On the Splitting of MO(2) over the Steenrod Algebra

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Abstract

We study the problem of whether the cohomology of the Thom space, $MO(2)$ can be split as a module over the Steenrod algebra by considering various finite-dimensional subalgebras with a basis of finite numbered operations. In order to solve this problem, we attempt to find a disjoint splitting, based on a minimal generating set for the cohomology of $MO(2)$. First, we solve the hit problem over the cohomology of $MO(2)$ in order to find a minimal generating set. Second, we use a basis for the subalgebra and find a splitting. The cohomology of $MO(2)$ is constructed by multiplying the symmetric algebra on two generators by ω, the Thom class of the universal two-dimensional vector bundle. Topologically, this corresponds to adding a common point at infinity to all the fibers of the vector bundle. We solve the hit problem over the cohomology $MO(2)$ in order to find a minimal generating set over a polynomial ring. The motivation for this problem is to better understand the real unoriented bordism spectrum MO. In this project, a splitting is first constructed for each of the cases based on one and two basis operations. For the cases of three and four operations, based on the generated sets, we conjecture that a splitting is not possible. From this result, we predict that $MO(2)$ is not split over the Steenrod algebra.

1 Introduction

The hit problem refers to a class of problems with the purpose of decomposing elements of a graded module in terms of elements in lower degrees. The terms which cannot be generated by lower degree elements are *non-hit elements.* A secondary goal of the hit problem is to find a pattern for these *non-hit elements.* The other elements, which are the hit elements, have the property of being submodule. An example of this is given by the polynomial ring $\mathbb{F}_2[x]$, which is a module over the ring $\mathbb{F}_2 + \mathbb{F}_2\theta$ (where x and θ have degree 1). The module action is given by

$$\theta \cdot x^n \equiv nx^{n+1} \pmod 2 \quad = \begin{cases} x^{n+1} & \text{if } n \text{ is odd} \\ 0 & \text{if } n \text{ is even} \end{cases} \pmod 2.$$

This means that the non-zero hit elements are the positive even powers of x (decomposable). These polynomials form a minimal generating set. In this example, odd powers of x together with $x^0 = 1$ form a minimal generating set. The motivation of this paper is to solve an extension of this example.

Described in 1947, the Steenrod algebra is a very powerful tool used in cohomology and stable homotopy problems, composed of stable cohomology operations called Steenrod squares, Sq^i. All stable operations are generated by these squares. [6] As a consequence, our main focus is on the properties of the Steenrod squaring operators.

The hit problem in algebraic topology focuses on modules over the Steenrod algebra. There has been much work in the past on modules which are polynomial rings; these arise from products of real projective spaces. The two variable case was solved by Peterson (1987) [1] and the three variable case was solved by Kameko (1990) [2]. The symmetric two variable case was solved by Janfada (1998) [3]. One version of the hit problem in the paper is to find a minimal generating subset of a graded module. We study the case of the (reduced) cohomology of the Thom space of the universal two-dimensional vector bundle. Our goal is determine whether a stable splitting is possible in order to expand our understanding of the real unoriented cobordism spectrum MO. In order to solve this, we first observe whether a splitting is possible for finite dimensioned subalgebras (of the module).

Section 2 is devoted to basic concepts/ideas of abstract algebra and the Steenrod algebra. This includes the Steenrod squares, Adem relations, and the Cartan formula. In Section 3, we explain the $MO(2)$ problem respectively. Sections 4.1, 4.2, 4.3 and 4.4 treat the $MO(2)$ problem for the $\mathcal{A}(0)$, $\mathcal{A}(1)$, $\mathcal{A}(2)$ and $\mathcal{A}(3)$ cases. Also included in this section, methods are shown to determine minimal generating sets for $MO(2)$ over certain subalgebras of the Steenrod algebra, which we use in subsequent sections to prove that $MO(2)$ splits over some of these algebras.

2 Definitions

2.1 Hit Element

Definition. An element m_j in a module M over a ring R is called a *hit element* if for some $m_i \in M$ and $r_i \in R$ there exists

$$m_j = \sum_i r_i m_i, \tag{2.1}$$

where the r_i have positive degrees and the sum is finite.

The study of the space of hit elements is commonly referred to as the hit problem. A major part of this problem is the determination of the dimensions of the spaces of hit elements in each degree given a particular ring and module.

2.2 Minimal Generating Set

Once the hit elements are determined, one can find a minimal generating set for the module as follows. Let $\mathfrak{U}_n$ denote the space of hit elements in degree n, and let $m_{n,1}, ..., m_{n,k_n}$ represent a basis for the quotient space $M_n/\mathfrak{U}_n$. Then the $m_{i,j}$ form a minimal generating set.

2.3 Steenrod Algebra

The Steenrod algebra is a graded ring, denoted by $\mathcal{A}$, where all coefficients are in modular two. This graded associative algebra is composed of linear operators called Steenrod Squares. An example of an element is $Sq^1Sq^5 + Sq^4Sq^2 + Sq^3Sq^3$. Steenrod subalgebras expressed by $\mathcal{A}(n)$ are generated by Steenrod squares with degree of powers of two, up to n. The Steenrod squares are natural transformations defined as

$$Sq^i : H^a(X) \to H^{a+i}(X),$$

where $H^a(X)$ is the a^{th} cohomology group of X. [3]

Theorem 2.1. *The Steenrod algebra is generated by the linear operators Sq^{2^n}, where $n \geq 0$.*

2.4 Symmetric Polynomials

We will be working with *symmetric* polynomials of two and three variables in this project. A symmetric polynomial has the property that the variable labels are interchangeable. Recall that symmetric polynomials are polynomials expressed in the *elementary symmetric polynomials* σ_i. For example, the elementary symmetric polynomial for degree two in three variables is $\sigma_2 = x_1x_2 + x_1x_3 + x_2x_3$.

In this project, the elementary symmetric variables for degree one and two in two variables are

$$\sigma_1 = x_1 + x_2 \quad \text{and} \quad \sigma_2 = x_1x_2,$$

respectively.

2.5 Adem Relations

The Adem relations are an essential tool for calculations in the Steenrod algebra. For $a, b \in \mathbb{N}$ and $a < 2b$ they say that

$$Sq^a Sq^b = \sum_i \binom{b-i-1}{a-2i} Sq^{a+b-i} Sq^i. \tag{2.2}$$

For example,

$$Sq^6 Sq^{13} = \binom{12}{6} Sq^{19} + \binom{11}{4} Sq^{18} Sq^1 + \binom{10}{2} Sq^{17} Sq^2 + \binom{9}{0} Sq^{16} Sq^3 = Sq^{17} Sq^2 + Sq^{16} Sq^3.$$

An important note is that all of the binomial coefficients are reduced modulo 2. An expression $Sq^{i_1} Sq^{i_2} ... Sq^{i_n}$ is called an *admissible monomial* if $i_j \geq 2i_{j+1}$, $(j = 1, 2, ..., n-1)$. The Adem relations cannot be applied to these expressions.

In fact, the Adem relations can be used to show that $\mathcal{A}$ is generated by the Sq^{2^i}, where $i = 0, 1, 2, 3, ...$ In this paper, we work with subalgebras $\mathcal{A}(n)$, which are generated by $Sq^1, Sq^2, Sq^4, ..., Sq^{2^n}$. By considering the dual of the Steenrod algebra, one can show that these algebras are finite; in fact, $\mathcal{A}(n)$ has dimension $2^{(n+1)(n+2)/2}$ (see Wood [6]).

Theorem 2.2. *The admissible monomials form an additive basis for the Steenrod Algebra.*

2.6 Cartan Formula

Modules over the Steenrod algebra that arise as the cohomology of spaces satisfy some additional conditions. They are commutative graded rings, and satisfy the Cartan formula:

$$Sq^n(ab) = \sum_{i+j=n} Sq^i(a) Sq^j(b). \tag{2.3}$$

For evaluating $Sq^i(x)$, some other rules apply.

(1) If the degree of x is equal to i, then $Sq^i(x) = x^2$.

(2) If j is greater than the degree of x, then $Sq^j(x) = 0$.

(3) The Steenrod square, Sq^0 is the identity, $Sq^0(x) = x$.

Janfada and Wood [3] showed that these facts allow us to evaluate any Steenrod square of any polynomial in variables of degree 1. An example of such a module is given by the polynomial algebra $\mathbb{F}_2[x_1, ..., x_n]$, which arises as the cohomology of the product of n copies of real projective space. The rules (1-3) and the Cartan formula completely determine the action of the Steenrod algebra. Since this action commutes with permutations of the variable labels, the subring $B(n)$ of symmetric polynomials is a submodule. This module arises as the cohomology of the base of the universal n-dimensional vector bundle.

2.7 Total Squaring Operation

To evaluate each Steenrod square operation we use the total squaring operation

$$Sq(x) = Sq^0(x) + Sq^1(x) + ... \tag{2.4}$$

This sum is finite since $Sq^i(x) = 0$ when $i > \deg(x)$. The total squaring operation is used to compute or break down expressions. The Cartan formula implies that Sq is a ring homomorphism. For example if we assume that x has degree one, we obtain the formula:

$$Sq^k(x^n) = \binom{n}{k} x^{n+k}. \tag{2.5}$$

3 The $MO(2)$ Minimal Generating Set

The $MO(2)$ module is similar to the symmetric two variable module $B(2) = \mathbb{F}_2[\sigma_1, \sigma_2]$, with the relation $MO(2)_{n+2} \cong B(2)_n$. The Steenrod squaring operations in $MO(2)$ are given by:

$$Sq^k(\omega\sigma_1^n\sigma_2^m) = \sum_{i+j=k} Sq^i(\omega)Sq^j(\sigma_1^n\sigma_2^m),$$

where ω is the Thom class of the universal two dimensional vector bundle over BO(2) with degree two. This creates the Thom complex is topologically obtained from the vector bundle by adding a common point at infinity to the vector space fibers (this is called *one point compactification*). The action of the Steenrod squares on ω is given as follows: the Sq^1 operation on ω is equal to $\omega \cdot \sigma_1$. The Sq^2 operation of ω is equal to $\omega^2 = \omega \cdot \sigma_2$. For $k > 1$,

$$Sq^k(\omega\sigma_1^n\sigma_2^m) = \omega\big[Sq^k(\sigma_1^n\sigma_2^m) + \sigma_1 Sq^{k-1}(\sigma_1^n\sigma_2^m) + \sigma_2 Sq^{k-2}(\sigma_1^n\sigma_2^m)\big]. \tag{3.1}$$

The action of the Steenrod squares on symmetric polynomials is given by:

$$Sq^k(\sigma_1^n\sigma_2^m) = \sum_{\substack{i+2j=n+2m+k \\ i\geq n, j\geq m}} \binom{m}{j-m}\binom{2m+n-j}{i-n}\sigma_1^i\sigma_2^j. \tag{3.2}$$

This can be proven by using the total squaring operation and using the axioms in Section 2.6 to compute that $Sq^1\sigma_2 = \sigma_1\sigma_2$.

In the formulas (3.1) and (3.2), let us define h as the sum of k, $2m$ and n. Let us also write $\omega\sigma_1^n\sigma_2^m$ as (n, m) for simplicity.

Conjecture 3.1. *For only $n = 0$ or 1, the $MO(2)$ module is decomposable as $MO(2) = M_1 + M_2$, where M_1 and M_2 are submodules of $MO(2)$ over the ring $\mathcal{A}(n)$ and $M_1 \cap M_2 = 0$ Thus $MO(2)$ is not decomposable over the Steenrod algebra is not decomposable.*

To prove this in a specific case, we find a minimal generating set for $MO(2)$ as a module over $\mathcal{A}(n)$. Then we divide the minimal generating set into two groups, α and β such that they generate two vector spaces, M_1 and M_2 whose sum is equal to $MO(2)$ and have an intersection of 0.

4 Results and Discussion

4.1 $\mathcal{A}(0)$ Problem for $MO(2)$

The most elementary case of the $MO(2)$ problem is with one operation, Sq^1, which is defined as:

$$Sq^1(\omega\sigma_1^n\sigma_2^m) = \omega\big[Sq^1(\sigma_1^n\sigma_2^m) + \sigma_1(\sigma_1^n\sigma_2^m)\big] = \omega(n+m+1)\sigma_1^{n+1}\sigma_2^m,$$

which means that if $n+m+1$ is even, the expression is zero, and if $n+m+1$ is odd, it is $\sigma_1^{n+1}\sigma_2^m$.

Lemma 4.1. *The non-hit elements are split into two cases:*

- *If h is odd, the non-hit elements are $(h-4a-2, 2a+1)$, where a is a non-negative integer.*
- *If h is even, the non-hit elements are $(h-4a, 2a)$, where a is a non-negative integer.*

Tables 1 and 2 are exemplified these two cases.

	$\omega\sigma_1^{10}\sigma_2^0$	$\omega\sigma_1^8\sigma_2^1$	$\omega\sigma_1^6\sigma_2^2$	$\omega\sigma_1^4\sigma_2^3$	$\omega\sigma_1^2\sigma_2^4$	$\omega\sigma_1^0\sigma_2^5$
$\omega\sigma_1^{11}\sigma_2^0$	1	0	0	0	0	0
$\omega\sigma_1^9\sigma_2^1$	0	0	0	0	0	0
$\omega\sigma_1^7\sigma_2^2$	0	0	1	0	0	0
$\omega\sigma_1^5\sigma_2^3$	0	0	0	0	0	0
$\omega\sigma_1^3\sigma_2^4$	0	0	0	0	1	0
$\omega\sigma_1^1\sigma_2^5$	0	0	0	0	0	0

Example when h is odd (11). The expressions on the top generate the values on the left when the entry is one, for example $Sq^1(\sigma_1^{10}\sigma_2^0) = \sigma_1^{11}\sigma_2^0$.

	$\omega\sigma_1^{11}\sigma_2^0$	$\omega\sigma_1^9\sigma_2^1$	$\omega\sigma_1^7\sigma_2^2$	$\omega\sigma_1^5\sigma_2^3$	$\omega\sigma_1^3\sigma_2^4$	$\omega\sigma_1^1\sigma_2^5$
$\omega\sigma_1^{12}\sigma_2^0$	0	0	0	0	0	0
$\omega\sigma_1^{10}\sigma_2^1$	0	1	0	0	0	0
$\omega\sigma_1^8\sigma_2^2$	0	0	0	0	0	0
$\omega\sigma_1^6\sigma_2^3$	0	0	0	1	0	0
$\omega\sigma_1^4\sigma_2^4$	0	0	0	0	0	0
$\omega\sigma_1^2\sigma_2^5$	0	0	0	0	0	1
$\omega\sigma_1^0\sigma_2^6$	0	0	0	0	0	0

Example when h is even (12). The expressions on the top generate the values on the left when the entry is one, for example $Sq^1(\sigma_1^9\sigma_2^1) = \sigma_1^{10}\sigma_2^1$.

For this case, it is fairly obvious that these two groups generate submodules that intersect trivially, since m is odd in the first group and m is even in the second group.

4.2 $\mathcal{A}(1)$ Problem for $MO(2)$

To have $\mathcal{A}(1)$ we add an additional operation, Sq^2:

$$\begin{aligned} Sq^2(\omega\sigma_1^n\sigma_2^m) &= \omega[Sq^2(\sigma_1^n\sigma_2^m) + \sigma_1(Sq^1(\sigma_1^n\sigma_2^m)) + \sigma_2(\sigma_1^n\sigma_2^m)] \\ &= \omega\left[\left(\binom{m}{1}+1\right)\sigma_1^{n+2}\sigma_2^m + \left(\binom{m+n}{1}+\binom{m+n}{2}\right)\sigma_1^n\sigma_2^{m+1}\right]. \end{aligned}$$

In this project, we used the Python programming language to determine the action of the operations. The method to most efficiently calculate the binomial coefficient $\binom{n}{m}$ in modulo 2 is to write n and m in binary and compare each pair of bits. If m has a one where n has a zero, the binomial coefficient will be even, which is zero modulo 2; otherwise, it will be odd. Example, for $\binom{6}{5}$:

$$6 = 110$$

$$5 = 101$$

The ones digit of 110 is zero while the ones digit of 101 is one, so $\binom{6}{5}$ is even, six in this case.

Lemma 4.2. *For the module $MO(2)$ over $\mathcal{A}(1)$, a minimal generating set is given by:*

$$(h-4-8a, 2+4a),\ \textit{when } h \equiv 0 \pmod 4$$
$$(0, h/2),\ \textit{when } h \equiv 0 \pmod 8$$
$$(h-8a, 4a),\ \textit{when } h \equiv 2 \pmod 4$$
$$(0, h/2),\ \textit{when } h \equiv 6 \pmod 8.$$

Proof. For $\mathcal{A}(1)$ the hit equation can be simplified to

$$x = Sq^1(y) + Sq^2(z),$$

where $\deg(y) = \deg(x) - 1$ and $\deg(z) = \deg(x) - 2$. For $MO(2)$, one has a convenient basis with degree $h+2$, given by the monomials $\omega\sigma_1^n\sigma_2^m$ with $n+2m = h$. To find the hit elements, one can form a matrix representing the map $Sq^1 \oplus Sq^2$, and preform column operations to put the matrix into column echelon form. The rows without leading ones then determine a minimal generating set. Some of these matrices are listed in the appendix. Inspecting the formulas for Sq^1 and Sq^2, one sees that they only depend (up to offsetting the exponents) on the values of n and m modulo 4. Thus these matrices follow a pattern which repeats with period 8. Thus we can see from the low degree cases that the above monomials will give a minimal generating set. □

Theorem 4.3. *$MO(2)$ splits as a module over $\mathcal{A}(1)$.*

Proof. We show that we can divide the above minimal generating set into two subsets which span submodules that intersect trivially. These two subsets are:

- $(4a, 2+4b)$ with $a, b \geq 0$
- $(0, 3+4a)$ with $a \geq 0$

and

- $(2+4a, 4b)$ with $a, b \geq 0$
- $(0, 4a)$ with $a \geq 0$.

The ring $\mathcal{A}(1)$ is generated by Sq^1 and Sq^2, and has a basis consisting of 1, Sq^1, Sq^2, Sq^3, Sq^2Sq^1, Sq^3Sq^1, $Sq^5 + Sq^4Sq^1$, and Sq^5Sq^1. We must apply these to the above elements. Rather than using variables a and b in the above, it is much more efficient to use a single template for each case. For example,

$$\begin{aligned}
Sq^0(0,2) &= (0,2) \\
Sq^1(0,2) &= (1,2) \\
Sq^2(0,2) &= (2,2) + (0,3) \\
Sq^3(0,2) &= (3,2) \\
Sq^2Sq^1(0,2) &= (1,3) \\
Sq^3Sq^1(0,2) &= (2,3) \\
(Sq^5 + Sq^4Sq^1)(0,2) &= (5,2) + (3,3) \\
Sq^5Sq^1(0,2) &= (4,3).
\end{aligned}$$

To compute the values for arbitrary a and b, one would simply take the above and add $(4a, 4b)$ to each label. Doing this for all four cases, using the templates $(0,2)$, $(0,3)$, $(2,0)$, and $(0,0)$, one sees that the span of the first group of elements above only involves the monomials (n, m) with $m \equiv 2$ or $3 \pmod 4$, while the span of the second group only involves monomials with $m \equiv 0$ or $1 \pmod 4$. $\square$

4.3 $\mathcal{A}(2)$ Problem for $MO(2)$

We add an additional operation, Sq^4:

The third operation is Sq^4:

$$\begin{aligned}
Sq^4(\omega\sigma_1^n\sigma_2^m) &= \omega\big[Sq^4(\sigma_1^n\sigma_2^m) + \sigma_1(Sq^3(\sigma_1^n\sigma_2^m)) + \sigma_2(Sq^2(\sigma_1^n\sigma_2^m))\big] \\
&= \omega\Bigg\{\left[\binom{m}{1} + \binom{m}{2}\right]\sigma_1^n\sigma_2^{m+2} \\
&\quad + \left[\binom{m}{2} + \binom{m}{1}\left(\binom{m+n-1}{1} + \binom{m+n-1}{2}\right)\right]\sigma_1^{n+2}\sigma_2^{m+1} \\
&\quad + \left[\binom{m}{3} + \binom{m}{4}\right]\sigma_1^{n+4}\sigma_2^m\Bigg\}.
\end{aligned}$$

Lemma 4.4. *For the $\mathcal{A}(2)$ case, there are five families of generators:*

$$(0, 8a)$$

$$(0, 2 + 8a)$$

$$(6 + 8a, 0)$$

$$(8a, 6 + 8b)$$

$$(4 + 8a, 2 + 8b).$$

Along with these families there is a special case: $(2, 0)$.

We determined these with the same method as the previous case.

Theorem 4.5. *The above elements form a minimal generating set for $MO(2)$ over $\mathcal{A}(2)$.*

Proof. We proceed as in the previous section. The hit equation in this case can be expressed as

$$x = Sq^1(w) + Sq^2(y) + Sq^4(z),$$

where $\deg(w) = \deg(x) - 1$, $\deg(y) = \deg(x) - 2$, $\deg(z) = \deg(x) - 4$. Thus the set of hit elements is the image of the map $Sq^1 \oplus Sq^2 \oplus Sq^4$. Inspecting the formulas for Sq^1, Sq^2 and Sq^4, we see that (up to offsetting the exponents) they only depend on n and m modulo 8. Thus, the relevant matrices follow patterns that repeat with period 16. The matrices are given in the appendix for low degrees. Again we preform column operations to put these matrices into column echelon form. The rows without leading ones determine a minimal generating set consisting of monomials. The special case of $(2, 0)$ arises because there is no contribution from Sq^4, since this would have to come from the degree with $h = -2$, which is zero. □

Conjecture 4.6. *$MO(2)$ cannot split as a module over $\mathcal{A}(2)$.*

Argument We show that we cannot divide the above minimal generating set into two subsets which span submodules that intersect trivially. The generating terms are:

- $(0, 8a)$
- $(0, 2 + 8a)$
- $(6 + 8a, 0)$
- $(8a, 6 + 8b)$
- $(4 + 8a, 2 + 8b)$
- $(2, 0)$

The ring $\mathcal{A}(2)$ is generated by Sq^1, Sq^2 and Sq^4, and has a basis consisting of sixty-four elements. We must apply these to the above elements. Again, we use a single template for each case. We first notice that all of the of the above generating sets are connected by common terms (we will always call these terms as "common terms"). Thus to make a disjoint group, we must add a term (hit or non-hit) and its set, comprised of 64 elements, to one of the sets generated by a generating terms and cancel out the common terms. We found that this is impossible because there is only one way to express the common term in with the same squaring operations. For example consider the set generated by $(0,8)$:

$$((0,8),(1,8),(1,9,3,8),(2,9),(0),(0,9),(0),(0),(1,11,5,9,7,8),$$
$$(2,11,6,9),(4,11))$$

The common term for this set is $(4,11)$, generated by $Sq^2Sq^1Sq^4Sq^2Sq^1$. Our goal is to find a term that can generate $(4,11)$ with the same operation so we work backwards, where x_a are the terms in each step (x_0 is the generating term):

$$x_5 = (4,11) = Sq^2[x_4], x_4 = (2,19) or (4,18)$$
$$x_4 = (2,19) or (4,18) = Sq^1[x_3], x_3 = (1,19)$$
$$x_3 = (1,19) = Sq^4[x_2], x_2 = (1,17)$$
$$x_2 = (1,17) = Sq^2[x_1], x_1 = (1,16)$$
$$x_1 = (1,16) = Sq^1[x_0], x_0 = (0,18)$$

For each of the six templates, there are common terms which cannot be canceled out. Thus making a splitting is impossible.

4.4 $\mathcal{A}(3)$ Problem for $MO(2)$

We add a fourth operation Sq^8:

$$\begin{aligned} Sq^8(\omega\sigma_1^n\sigma_2^m) &= \omega\big[\sigma_2 Sq^6(\sigma_1^n\sigma_2^m)+\sigma_1 Sq^7(\sigma_1^n\sigma_2^m)+Sq^8(\sigma_1^n\sigma_2^m)\big] \\ &= \omega\Bigg\{\left[\binom{n+m}{8}+\binom{n+m}{7}\right]\sigma_1^{n+8}\sigma_2^m \\ &\quad+\left[m\left(\binom{n+m-1}{6}+\binom{n+m-1}{5}\right)+\binom{n+m}{6}\right]\sigma_1^{n+6}\sigma_2^{m+1} \\ &\quad+\left[\binom{m}{2}\left(\binom{n+m-2}{4}+\binom{n+m-2}{3}\right)+m\binom{n+m-1}{4}\right]\sigma_1^{n+4}\sigma_2^{m+2} \\ &\quad+\left[\binom{m}{3}\left(\binom{n+m-3}{2}+\binom{n+m-3}{1}\right)+\binom{m}{2}\binom{n+m-2}{2}\right]\sigma_1^{n+2}\sigma_2^{m+3} \\ &\quad+\left[\binom{m}{4}+\binom{m}{3}\right]\sigma_1^n\sigma_2^{m+4}\Bigg\}. \end{aligned}$$

For the $\mathcal{A}(3)$ case, there are six families of generators:

$$(16a-4,2)$$

$$(16a-2,0)$$

$$(16a,14+16b)$$

$$(0,6+16a)$$

$$(8+16a,6+16b)$$

$$(0,2+16a).$$

Along with these families there is a special case: $(4,2)$, $(6,0)$, $(2,0)$, $(0,0)$.

We proceed as in the previous section. The hit equation in this case can be expressed as

$$x = Sq^1(v)+Sq^2(w)+Sq^4(y)+Sq^8(z),$$

where $\deg(v)=\deg(x)-1$, $\deg(w)=\deg(x)-2$, $\deg(y)=\deg(x)-4$, $\deg(z)=\deg(x)-8$. Thus the set of hit elements is the image of the map $Sq^1\oplus Sq^2\oplus Sq^4\oplus Sq^8$. Inspecting the formulas for Sq^1, Sq^2, Sq^4 and Sq^8, we see that (up to offsetting the exponents) they only depend on n and m modulo 16. Thus, the relevant matrices follow patterns that repeat with period 32. Again we preform column operations to put these matrices into column echelon form. The rows without leading ones determine a minimal generating set consisting of monomials.

Conjecture 4.7. *$MO(2)$ cannot split as a module over $\mathcal{A}(3)$.*

Argument The ring $\mathcal{A}(3)$ is generated by Sq^1, Sq^2, Sq^4 and Sq^8, and has a basis consisting of 1024 elements. We must apply these to the above elements. Again, we use a single template for each case. We first notice that all of the of the above generating sets are connected by common terms. Thus to make a disjoint group, we must add a term (hit or non-hit) and its set, comprised of 1024 elements, to one of the sets generated by a generating terms and cancel out the common terms. Using the same method as $\mathcal{A}(2)$, we found that this is impossible because there is only one way to express the common term in with the same squaring operations.

5 Conclusion

The hit problem over the Steenrod algebra that arises from cohomology theory asks for a criterion of the the minimal generating set. In addition, we ask whether the generated terms can be split into two independent subsets based in the minimal generating set. To answer this questions, we look at Steenrod subalgebras which are generated by $n+1$ Steenrod Square operations with degree 2^k, $0 \leq k \leq n$. We have proved that for particular cases, there is a splitting of $MO(2)$ into two submodules, but not the subalgebra $\mathcal{A}(2)$ and $\mathcal{A}(3)$. On those cases, we conjecture that a splitting is not possible which means that $MO(2)$ over the Steenrod algebra cannot be split into two disjoint groups. We also observe that the size of the algebra $\mathcal{A}(n)$ increases rapidly with n; when the number of operators ranges from one to four, the number of dimensions of the corresponding algebra grows from 2, 8, 64 to 1024.

6 Future Work

Future work includes a proof of the conjecture that $MO(2)$ cannot be split into two submodules over the Steenrod algebra. This includes the determination of whether $\mathcal{A}(4)$, which contains 2^{16} basis elements, can be spilt into two submodules. Another goal is to find whether $MO(3)$ splits over the Steenrod algabra as two submodules. Another possibility is to find whether $MO(4)$ (four variables) can be split. We also plan to generalize our current work to find a solution for $MO(n)$ over the Steenrod algebra.

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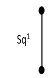
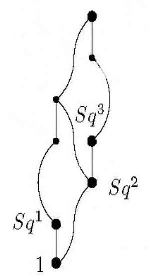
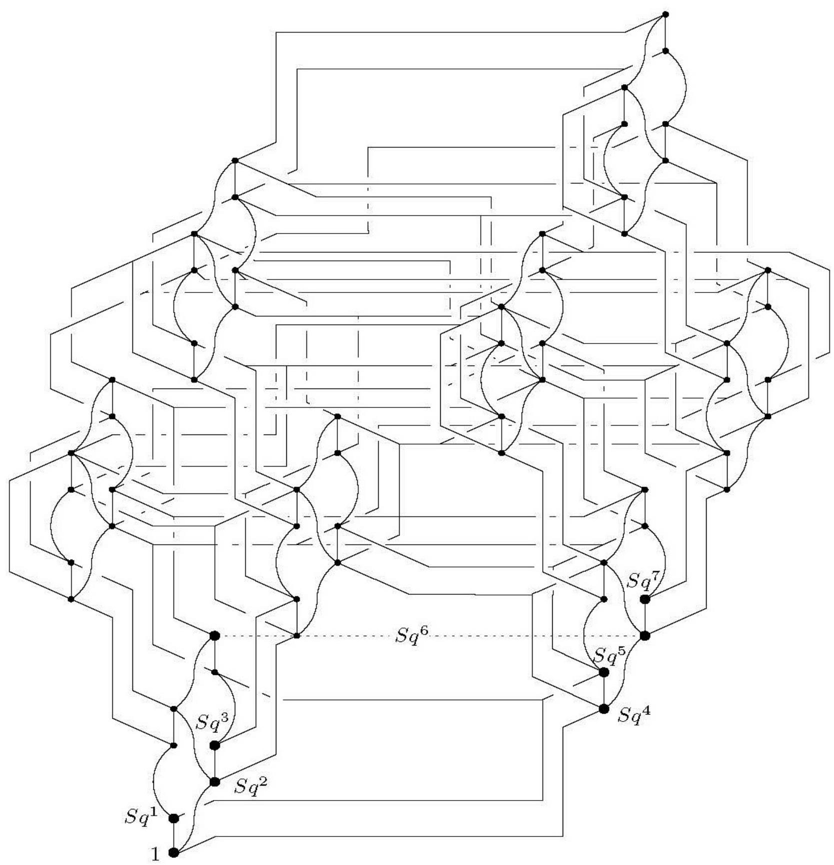
A Appendix: Visual Graphs of $\mathcal{A}(n)$ subalgebras



The diagram of $\mathcal{A}(0)$ (taken from [?])



The diagram of $\mathcal{A}(1)$. A straight line means (left) multiplication by Sq^1, a curved line means multiplication by Sq^2 (taken from [?])



The diagram of $\mathcal{A}(2)$. A straight line means (left) multiplication by Sq^1, a curved line means multiplication by Sq^2, and a bent line means multiplication by Sq^4 (taken from [?])

B Appendix: Matrices of Various Powers

The following are the combined matrices of the hit elements for Sq^1, Sq^2, Sq^4 and Sq^8. The relevant basis elements are the monomials (n, m) of appropriate degree listed in order of increasing m.

For $h = 1$:

$$(1)$$

For $h = 2$:

$$\left(\begin{array}{c|c} 0 & 0 \\ 0 & 1 \end{array}\right)$$

For $h = 3$:

$$\left(\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 0 & 1 \end{array}\right)$$

For $h = 4$:

$$\left(\begin{array}{cc|cc|c} 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

For $h = 5$:

$$\left(\begin{array}{ccc|cc|c} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{array}\right)$$

For $h = 6$:

$$\left(\begin{array}{ccc|ccc|cc} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \end{array}\right)$$

For $h = 7$:

$$\left(\begin{array}{cccc|ccc|cc} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \end{array}\right)$$

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