $R_{m,l}$ we have that

$$N_i(R_{m,l}) = M_i(R_{m,l})/M_{i+1}(R_{m,l}) = M_i(A/I_{m,l})/M_{i+1}(A/I_{m,l}),$$

which is equivalent to $N_i(R_{m,l}) = \left(M_i(A) + I_{m,l}\right) / \left(M_{i+1}(A) + I_{m,l}\right)$. If $I_{m,l}$ is a subspace of the two-sided ideal $M_j(A)$ for some j, then $M_j(A) + I_{m,l} = M_j(A)$. From Equation (1) we have that $I_{m,l} \subset M_j(A)$ for $j \leq m+l-r$. Hence, we obtain isomorphic elements M_j and consecutively isomorphic quotients; so $N_i \cong N_i(R_{m,l})$ for $i \leq m+l-r$. The last argument completes the proof. ■

There are some specific cases when we can strengthen the known results for the properties of null pairs. For instance $(2, 2)$ is not a null pair, however, the following statement holds

Proposition 5.4. *The following inclusions are satisfied*

$$(M_2(A_2))^2 \subset M_3(A_2) \text{ and } (M_2(A_3))^2 \subset M_3(A_3).$$

Proof. Let us consider the free associative algebra A_n with $n \leq 3$. Note that the ideal $M_2(A_n)$ is generated as a two-sided ideal by the elements of the form $[x_i, x_j]$ (see [6]). Therefore, all the elements of the form

$$a[x_i, x_j]b[x_k, x_l]c$$

generate our space $M_2(A_n)$, where a, b and c are arbitrary elements in our algebra. Furthermore, as a two sided ideal, we span the space $M_2(A_n)$ simply by the elements in the generating set $G = \{[x_i, x_j]r[x_k, x_l] | r \in A_n\}$. Let us take an element

$$g = [x_i, x_j]r[x_k, x_l]$$

in the generating set G. Due to the fact that n is less than four, we have that two of the indices of the variables are equal. If $i = j$ or $k = l$, then g equals zero and it is in the ideal $M_3(A_n)$. If not, without loss of generality, we consider $i = k$. Recall the identity

$$[b, cd] + c[d, b] = [b, c]d \tag{2}$$

for arbitrary elements a, b, c and d in A_n. We apply Equation (2) for the first two factors in the element g to get

$$g = [x_i, x_j r][x_k, x_l] + x_j[r, x_i][x_k, x_l].$$

Now, we have that $-g = [x_j r, x_i][x_k, x_l] - x_2[r, x_i][x_k, x_l]$ and we apply the identity

$$[a, b][b, c] = 3[ab, b, c] - 3[a, b, c]b + [ac, b, b] - a[c, b, b] - [a, b, b]c$$

to obtain $-g$ as a linear combination of elements in $M_3(A_n)$. The fact that g is in the generating set G for $M_2(A_n)$ completes the proof. ■

We extend the same idea for more classes of PI-algebras. Using similar arguments as in Theorem 5.3 we derive the following result.

Theorem 5.5. *The space $N_i(S_{m,l})$ is isomorphic to $N_i(A)$ for $i \leq m+l-r$, where r is two in the general case and one in the case of (m, l) being a null pair.*

Proof. Let us take an element m of the ideal $A[L_m, L_l]$. Suppose $m = a[B, C]$ where $a \in A$, $B \in L_m$ and $C \in L_l$. We expand the commutator to get $m = a \cdot B \cdot C - a \cdot C \cdot B$. From Theorem 5.1 we have that $a \cdot B \cdot C$ and $a \cdot C \cdot B$ are in the ideal M_{m+l-2}. Furthermore, Theorem 5.2 states that if (m, l) is a null pair, $a \cdot B \cdot C$ and $a \cdot C \cdot B$ are in M_{m+l-1}. This means that $m \in M_{m+l-r}$, where r is two in the general case and one if the pair (m, l) is a null one. Now, similarly to the proof of Theorem 5.3, we consider

$$N_i(S_{m,l}) = M_i(S_{m,l})/M_{i+1}(S_{m,l}) = (M_i(A) + A[L_m, L_l])/(M_{i+1}(A) + A[L_m, L_l]),$$

and from the above considerations we obtain

$$(M_i(A) + A[L_m, L_l])/(M_{i+1}(A) + A[L_m, L_l]) = M_i(A)/M_{i+1}(A) = N_i(A).$$

The proof is completed. ■

5.1. **Isomorphism between the B-series and the N-series.** One of the most important observations we made for the PI-algebras under consideration is that $B_i \cong N_i$ for i large enough. Below we provide a proof of this statement. First, we state a result by Bapat and Jordan [1].

Theorem 5.6 (Bapat and Jordan, 2010)**.** *For l odd and k arbitrary we have that*

$$[M_l(A), L_k(A)] \in L_{k+l}(A).$$

We apply this theorem to prove the next lemma.

Lemma 5.7. *The following holds*

$$L_i(A) + M_j(A) \cdot M_2(A) = M_i(A) + M_j(A) \cdot M_2(A),$$

for all even positive integers i such that $i \geq j + 1$.

Proof. The inclusion $L_i(A) \subset M_i(A)$ implies that

$$L_i(A) + M_j(A) \cdot M_2(A) \subset M_i(A) + M_j(A) \cdot M_2(A).$$

Since we know that the two-sided ideal $M_i(A)$ is actually a one-sided ideal, we have $M_i(A) = [A, L_{i-1}(A)] \cdot A$. Let $[s, C]t$ be an arbitrary element of this ideal, where $s, t \in A$ and $C \in L_{i-1}(A)$. We apply the identity $[b, cd] + c[d, b] = [b, c]d$ with $b = s$, $c = C$ and $d = t$ and obtain

$$[s, C]t = [s, Ct] + C[t, s].$$

Suppose $i \geq j + 1$. Since $C \in M_{i-1}(A)$, we have that $C[t, s] \in M_j(A) \cdot M_2(A)$.

Moreover, $[s, Ct]$ is in $[L_1, M_{i-1}]$. Suppose i is even. From Theorem 5.6 we get that $[s, Ct] \in L_i$. Therefore,

$$[s, Ct] + C[t, s] + M_j(A) \cdot M_2(A) \in L_i(A) + M_j(A) \cdot M_2(A),$$

which implies

$$L_i(A) + M_j(A) \cdot M_2(A) \supset M_i(A) + M_j(A) \cdot M_2(A).$$

This completes the proof. ■

We use this result to prove a stronger statement.

Theorem 5.8. *The following equality is satisfied*

$$L_i(A) + M_j(A) \cdot M_2(A) = M_i(A) + M_j(A) \cdot M_2(A)$$

for all positive integers $i \geq t$, where $t = 2\left\lceil \frac{j+1}{2} \right\rceil$.

Proof. We have that $L_i(A) + M_j(A) \cdot M_2(A) \subset M_i(A) + M_j(A) \cdot M_2(A)$. We use induction on the index i. Note that t is the smallest even number such that $t \geq j+1$. Hence, from Lemma 5.7 we have that the theorem is true for $i = t$. This is our base case.

Now, suppose that the statement holds for every j' such that $j' \leq i$ and $i \geq t$. Take $m = [b, C]a \in M_{i+1}(A)$, where C is a commutator in L_i. Therefore,

$$[b, C]a = [b, Ca] - C[b, a].$$

We have that $C[b, a] \in M_j(A) \cdot M_2(A)$. Consider $[b, Ca]$, where Ca is in $M_i(A)$. We use the induction hypothesis and present Ca as $d + m'$, where $d \in L_i$ and $m' \in M_j(A) \cdot M_2(A)$. Now, using the bilinearity of the Lie brackets we get

$$[b, Ca] = [b, d] + [b, m'].$$

Furthermore, $[b, d] \in L_{i+1}(A)$ and $[b, m'] \in M_j(A) \cdot M_2(A)$. Hence, the statement is true for $i + 1$ as well. This completes the induction and the proof is finished. ■

We conclude the section with the Isomorphism Property statement which is the main result.

Theorem 5.9 (Isomorphism Property). *The following holds*

$$B_i(R_{j,2}) \cong N_i(R_{j,2})$$

for all positive integers $i \geq t$, *where* $t = 2\left\lceil \frac{j+1}{2} \right\rceil$.

Proof. By definition

$$\begin{aligned} B_i(R_{j,2}) &= L_i(R_{j,2})/L_{i+1}(R_{j,2}) \\ &= (L_i(A) + M_j(A) \cdot M_2(A))/(L_{i+1}(A) + M_j(A) \cdot M_2(A)). \end{aligned}$$

and

$$\begin{aligned} N_i(R_{j,2}) &= M_i(R_{j,2})/M_{i+1}(R_{j,2}) \\ &= (M_i(A) + M_j(A) \cdot M_2(A))/(M_{i+1}(A) + M_j(A) \cdot M_2(A)). \end{aligned}$$

From Theorem 5.8 we have that $L_i(A)+M_j(A)\cdot M_2(A) = M_i(A)+M_j(A)\cdot M_2(A)$. We use this to complete the proof. ■

6. Complete description of $N_r(R_{2,2})$.

In this section we consider the N-series of the free metabelian associative algebra. First we start with a result about the structure of $R_{2,2}$.

We use techniques similar to those used by Drensky who describes in his book [5] a basis over a field of characteristic zero for $R_{2,2}$.

Theorem 6.1 (Drensky, 1999). *The elements in the set*

$$E := \{x_1^{a_1} \cdots x_n^{a_n}[x_{i_1}, \ldots, x_{i_r}]\}$$

form a basis for $R_{2,2}$, *where* $a_j \geq 0$ *for* $1 \leq j \leq n$, $i_1 > i_2$, $i_2 \leq \cdots \leq i_r$ *and* $r = 1, 2, 3, \ldots$

Definition 6.2. For an element m of the form $m = a \cdot C \cdot b$, where a and b are monomials in $x_1, x_2, \ldots, x_n$ and C is a commutator, let $l(m)$ denote the length of the commutator C. We treat monomials in n variables as commutators of length one.

For instance, $l(x_1^2x_2) = 1$; $l(x_1[x_1, x_2]x_1^3x_2) = 2$; $l(x_1[x_1, x_2, x_3]) = 3$. We present properties of the identities in the free metabelian associative algebra:

$$xy[a, b] - yx[a, b] = [x, y][a, b] = 0.$$

This leads us to $xy[a, b] = yx[a, b]$ and it means that we can always order elements which multiply the commutator to the left. Next, we consider the identity

$$Cx = xC + [C, x].$$

If C is a commutator we get that we can transform every right multiplication as a sum of a left multiplication and a longer commutator. We note that

$$[a, bc] = b[a, c] + [a, b]c,$$

which helps us reduce the degrees of certain monomials. Moreover, in the free metabelian associative algebra

$$0 = [a,b][c,d] - [c,d][a,b] = [[a,b],[c,d]] = [a,b,c,d] - [a,b,d,c].$$

From this identity we get $[a,b,r_{\sigma(1)},\ldots,r_{\sigma(m)}] = [a,b,r_1,\ldots,r_m]$, where $\sigma \in S_n$ is a permutation in the symmetric group S_n. This allows us to freely permute the elements after the first two. The following lemma is important for the proofs of the arguments in this section.

Lemma 6.3. *Every element m of the form*

$$m = b[[x_{i_1},\ldots,x_{i_r}],c],$$

can be presented as a linear combination of elements e_i of the basis E with length $l(e_i) \geq l(m)$, where $i_1 > i_2$ and $i_2 \leq \cdots \leq i_r$.

Proof. Due to the bilinearity of the Lie bracket and the distributive property of the polynomials, without loss of generality, we set b and c to be monomials. Now, suppose $c = x_{i_1}^{a_1}\cdots x_{i_m}^{a_m}$. We prove the statement via induction on the total degree $a_1 + \cdots + a_m$.

For $\deg c = 1$ we have that $c = x_j$ is a variable in the set of n variables which generate the free associative algebra. Therefore, we have that $m = b[x_{i_1},\ldots,x_{i_r},x_j]$. Now, we use the permutation properties of these brackets and the Jacobi identity combined with the anticommutative law to present m of the form

$$m = b'[x_{j_1},\ldots,x_{j_{r+1}}],$$

where $j_1 > j_2$ and $j_2 \leq \cdots \leq j_{r+1}$. The base of the induction is proven.

Suppose that we have proven the statement for all monomials c' with $\deg c' \leq i$ and $i \geq 1$. Consider the monomial $c = x_{j_1}^{a_1}\cdots x_{j_h}^{a_h}$ with $\deg c = i+1$. We have that

$$\begin{aligned} m &= b[[x_{i_1},\ldots,x_{i_r}],x_{j_1}^{a_1}\cdots x_{j_h}^{a_h}] \\ &= b\cdot x_{j_1}^{a_1}\cdots x_{j_h}^{a_h-1}[[x_{i_1},\ldots,x_{i_r}],x_{j_h}] + b[[x_{i_1},\ldots,x_{i_r}],x_{j_1}^{a_1}\cdots x_{j_h}^{a_h-1}]x_{j_h}. \end{aligned}$$

With proper permutations of the elements in the commutator we can present the first element in the sum of the desired form. Hence,

$$l(b\cdot x_{j_1}^{a_1}\cdots x_{j_h}^{a_h-1}[[x_{i_1},\ldots,x_{i_r}],x_{j_h}]) = l(m).$$

Now, for the second summand we have

$$\begin{aligned} b[[x_{i_1},\ldots,x_{i_r}],x_{j_1}^{a_1}\cdots x_{j_h}^{a_h-1}]x_{j_h} &= b\cdot x_{j_h}[[x_{i_1},\ldots,x_{i_r}],x_{j_1}^{a_1}\cdots x_{j_h}^{a_h-1}] \\ &\quad + b[[x_{i_1},\ldots,x_{i_r}],x_{j_1}^{a_1}\cdots x_{j_h}^{a_h-1},x_{j_h}]. \end{aligned}$$

For the first element in the sum we use the induction hypothesis because

$$\deg x_{j_1}^{a_1}\cdots x_{j_h}^{a_h-1} = i.$$

For the second summand we permute the last two elements in the commutator and use the induction hypothesis again. Thus, we prove the statement for $\deg c = i+1$ as well, which completes our induction and proof respectively. ■

The next result is crucial for the proof of the main theorem (Theorem 6.5) in this paper.

Lemma 6.4. *The following holds*

$$M_i(R_{2,2}) \subset \operatorname{span}(x_1^{a_1} \cdots x_n^{a_n}[x_{i_1}, \dots, x_{i_r}])$$

for every $r \geq i$ *and* $a_j \geq 0$ *for* $j = 1, \dots, n$.

Proof. We use induction on the index i. Note that $M_1(R_{2,2}) = R_{2,2}$. For $r \geq 1$ we have that $\operatorname{span}(x_1^{a_1} \cdots x_n^{a_n}[x_{i_1}, \dots, x_{i_r}]) = R_{2,2}$ due to Theorem 6.1. Hence, the statement is true for $i = 1$. We take an element $m \in M_2(R_{2,2})$ and present it as a unique linear combination of elements of the basis E in the following manner

$$m = \sum_j \alpha_j e_j,$$

where $\alpha_j \in k$. We know that as a two-sided ideal $M_2(R_{2,2})$ is generated by $\{[x_s, x_t]\}$ (see [6]). Hence, m is of the form $\sum_{s,t} a \cdot [x_s, x_t] \cdot b$, where a and b are monomials. Moreover, the sum of the element e_j with $\alpha_j \neq 0$ and $l(e_j) \geq 2$ is also of the form $\sum_{s,t} a \cdot [x_s, x_t] \cdot b$ because these elements are in $M_2(R_{2,2})$. If we suppose that there are elements e_j with $l(e_j) = 1$ we get that their sum should be of the form $\sum_{s,t} a \cdot [x_s, x_t] \cdot b$, which is a contradiction. Therefore, the statement is true for $i = 2$ as well. This case is the base case for the induction.

Suppose that we have proven the proposition for all j, such that $j \leq i$ and $i \geq 2$. Let us take $m \in M_{i+1}(R_{2,2})$. Without loss of generality, we assume this element is of the form $m = a[b, C]$, where C is a commutator with $l(C) = i$ and a, b being monomials. We multiply by negative one and consider m as $a'[C, b]$. We have that $C \in M_i(R_{2,2})$ and we use the induction hypothesis to present it as a linear combination of elements of the basis with length greater than or equal to i. Thus

$$m = a' \left[\sum_j a_j C_j, b \right],$$

where $\{a_j\}$ are monomials in the variables $x_1, \dots, x_n$ and $\{C_j\}$ are the desired commutators. Once again, due to bilinearity, we consider only the case

$$m = a'[a''C', b] = a'a''[C', b] + a'[a'', b]C'.$$

The second summand is zero in the free metabelian associative algebra $R_{2,2}$ because $l(C') \geq 2$. For the first summand we have that $l(a'a''[C', b]) \geq i+1$. We use Lemma 6.3 for $a'a''[C', b]$ to complete the induction step and thus the proof of the lemma. ■

We continue with the most important result in this paper.

Theorem 6.5 (The Structure Theorem)**.** *For a fixed* $r \geq 1$ *a basis for the elements* $N_r(R_{2,2})$ *is*

$$\{x_1^{a_1} \cdots x_n^{a_n}[x_{i_1}, \dots, x_{i_r}]\},$$

where $a_j \geq 0$ *for* $1 \leq j \leq n$*,* $i_1 > i_2$ *and* $i_2 \leq \cdots \leq i_r$ *in the commutators of length* r*.*

Proof. Consider the filtration of the elements M_i:

$$R_{2,2} \supset M_2(R_{2,2}) \supset M_3(R_{2,2}) \cdots .$$

We introduce the subspaces Q_j, where $Q_j := \operatorname{span}(x_1^{a_1}\cdots x_n^{a_n}[x_{i_1},\ldots,x_{i_h}] \mid h \geq j)$, i.e. the span of the basis elements with commutator length greater than or equal to j. Consider the filtration

$$R_{2,2} \supset Q_2(R_{2,2}) \supset Q_3(R_{2,2}) \cdots .$$

On the one hand, $Q_i(R_{2,2}) \subset M_i(R_{2,2})$ because every commutator of length greater than or equal to i is in $M_i(R_{2,2})$. On the other hand, from Lemma 6.4 we get that $M_i(R_{2,2}) \subset Q_i(R_{2,2})$. Thus $Q_i(R_{2,2}) = M_i(R_{2,2})$ and we have that the filtration of the elements Q_i is compatible with the one of the elements M_i. The definition of $N_i(R_{2,2})$ implies

$$N_i(R_{2,2}) = M_i(R_{2,2})/M_{i+1}(R_{2,2}) = Q_i(R_{2,2})/Q_{i+1}(R_{2,2}).$$

The statement of the theorem follows from the fact that a basis for $Q_j(R_{2,2})$ is $\{x_1^{a_1}\cdots x_n^{a_n}[x_{i_1},\ldots,x_{i_r}]\}$, for $i_1 > i_2$, $i_2 \leq \cdots \leq i_r$ and $r = j, j+1, j+2, \ldots$ ■

The elements N_r exhibit a natural grading of the form

$$N_r(R_{2,2}) = \bigoplus_{d\geq 0} N_r(R_{2,2})[d],$$

where the subspace $N_r(R_{2,2})[d]$ is spanned by all of the elements e of the basis E with $\deg e = d$.

This way we may try to find the structure of the elements N_r in terms of finite dimensions. The following combinatorial results confirm the conjectures of the patterns for the free metabelian associative algebra.

Corollary 6.6. *For the behavior of the elements $N_r(R_{2,2})$ we have that*

$$\dim(N_r(R_{2,2})[d]) \sim c_{r,n} d^{n-1},$$

where $c_{r,n}$ is a constant as d tends to infinity.

Proof. We count the number of elements of degree d of the form

$$x_1^{a_1}\cdots x_m^{a_n}[x_{i_1},\ldots,x_{i_r}],$$

where $r \geq 1$, $a_j \geq 0$ for $1 \leq j \leq n$, $i_1 > i_2$ and $i_2 \leq \cdots \leq i_r$ in the commutators of fixed length r. For this purpose, we consider a vector space with proper grading. It generates a Hilbert series which leads to the desired result. Let V be the vector space over a field of characteristic zero with basis all the commutators $[x_{i_1},\ldots,x_{i_r}]$, where $i_1 > i_2$ and $i_2 \leq \cdots \leq i_r$ for all $r = 1, 2, \ldots$ The space V has a natural multigrading if we take into account the degree of each variable in the commutators: $V = \bigoplus_t V^{(t_1,\ldots,t_n)}$. The Hilbert series of V is

$$\operatorname{Hilb}(V, z_1, \ldots, z_n) = \left(\sum_{i=1}^{n} z_i - 1\right) \prod_{j=1}^{n} \frac{1}{1-z_j}.$$

Therefore, for normal grading we get $\operatorname{Hilb}(V, z) = (nz-1)/(1-z)^n$ and since the degree of the commutator is r,

$$\dim(N_r(R_{2,2})[d]) = \left([z^r]\frac{nz-1}{(1-z)^n}\right)\binom{d-r+n-1}{n-1}, \tag{3}$$

where $\left([z^r]\frac{nz-1}{(1-z)^n}\right)$ is the coefficient in front of the n-th power of the formal power series $(nz-1)/(1-z)^n$. Now, we estimate Equation (3) asymptotically to complete the proof. ■

Note that this result is compatible with the conjectures for the cases of two and three variables. One just has to expand the formal power series $(nz-1)/((1-z)^n)$.

7. UNIVERSAL HYPOTHESIS ABOUT THE BEHAVIOR OF THE LOWER CENTRAL SERIES QUOTIENTS OF PI-ALGEBRAS

In [7] Feigin and Shoikhet consider the action of W_n on the lower central series quotients of the algebra A_n. Moreover, these quotients are W_n-modules which are finite in length. We can describe the structure of the elements $N_r(R_{i,j})$ in terms of irreducible representations appearing in the Jordan-Hölder decomposition of N_r.

The algebras $R_{i,j}$ and $S_{i,j}$ are interesting because the structure of their commutator ideals allows the action of W_n to descend to an action on the lower central series quotients for $R_{i,j}$ and $S_{i,j}$ as well. From basic facts of the representation theory of W_n we have that for a fixed r, the dimensions of the quotient components $B_r[d]$ and $N_r[d]$ are polynomials in d for d large enough. This, however, is not always true for small values of d.

From Theorem 6.5 we get that

$$N_r(R_{2,2}) = \mathcal{F}_{(r-1,1,0,\dots,0)}$$

as a W_n-module, where $\mathcal{F}_{(r-1,1,0,\dots,0)}$ is a single irreducible representation of W_n of rank r. This confirms the observations in Table 1 and Table 2. As one can see, the sequences in the rows behave like arithmetic progressions. We determine the irreducible module by considering the degrees of the basis elements described in Theorem 6.5. In Appendix B we present tables for several other algebras. It would be interesting to prove the following conjecture, about the universal behavior of the elements $N_r(R_{m,l})$.

Conjecture 7.1 (Universal Behavior)**.** *Given $r > m+l$, the Jordan-Hölder series for $N_r(R_{m,l})$ consists only of irreducible W_n-modules of rank r.*

8. ADDITIONAL RESULTS. REPRESENTATION THEORY OF $GL_n(\mathcal{K})$

In this section we present additional results on the structure of the N-series of the algebras $R_{i,j}$. First, we need to introduce some new terminology of polynomial identities.

8.1. **Some more PI-theory.**

Let us consider $\{f_i(x_1,\dots,x_n) \in A_n\}$ — a set of polynomials in the free associative algebra A_n. We denote with a *variety defined (determined) by the system of polynomial identities* $\{f_i \mid i \in I\}$ the class $\mathfrak{D}$ of all associative algebras satisfying all of the identities $f_i = 0, i \in I$. We call a variety $\mathfrak{M}$ a *subvariety* if $\mathfrak{M} \subset \mathfrak{D}$.

Definition 8.1. We denote with $T(\mathfrak{D})$ the set of all identities satisfied by the variety $\mathfrak{D}$ and call it the *T-ideal* of $\mathfrak{D}$.

Hence, we can see that the elements $M_i(A)$ $(i \geq 2)$ are T-ideals of the corresponding classes (varieties) $\mathfrak{M}_i$ of all algebras satisfying the identity $[a_1,\dots,a_i] = 0$. Furthermore, their products $M_i(A)\cdot M_j(A)$ are also T-ideals of the corresponding varieties $\mathfrak{M}_{i,j}$ satisfying $[a_1,\dots,a_i][b_1,\dots,b_j] = 0$. Thus we can translate results on T-ideals in the language of the M-ideals. This will be particularly useful, since the algebras we consider, $R_{i,j}$, have the structure of the free associative algebra factored by a specific T-ideal.

Definition 8.2. For the generating set Y the algebra $F_Y(\mathfrak{D})$ in the variety $\mathfrak{D}$ is called a *relatively free algebra* of $\mathfrak{D}$ (or a $\mathfrak{D}$-free algebra) if $F_Y(\mathfrak{D})$ is free in the class $\mathfrak{D}$ (and is freely generated by Y).

Such algebras exist and are comprehensively studied in [5]. Two relatively free algebras of the same rank are isomorphic. Thus, if the rank of $F_Y(\mathfrak{D})$ is n, we may simply write $F_n(\mathfrak{D})$. It is known that the free metabelian associative algebra $R_{2,2}$ is the algebra $F_n(\mathfrak{M}_{2,2})$ — the relatively free algebra of the variety $\mathfrak{M}_{2,2}$ of algebras satisfying $[a,b][c,d] = 0$. Moreover, the free algebras $F_n(\mathfrak{M}_{i,j})$ are equivalent to $R_{i,j}$ in our language.

Definition 8.3. A polynomial f in the free associative algebra A_n is called *proper* if it is a linear combination of products of commutators. We denote with $P_n(A)$ the set of all proper polynomials in A.

Let us take a PI-algebra $R_{i,j}$. In a similar way, we define $P_n(R_{i,j})$ to be the image in $R_{i,j}$ of the vector subspace $P_n(A)$. When we know the number of variables, we may simply write $P(R_{i,j}) = P_n(R_{i,j})$ — the set of all proper polynomials in the PI-algebra $R_{i,j}$.

We continue with an important theorem which we will use throughout the paper. The original statement is in the language of relatively free algebras but we translate it in the language of the algebras $R_{i,j}$.

Theorem 8.4 (Drensky, 1999)**.** *For the PI-algebra $R_{i,j}$ we have that*

$$R_{i,j} \cong \mathcal{K}[x_1, \ldots, x_n] \otimes P_n(R_{i,j}).$$

For more information on relatively free algebras and T-ideals see [5].

8.2. **Representation theory of the general linear group.** In this subsection we consider basic results on the representation theory of the general linear group $GL_n(\mathcal{K})$, acting on A_n. For the purposes of this paper we shall describe GL_n's irreducible modules in terms of Schur functions and, thus, Young diagrams only. For more information on the representation theory of finite and nonfinite groups see [15]. The representation theory of GL_n is connected with the representation theory of the symmetric group S_n. The action of GL_n, on the other hand, is connected with the action of W_n. Therefore, it would be interesting to describe the results in this paper in the language of representations of S_n and W_n too. For convenience, however, here we shall concentrate on GL_n-representations.

Consider π — a finite dimensional representation of the group $GL_n(\mathcal{K})$, i.e.

$$\pi : GL_n(\mathcal{K}) \longrightarrow GL_r(\mathcal{K})$$

for a given r.

Definition 8.5. The representation π is *polynomial* if the entries $(\pi(g))_{ij}$ of the corresponding $r \times r$ matrix $\pi(g)$ are polynomials of the entries a_{kl} of g, given $g \in GL_n$. The polynomial representation π is called *homogeneous* of degree d if its entries are of degree d.

Similarly, we say that the $GL_n(\mathcal{K})$-modules are correspondingly polynomial modules and homogeneous polynomial modules. We want to see the action of GL_n on the free associative algebra A_n. More specifically, we can extend the action of

GL_n on the vectors space V_n, with generators $x_1, \dots, x_n$, diagonally on the free associative algebra A_n by

$$g(x_{r_1} \cdots x_{r_m}) = g(x_{r_1}) \cdots g(x_{r_m}),$$

where $g \in GL_n, x_{r_1} \cdots x_{r_m} \in A_n$. This way, A_n becomes a left GL_n-module which is a direct sum of the submodules $A_n^{(r)}$, for $r = 0, 1, 2, \dots$ of the grading of A_n.

Theorem 8.6. *All polynomial representations of GL_n are direct sums of irreducible homogeneous polynomial subrepresentations. Moreover, all irreducible homogeneous polynomial GL_n-modules of degree d are isomorphic to a submodule of $A^{(d)}$.*

The irreducible homogeneous polynomial GL_n-representations are known.

Definition 8.7. We denote with $s_\lambda = s_\lambda(X_d)$ the Schur function which is a quotent of Vandermonde type determinants

$$s_\lambda(X_d) = \frac{V'(\lambda + \delta, X_d)}{V'(\delta, X_d)},$$

where $\lambda = (\lambda_1, \dots, \lambda_d)$ is a partition, $\delta = (d-1, \dots, 0)$ and

$$V'(\mu, X_d) = \begin{vmatrix} x_1^{\mu_1} & x_2^{\mu_1} & \cdots & x_d^{\mu_1} \\ x_1^{\mu_2} & x_2^{\mu_2} & \cdots & x_d^{\mu_2} \\ \vdots & \vdots & & \vdots \\ x_1^{\mu_d} & x_2^{\mu_d} & \cdots & x_d^{\mu_d} \end{vmatrix},$$

where $\mu = (\mu_1, \dots, \mu_d)$ and X_d is a set of variables.

The following theorem is fundamental to our representation theory considerations in this paper. It can be found in [15] and [5].

Theorem 8.8. *The pairwise nonisomorphic irreducible homogeneous polynomial GL_n-representations of degree $d \geq 0$ are in 1-1 correspondence with partitions $\lambda = (\lambda_1, \dots, \lambda_r)$ of d.*

Definition 8.9. Let $Y_n(\lambda)$ be the irreducible GL_n-module related to λ.

Due to the correspondence with partitions, we have that the dimension of $Y_n(\lambda)$ for a fixed λ is equal to the number of semi-standard Young λ-tableaux of content $\lambda = (\lambda_1, \dots, \lambda_r)$. For more information on Young diagrams and Young tableaux see [14].

8.3. **Additional results.** Now we are ready to translate Theorem 6.5 (The Structure Theorem for the free metabelian associative algebra) in the language of representation theory of GL_n. The action of the general linear group on A_n translates on $R_{i,j} \subset A_n$ and thus on the quotients $N_r(R_{i,j})$.

Theorem 8.10. *We have the following module structure*

$$N_r(R_{2,2}) \cong \left(\sum_{j=0}^{\infty} Y_n(j) \right) \otimes Y_n(r-1, 1),$$

where $Y_n(\lambda)$ is the irreducible GL_n-module related to the partition λ.

Proof. First of all, from Theorem 6.5 we have that a basis for $N_r(R_{2,2})$ is

$$\{x_1^{\alpha_1}\cdots x_n^{\alpha_n}[x_{i_1},\ldots,x_{i_r}]\}$$

for a fixed r, where $i_1 > i_2 \leq \cdots \leq i_r$. This means that

$$N_r(R_{2,2}) \cong \mathcal{K}[X_n] \otimes P_n^{(r)}(R_{2,2}),$$

where $P_n^{(r)}(R_{2,2})$ is the vector space of proper elements in $R_{2,2}$ of degree r (with basis $\{[x_{i_1},\ldots,x_{i_r}]\}$ for $i_1 > i_2 \leq \cdots \leq i_r$). Now, since the Hilbert series of all homogeneous polynomials of a fixed degree j is the complete symmetric function

$$\mathrm{H}(t_1,\ldots,t_n) = \sum t_1^{\alpha_1}\cdots t_n^{\alpha_n}, \text{ where } \alpha_1+\cdots+\alpha_n = j,$$

we have that this series is equal to the Schur function $s_{(j)}(t_1,\cdots,t_n)$ which gives us the isomorphism

$$\mathcal{K}[X_n] \cong \sum_{j=0}^{\infty} Y_n(j).$$

Consider the mapping

$$[x_{i_1},\ldots,x_{i_r}] \mapsto \begin{array}{|c|c|c|c|c|}\hline i_2 & i_3 & i_4 & \cdots & i_r \\ \hline i_1 \\ \cline{1-1}\end{array},$$

where $i_1 > i_2 \leq \cdots \leq i_r$. This mapping is a bijection between the elements of degree r in $P_n(R_{2,2})$ and the semi-standard Young tableaux. Thus the Hilbert series of $P_n^{(r)}(R_{2,2})$ is equal to $s_{(r-1,1)}(t_1,\ldots t_n)$ and this consequently leads us to the isomorphism

$$P_n^{(r)}(R_{2,2}) \cong Y_n(r-1,1).$$

The last argument completes the proof. ■

The GL_n-structure, we presented, would give us the same dimensions we obtained for the gradings of the N-elements in Corollary 6.6. However, the structure is of the form of a tensor product of irreducible modules. It would be interesting to present it of the form of a direct sum of Y_n-modules. This is possible via the Littlewood-Richardson rule ([15]). However, here the modules are simpler, so we shall use the Young rule.

Theorem 8.11 (Young Rule). *We have that*

$$Y_n(j) \otimes Y_n(\lambda_1,\ldots,\lambda_n) \cong Y_n(\lambda_1+p_1,\ldots,\lambda_n+p_n),$$

where the summation is over all p_i, $i = 0,1,\ldots,n$, such that $\lambda_i + p_i \leq \lambda_{i-1}$ for $i = 0,1,2,\ldots,n$. Moreover,

$$Y_n(1^q) \otimes Y_n(\lambda_1,\ldots,\lambda_n) \cong Y_n(\lambda_1+\varepsilon_1,\ldots,\lambda_n+\varepsilon_n),$$

where the summation is over all $\varepsilon_i = 0,1$, such that $\varepsilon_1+\cdots+\varepsilon_n = q$ and $\lambda_i+\varepsilon_i \leq \lambda_{i-1}+\varepsilon_{i-1}$ for $i = 0,1,2,\ldots,n$.

We note that (1^q) stands for the partition $(1,\ldots,1)$ of q with exactly q 1s. It is sufficient to consider $Y_n(j)\otimes Y_n(r-1,1)$. The Young rule gives us

$$Y_n(j)\otimes Y_n(r-1,1) \cong \sum_{k=0}^{\min\{j,r-2\}} Y_n(r+j-k-1,k+1).$$

Therefore, we obtain the structure

$$N_r(R_2,2) \cong \sum_{j=0}^{\infty} \sum_{k=0}^{\min\{j,r-2\}} Y_n(r+j-k-1,k+1).$$

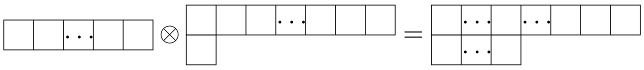

Table 3. Diagrammatic version of the product of two irreducible modules via the Young rule.

We can also calculate the dimensions of these modules using the Weyl character formula (see [15]).

Theorem 8.12 (Weyl character formula). *We have that the representation $Y_n(\lambda)$ is zero if and only if $n < p$ for p — number of parts in the partition λ. If $n \geq p$ we have*

$$\dim Y_n(\lambda) = \prod_{1\leq i<j\leq n} \frac{\lambda_i - \lambda_j + j - i}{j-i}.$$

For instance, for the lowest degree part $(r-1,1)$ we have the dimension

$$(r-1+1-1)/1 = r-1.$$

This is compatible with our MAGMA calculations.

So far in this paper we presented a method for obtaining the structure of the N-series of the free metabelian associative PI-algebra $R_{2,2}$. In the following lines we will try to generalize this method for two variables x_1, x_2. First we extend some of the previous results.

Theorem 8.13. *For the ideals $M_j(A_2)$ and $M_2(A_2)$ we have that*

$$M_j(A_2) \cdot M_2(A_2) \in M_{j+1}(A_2)$$

for any j greater than one.

Proof. First of all, note that $M_2(A_2)$ is generated by $[x_1,x_2]$ as a two-sided ideal. Hence, $M_j(A_2) \cdot M_2(A_2)$ as a two-sided ideal is generated by all elements of the form

$$\{[C,\bar{x}]r[x_1,x_2]\},$$

where C is a commutator of length $j-1$ (meaning that $l(C) = j-1$), r is a monomial in A_2, and $\bar{x} \in \{x_1,x_2\}$. Without loss of generality, we consider only the case $\bar{x} = x_1$. Thus we are interested in

$$[C,x_1]r[x_1,x_2].$$

Having in mind that

$$-[x_1,C]r[x_1,x_2] = C[x_1,r][x_1,x_2] - [x_1,Cr][x_1,x_2],$$

$[x_1,r][x_1,x_2] \in M_3(A_2)$ (from Proposition 5.4), $C \in M_{j-1}(A_2)$ and $M_{j-1}(A_2) \cdot M_3(A_2) \in M_{j+1}(A_2)$ (since 3 is odd) we get that $C[x_1,r][x_1,x_2]$ is in $M_{j+1}(A_2)$, as intended.

Due to the above considerations, we need to consider only $[Cr, x_1][x_1, x_2]$, but here we may apply identity

$$[a, b][b, c] = 3[ab, b, c] - 3[a, b, c]b + [ac, b, b] - a[c, b, b] - [a, b, b]c$$

to obtain

$$3[Crx_1, x_1, x_2] - 3[Cr, x_1, x_2]x_1 + [Crx_2, x_1, x_1] - Cr[x_2, x_1, x_1] - [Cr, x_1, x_1]x_2.$$

Moreover, take the elements $3[Cr, x_1, x_2]x_1$, $[Crx_2, x_1, x_1]$, and $[Cr, x_1, x_1]x_2$. The commutators Cr and Crx_2 both belong $M_{j-1}(A_2)$. Since x_1 is in $L_1(A_2)$, from Theorem 5.6 we get that $[Cr, x_1]$, $[Crx_2, x_1]$ and $[Cr, x_1]$ are in $L_j(A_2)$. This automatically means that $3[Cr, x_1, x_2]x_1$, $[Crx_2, x_1, x_1]$, and $[Cr, x_1, x_1]x_2)$ are in M_{j+1}. Now, we consider only $Cr[x_2, x_1, x_1] \in M_{j-1}(A_2) \cdot M_3(A_2) \in M_{j+1}(A_2)$.

The proof is completed. ■

We additionally modify Lemma 6.3.

Lemma 8.14. *Every element m of the form*

$$m = b[[x_{i_1}, \dots, x_{i_r}], c],$$

can be presented as a linear combination of elements

$$b'[x_{j_1}, \dots, x_{j_{r+v}}],$$

where v is greater than zero.

Proof. Due to the bilinearity of the Lie bracket and the distributive property of the polynomials, without loss of generality, we set b and c to be monomials. Now, suppose $c = x_{i_1}^{a_1} \cdots x_{i_m}^{a_m}$. We prove the statement via induction on the total degree $a_1 + \cdots + a_m$.

For $\deg c = 1$ we have that $c = x_j$ is a variable in the set of n variables which generate the free associative algebra. Therefore, we have that $m = b[x_{i_1}, \dots, x_{i_r}, x_j]$. The base of the induction is proven.

Suppose that we have proven the statement for all monomials c' with $\deg c' \leq i$ and $i \geq 1$. Consider the monomial $c = x_{j_1}^{a_1} \cdots x_{j_h}^{a_h}$ with $\deg c = i + 1$. We have that

$$\begin{aligned} m &= b[[x_{i_1}, \dots, x_{i_r}], x_{j_1}^{a_1} \cdots x_{j_h}^{a_h}] \\ &= b \cdot x_{j_1}^{a_1} \cdots x_{j_h}^{a_h - 1}[[x_{i_1}, \dots, x_{i_r}], x_{j_h}] + b[[x_{i_1}, \dots, x_{i_r}], x_{j_1}^{a_1} \cdots x_{j_h}^{a_{h-1}}]x_{j_h}. \end{aligned}$$

With proper permutations of the elements in the commutator we can present the first element in the sum of the desired form. Hence,

$$l(b \cdot x_{j_1}^{a_1} \cdots x_{j_h}^{a_h - 1}[[x_{i_1}, \dots, x_{i_r}], x_{j_h}]) = l(m).$$

Now, for the second summand we have

$$\begin{aligned} b[[x_{i_1}, \dots, x_{i_r}], x_{j_1}^{a_1} \cdots x_{j_h}^{a_{h-1}}]x_{j_h} &= b \cdot x_{j_h}[[x_{i_1}, \dots, x_{i_r}], x_{j_1}^{a_1} \cdots x_{j_h}^{a_{h-1}}] \\ &\quad + b[[x_{i_1}, \dots, x_{i_r}], x_{j_1}^{a_1} \cdots x_{j_h}^{a_{h-1}}, x_{j_h}]. \end{aligned}$$

For the first element in the sum we use the induction hypothesis because

$$\deg x_{j_1}^{a_1} \cdots x_{j_h}^{a_{h-1}} = i.$$

For the second summand we permute the last two elements in the commutator and use the induction hypothesis again. Thus, we prove the statement for $\deg c = i + 1$ as well, which completes our induction and proof respectively. ■

We continue with the modification of Lemma 6.4. Note that we work in two variables.

Lemma 8.15. *When working in two variables, for the ideal* $M_r(R_{2,3})$ *we have that*

$$M_r(R_{2,3}) \cong \mathcal{K}[x_1, x_2] \otimes \operatorname{span}\{[x_{i_1}, \dots, x_{i_j}], [x_{k_1}, \dots, x_{k_{j'}}][x_1, x_2]\},$$

where $i_1, \dots, i_j$ *are either 1 or 2,* $j \geq r$*,* $k_1, \dots, k_j$ *are either 1 or 2, and* $j' \geq r-1$.

Proof. First, note that

$$ab[c, d, e] - ba[c, d, e] = [a, b][c, d, e] = 0.$$

This means that our ideal is a left $\mathcal{K}[x_1, x_2]$-module.

We proceed by induction on r. Since $M_1(R_{2,3}) = R_{2,3}$, for $r = 1$ the statement is satisfied. Furthermore, $M_2(R_{2,3}) = \operatorname{span}\{a[x_1, x_2]b\}$ which also satisfies the conditions. Now, as a two sided ideal, $M_3(R_{2,3})$ is generated by (see [5])

$$[x_{i_1}, x_{i_2}, x_{i_3}],\ [x_{i_1}, x_{i_2}][x_{i_3}, x_{i_4}] + [x_{i_2}, x_{i_3}][x_{i_1}, x_{i_4}],$$

which again correspond to the statement.

Now, suppose we have proven things for all $3 \leq j \leq i$ and consider $r = i + 1$. We take $m \in M_{i+1}(R_{2,3})$. Without loss of generality, $m = a[C, b]$ where a and b are monomials in A_2 and $C \in L_i(A_2) \subset M_i(A_2)$. Thus $C \in M_i(A_2)$ and we use the induction hypothesis to present m as a sum of two independent sums

$$m' = a' \left[\sum b'[x_{i_1}, \dots, x_{i_j}], c'\right],$$

and

$$m'' = a'' \left[\sum b''[x_{k_1}, \dots, x_{k_{i-1}}][x_1, x_2], c''\right],$$

where $j \geq i$, all of the indices are either 1 or 2, $a', a'' \in \mathcal{K}[x_1, x_2]$, and $b', b'', c', c'' \in A_2$ are monomials.

Due to the bilinearity, for m' we consider only

$$\begin{aligned} m' &= a' \left[b'[x_{i_1}, \dots, x_{i_j}], c'\right] \\ &= a'b'[x_{i_1}, \dots, x_{i_j}, c'] + a'[b', c'][x_{i_1}, \dots, x_{i_j}]. \end{aligned}$$

For the first summand we use Lemma 8.14. The second element in the sum is zero, because $l([x_{i_1}, \dots, x_{i_j}]) \geq 3$.

Due to the bilinearity, for m'' we consider only

$$\begin{aligned} m'' &= a'' \left[b''[x_{k_1}, \dots, x_{k_{i-1}}][x_1, x_2], c''\right] \\ &= a''b'' \left[[x_{k_1}, \dots, x_{k_{i-1}}][x_1, x_2], c''\right] + a'' \left[b'', c''\right] [x_{k_1}, \dots, x_{k_{i-1}}][x_1, x_2] \\ &= a''b'' \left[x_{k_1}, \dots, x_{k_{i-1}}\right] [x_1, x_2, c''] + a''b'' \left[x_{k_1}, \dots, x_{k_{i-1}}, c''\right] [x_1, x_2]. \end{aligned}$$

We see that $a''b''[x_{k_1}, \dots, x_{k_{i-1}}][x_1, x_2, c''] = 0$ since the first commutator is longer than 2. The second summand satisfies the statement by Lemma 8.14 for the first commutator. Now we use the left $\mathcal{K}[x_1, x_2]$-action to complete the induction and thus the proof. ■

We would like to find the GL_n-module structure of the algebra $R_{2,3}$. Theorem 8.4 gives us that

$$R_{2,3} = \mathcal{K}[x_1, \dots, x_n] \otimes P_n(R_{2,3}).$$

Thus, it is sufficient to consider the vector space of proper polynomials $P_n(R_{2,3})$. Before we calculate that, note that it is easy to see that $P_n(A/M_2(A)) = \mathcal{K}$ since

$A/M_2(A)$ is the abelianization of A_n. The algebra $A/M_3(A)$ is the so called Grassman algebra and from [5] we have that

$$P_n(A/M_3(A)) \cong \sum_{k=0}^{\infty} Y_n\left(1^{2k}\right).$$

We know that $M_2(A) \cdot M_3(A)$ is a product of two T-ideals. Formanek [8] presented the Hilbert series of quotients of the form $A/(U \cdot V)$ in terms of the Hilbert series of the quotient A/U and A/V, where U and V are two T-ideals. Here we translate this result in the language of GL_n-representations.

Theorem 8.16 (Formanek, 1985)**.** *For $P_n(A/(T \cdot V))$, the space of proper polynomials in $A/(T \cdot V)$, we have*

$$P_n(A/(U{\cdot}V)) \cong P_n(A/U) \oplus P_n(A/V) \oplus (Y_n(1)-1) \otimes P_n(A/U) \otimes P_n(A/V) \otimes \mathcal{K}[x_1, x_2],$$

where U and V are T-ideals.

Based on Lemmma 8.15 and Theorem 8.16 we state the following conjecture

Conjecture 8.17. *For the algebra $R_{2,3}(A_2)$ we have that*

$$N_r(R_{2,3}(A_2)) \cong \left(\sum_{i=0}^{\infty} Y_2(i)\right) \otimes \left(Y_2(r-1,1) \oplus Y_2(r-2,2)\right),$$

for r greater than four.

The final step of the proof would be to modify the Structure Theorem for the case $R_{2,3}$ and "adjust" the indices of the irreducible modules. Our MAGMA calculations also agree with the conjecture.

9. Conclusion

We studied the lower central series of algebras with polynomial identities. Using approaches from PI theory, linear algebra, and representation theory, we considered some general properties for the algebras $R_{i,j}$. We gave a comprehensive classification of the structure of the N-series for the free metabelian associative algebra $R_{2,2}$. Furthermore, we formulated a conjecture about the general behavior of the lower central series of a class of PI-Algebras. Studying the representation theory of GL_n, S_n, and W_n we may generalize the method, which we present in this paper, for more PI-algebras of the form $R_{i,j}$.

Acknowledgments

I would like to sincerely acknowledge my mentor Mr. Nathan Harman for the useful resources and insightful discussions. My deepest gratitude goes to Professor Pavel Etingof for suggesting the topic and supervising it. I am extremely grateful to my tutor Mr. Antoni Rangachev for the constant encouragement and advice, as well as the constructive criticism, which taught me how to write in a more coherent mathematical language. Many thanks to Doctor Tanya Khovanova and Doctor Slava Gerovitch for supporting the professional development of this project. I am grateful to Professor Vesselin Drensky for introducing me to the field of PI theory. I would like to thank Doctor Jenny Sendova for helping me make this paper more reader-friendly. I also acknowledge the useful conversations with Kati Velcheva, Katrina Evtimova, Jacob McNamara and Katherine Cordwell. I thank the supportive community of the Bulgarian RSI alumni: Kalina Petrova and Stanislav Atanasov. I would like to acknowledge "St. Cyril and St. Methodious International Foundation" and "Foundation America for Bulgaria" for giving me the opportunity to participate in RSI, where this project was started. I express my gratitude to the "Center of Excellence in Education", the staff of the Research Science Institute and the Mathematics Department of Massachusetts Institute of Technology for the support and warm atmosphere during my first six weeks of work on this paper.

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Appendix A. Basic definitions

Definition A.1. An *algebra* (or $\mathcal{K}$-algebra) is a vector space A over a field $\mathcal{K}$, equipped with a binary operation $* : (A, A) \to A$, which we call *multiplication*, such that for every $a, b, c \in A$ and $\alpha \in \mathcal{K}$ we have:

$$(a+b) * c = a * c + b * c,$$
$$a * (b+c) = a * b + a * c,$$
$$\alpha(a * b) = (\alpha a) * b = a * (\alpha b).$$

Definition A.2. A k-algebra $\mathfrak{g}$ with multiplication $[,] : (\mathfrak{g}, \mathfrak{g}) \to \mathfrak{g}$ is a *Lie algebra* if the following holds:

$$[a,a] = 0 \text{ (the anticommutative law)},$$
$$[[a,b],c] + [[b,c],a] + [[c,a],b] = 0 \text{ (Jacobi identity)}.$$

We also recursively define $[r_1, \ldots, r_n] = [[r_1, \ldots, r_{n-1}], r_n]$ for elements in $\mathfrak{g}$.

Definition A.3. A vector space A is *graded*, if it is a direct sum of the following type

$$A = \bigoplus_{i \geq 0} A^{(i)},$$

where the subspaces $A^{(i)}$ are *homogeneous components of degree i of A.*

In a similar way we consider multigrading of $A = \bigoplus_i V^{(i_1,\dots,i_n)}$, where we have homogeneous components of a multiindex degree. An algebra A with the property $A^{(i)} \cdot A^{(j)} \subset A^{(i+j)}$ is a graded algebra.

Definition A.4. For two vector spaces V and W with corresponding bases $\{v_i \mid i \in I\}$ and $\{w_j \mid j \in J\}$ we define their tensor product $V \otimes W$ to be the vector space with basis

$$\{v_i \otimes w_j \mid i \in I, j \in J\}.$$

The multiplication of linear combinations is defined as

$$\left(\sum_{i \in I} \alpha_i v_i \right) \otimes \left(\sum_{j \in J} \alpha_j w_j \right) = \sum_{i \in I} \sum_{j \in J} \alpha_i \beta_j (v_i \otimes w_j),$$

where $\alpha_i, \beta_j \in \mathcal{K}$.

Definition A.5. For $\dim A^i < \infty$, where $i \in \mathbb{N}_0$, we denote with the *Hilbert-Poincaré series* of the algebra A the formal power series

$$H(A, z) = \mathrm{Hilb}(A, z) = \sum_{i \geq 0} \dim A^{(i)} z^i.$$

For multigrading we take formal power series in more variables. Thus

$$H(A, z_1, \dots, z_n) = \mathrm{Hilb}(A, z_1, \dots, z_n) = \sum_i \mathrm{dim} A^{(i_1,\dots,i_n)} z_1^{i_1} \cdots z_n^{i_n}.$$

APPENDIX B. COLLECTED DATA VIA MAGMA

Element in the series:	Degrees of grading:									
$B_i[d]$	0	1	2	3	4	5	6	7	8	9
$B_2[d]$	0	0	1	2	3	4	5	6	7	8
$B_3[d]$	0	0	0	2	4	6	8	10	12	14
$B_4[d]$	0	0	0	0	3	6	9	12	15	18
$B_5[d]$	0	0	0	0	0	4	8	12	16	20
$B_6[d]$	0	0	0	0	0	0	5	10	15	20
$B_7[d]$	0	0	0	0	0	0	0	6	12	18
$B_8[d]$	0	0	0	0	0	0	0	0	7	14
$B_9[d]$	0	0	0	0	0	0	0	0	0	8

TABLE 4. Calculations for $A_2/(M_2(A_2) \cdot M_2(A_2))$.

Element in the series:	Degrees of grading:						
$N_i[d]$	0	1	2	3	4	5	6
$N_2[d]$	0	0	3	9	18	30	45
$N_3[d]$	0	0	0	8	24	48	80
$N_4[d]$	0	0	0	0	15	45	90
$N_5[d]$	0	0	0	0	0	24	72
$N_6[d]$	0	0	0	0	0	0	35

TABLE 5. Calculations for $A_3/(M_2(A_3) \cdot M_2(A_3))$.

B.1. **Data for the algebra $A/(M_2(A) \cdot M_2(A))$.** Let us consider the elements B_i for two variables (Table 4). In each row we observe arithmetic progressions with starting element $B_i[i] = i - 1$ and common difference $i - 1$. For instance, the sequence $(4, 8, 12, 16, 20, \dots)$ follows this pattern.

The elements N_i for three variables (Table 5) exhibit the following structure. In each row we observe a sequence with a starting element $B_i[i] = i^2 - 1$. The differences between consecutive elements form an arithmetic progression with starting element $2(i^2 - 1)$ and difference $i^2 - 1$. For example, for $i = 3$ consider the sequence $(8, 24, 48, 80, \dots)$. The differences form $(16, 24, 32, \dots)$, which is compatible with our conjecture.

Element in the series:	Degrees of grading:									
$B_i[d]$	0	1	2	3	4	5	6	7	8	9
$B_2[d]$	0	0	1	2	3	4	5	6	7	8
$B_3[d]$	0	0	0	2	4	6	8	10	12	14
$B_4[d]$	0	0	0	0	3	6	9	12	15	18
$B_5[d]$	0	0	0	0	0	6	12	18	24	30
$B_6[d]$	0	0	0	0	0	0	8	16	24	32
$B_7[d]$	0	0	0	0	0	0	0	10	20	30
$B_8[d]$	0	0	0	0	0	0	0	0	12	24
$B_9[d]$	0	0	0	0	0	0	0	0	0	14

TABLE 6. Calculations for $A_2/(M_2(A_2) \cdot M_3(A_2))$.

Element in the series:	Degrees of grading:						
$N_i[d]$	0	1	2	3	4	5	6
$N_2[d]$	0	0	3	9	18	30	45
$N_3[d]$	0	0	0	8	30	66	116
$N_4[d]$	0	0	0	0	18	54	108
$N_5[d]$	0	0	0	0	0	48	144

TABLE 7. Calculations for $A_3/(M_2(A_3) \cdot M_3(A_3))$.

B.2. **Data for the algebra $A/(M_2(A) \cdot M_3(A))$.** As one can see, the information presented in Table 6 gives us arithmetic progressions with starting elements $B_i[i]$ and difference $B_i[i]$. The sequence of the first nonzero elements

$$(1, 2, 3, 6, 8, 10, 12, 14, \dots)$$

stabilizes after the element 3 and behaves like an arithemtic progression with common difference two.

We consider the elements $N_i(A/(M_2(A) \cdot M_3(A))$ (Table 7). For a sequence in a row, the differences between consecutive elements form an arithmetic progression. For example, consider the sequence with nonzero elements for N_3

$$(18, 54, 108, 180, \dots).$$

The differences form $(36, 54, 72, \dots)$ which is an arithmetic progression with starting element 36 and common difference 18.

Element in the series:	Degrees of grading:									
$N_i[d]$	0	1	2	3	4	5	6	7	8	9
$N_2[d]$	0	0	1	2	3	4	5	6	7	8
$N_3[d]$	0	0	0	2	5	8	11	14	17	20
$N_4[d]$	0	0	0	0	3	6	9	12	15	18
$N_5[d]$	0	0	0	0	0	4	8	12	16	20
$N_6[d]$	0	0	0	0	0	0	5	10	15	20
$N_7[d]$	0	0	0	0	0	0	0	6	12	18
$N_8[d]$	0	0	0	0	0	0	0	0	7	14
$N_9[d]$	0	0	0	0	0	0	0	0	0	8

TABLE 8. Calculations for $A_2/(A_2[L_2(A_2), L_2(A_2)])$.

Element in the series:	Degrees of grading:						
$B_i[d]$	0	1	2	3	4	5	6
$B_2[d]$	0	0	3	8	15	24	35
$B_3[d]$	0	0	0	8	24	48	80
$B_4[d]$	0	0	0	0	15	45	90
$B_5[d]$	0	0	0	0	0	24	72
$B_6[d]$	0	0	0	0	0	0	35

TABLE 9. Calculations for $A_3/(A_3[L_2(A_3), L_2(A_3)])$.

B.3. **Data for the algebra** $A/(A[L_2(A), L_2(A)])$**.** In this case the table for

$$N_i(A/(A[L_2(A), L_2(A)]))$$

(Table 8) copies the information in Table 4, except for the row for N_3. The respective sequence is an arithmetic progression with starting element 2 and common difference 3.

The elements $B_i(A/(A[L_2(A), L_2(A)]))$ follow the same pattern as in Table 5.

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