On the Column Extremal Functions of Forbidden 0-1 Matrices

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Abstract

A 0-1 matrix is a matrix in which every element is either 0 or 1. The weight extremal function $ex(n, P)$ counts the maximum number of 1's in an $n \times n$ matrix which avoids a pattern matrix P. The column extremal function $ex_k(m, P)$ counts the maximum number of columns that a matrix with m rows and k 1's per column can contain such that the matrix avoids P. Set weight and column extremal functions count maximum numbers of 1's and columns respectively of matrices which avoid a given collection of patterns.

We find bounds on the column extremal function for elementary operations on one or two patterns. Using visibility representations, we determine linear bounds on the column extremal functions of patterns with 1's in the same row crossing 1's in the same column and linear bounds on the set extremal functions of a related class of pattern sets. We prove that for any $r \times c$ rectangular configuration, the column extremal function is $\theta(m^r)$. To improve and find new bounds on the weight extremal function, we determine the relation $ex(m, n, P) \leq k(ex_k(m, P) + n)$ for range-overlapping patterns P. We define a new pattern and use bounds on extremal functions of letter sequences coupled with matrix-sequence transformations to bound its column extremal function for $k = 4$ and 5. Finally, we find an upper bound on its weight extremal function by applying our inequality for range-overlapping patterns.

1 Introduction

We define a ***0-1 matrix*** as a matrix consisting solely of 0's and 1's. A ***submatrix*** of matrix M is a matrix formed by selecting certain rows and columns of M. Matrix M ***contains*** or ***represents*** P if and only if a submatrix of M can be transformed into P by changing any number of 1's to 0's. Conversely, matrix M ***avoids*** P if M does not contain P. In Figure 1, A contains B because we can take the submatrix consisting of the first two rows and last two columns and delete the 1-entry in the top right corner to obtain B. On the other hand, A avoids C because no submatrix of A can be transformed into a copy of C by changing some number of 1's to 0's.

$$\underset{A}{\begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix}} \quad \underset{B}{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}} \quad \underset{C}{\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}}$$

Figure 1: A contains B but avoids C.

1.1 Definitions and Methodology

Call the number of 1's in a matrix its ***weight***. A ***pattern*** P is a fixed 0-1 matrix in which we are interested. Here, we define functions on such patterns.

Definition 1.1. The ***weight extremal function*** $ex(n, P)$ is defined as the maximum number of 1-entries that an $n \times n$ matrix can sustain without containing the pattern P. A generalized form of the weight extremal function, $ex(m, n, P)$ gives the maximum weight of an $m \times n$ matrix which avoids P.

Definition 1.2. For a collection S of 0-1 matrices, let the ***set weight extremal function*** $exs(n, S)$ denote the maximum number of 1-entries in an $n \times n$ matrix which avoids every element in S.

Definition 1.3. The ***column extremal function*** $ex_k(m, P)$ gives the maximum number of columns, each with at least k 1-entries, in a 0-1 matrix with m rows which avoids P.

Definition 1.4. For a collection S of 0-1 matrices, let the ***set column extremal function*** $exs(n, S)$ denote the maximum number of columns, each with at least k 1-entries, in a matrix with m rows which avoids every element in S .

We now take a look at some of the types of patterns discussed in this paper.

Definition 1.5. Call a pattern ***linear*** if its weight extremal function is linear in n or its column extremal function is linear in m.

Definition 1.6. We define a ***rectangular pattern*** as a matrix filled with 1's. Let $P_{r,c}$ denote an $r \times c$ rectangular pattern.

Definition 1.7. For each column c in a pattern P, draw a segment connecting the topmost and bottommost 1-entries. Call P ***range-overlapping*** if, for every pair of columns c_1 and c_2, there exists a horizontal line passing through the corresponding segments of both columns.

Representing 1-entries with boxes, Figure 2 shows examples of range-overlapping and nonrange-overlapping patterns. These patterns provide a crucial link between weight and column extremal functions.

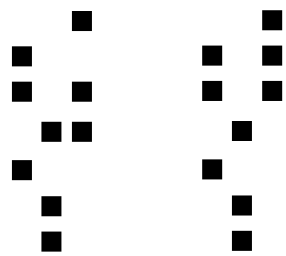

Figure 2: The pattern on the left is range-overlapping. The pattern on the right is nonrange-overlapping because its final two columns have disjoint ranges.

Next, we discuss some miscellaneous terms that will figure into the methods and techniques we use to prove our results.

Definition 1.8. A ***bar visibility representation*** or ***visibility representation*** of a planar graph is a drawing where vertices are represented as finite, disjoint horizontal bars and edges are drawn as vertical segments which may not cross any bar.

Visibility representations will be used to prove linear bounds on the column extremal functions of L_1 and L_2, patterns with linear weight extremal functions. More generally, we define bar-s visibility representations to find linear bounds on the set extremal functions of collections $T_{r,s}$, which we will describe in section 4.

Definition 1.9. A ***bar s-visibility representation*** of a bar s-visibility hypergraph is a drawing where vertices are represented as finite, disjoint horizontal bars and edges are drawn as vertical segments intersecting only the $s+2$ vertex bars constituting the edge. The topmost bar in an edge "sees" through s other bars to view the bottommost bar.

Definition 1.10. The ***height*** of a matrix is the number of nonempty rows present in the matrix.

2 Background

2.1 Motivation for the Weight Extremal Function

Fulek [6] briefly discusses the applications associated with the weight extremal function. The motivation for seeking bounds on $ex(n, P)$ stems from the problem of determining the complexities of computer algorithms that avoid specific rectilinear obstacles and minimize path distance, as described by Mitchell [11]. Finding the weight extremal functions of patterns provides upper bounds on the complexities of corresponding algorithms. Mitchell also describes the translation of the shortest path problem to wire-routing, and a later paper by Lee et al. [13] adds motion planning in robotics to the list.

The problem has mathematical applications in discrete geometry as well as graph theory. We can relate 0-1 matrices to bipartite graphs by allowing rows and columns to represent the two disjoint sets of vertices and letting 1's represent edges. Keszegh [7] notes that the problem of determining $ex(n, P)$ is a variation of Turán extremal graph theory for bipartite graphs. Avoiding a given pattern matrix is equivalent to avoiding a corresponding subgraph.

Forbidden patterns can also be connected to forbidden subsequences in sequences of letters. Pettie [10] used bounds on $ex(n, P)$ to derive bounds on the maximum lengths of sequences on 3 or more letters that must avoid certain subsequences. He has developed matrix-sequence transformations that relate the extremal functions of matrices and sequences.

2.2 Previous Work

Tardos [4] proved that all nontrivial patterns have weight extremal functions that are at least linear in order. His result reveals the significance of linear patterns as a set of fundamental patterns and justifies their being extensively studied.

The Stanley-Wilf conjecture, which states that for every permutation Π, there exists a constant C such that the number of permutations of length n which avoid Π is at most C^n, was proposed around 1992 and remained a major open problem for over 10 years. Füredi and Hajnal [5] then conjectured that permutation matrices are linear. Several years later, Klazar [2] showed that Füredi and Hajnal's conjecture, if validated, would imply the Stanley-Wilf conjecture. Finally, in 2004, Marcus and Tardos [1] proved the Füredi-Hajnal conjecture, indirectly solving the Stanley-Wilf conjecture.

Keszegh [3,7], Tardos [4], Füredi and Hajnal [5], among others, proposed elementary operations which alter patterns in such ways that their new weight extremal functions can still be bounded above using function values of the original patterns. Many of these operations preserve the orders of weight extremal functions with respect to n. Operations allow us to both generate larger patterns with bounded functions and find bounds on complex patterns.

Fulek [6] found bounds on the weight extremal functions of L_1 and L_2, shown in Figure 1, using visibility representations constructed by treating rows of a given $n \times n$ matrix as vertices and projections of 1-entries on lower rows as edges. He bounded the number of edges, and, in turn, the number of 1-entries of the square matrix by limiting the multiplicity of edges in the visibility representation.

$$\underset{L_1}{\begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{pmatrix}} \quad \underset{L_2}{\begin{pmatrix} 0 & 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix}}$$

Figure 3: Patterns L_1 and L_2

2.3 The Column Extremal Function, $ex_k(m, P)$

Nivasch [9] improved early bounds on maximum sequence length by blocking sequences into groups of unique symbols. He defined a function $Ex_k(m, s)$ which gives the maximum number of letters in a sequence with m blocks, avoiding an alternation $abab...$ of length s, such that each letter occurs at least k times. Using matrix-sequence transformations, these sequence extremal functions can once again be related to matrix extremal functions. In their paper, Cibulka and Kynčl [8] defined what we call column extremal functions. Cibulka and Kynčl used these functions to assess situations in which matrices avoid a set of patterns.

2.4 Contents and Results

Section 1 provides a basic explanation of the topic and a selection of key terms and methodology. In section 2, we give the historical context of extremal functions, discuss previous developments, and state our own results. In section 3, we determine upper bounds on the column extremal functions of patterns undergoing elementary operations. Section 4 uses visibility representations and bar s-visibility representations to restrict matrices avoiding patterns and classes of patterns. Extending Fulek's [6] argument, we prove linear bounds on the column extremal functions of patterns L_1, L_2, and L_3 and on the set extremal functions of $T_{r,s}$. The pattern L_3 and the collections $T_{r,s}$ will be described in detail later. Section 5 investigates the column extremal functions of patterns with rectangular configurations of r rows and c columns. For $k \geq r$, we show $ex_k(m, P_{r,c}) = \theta(m^r)$ [1]. By a proof analogous to Nivasch's [8], we relate the column and weight extremal functions of range-overlapping patterns in section 6. In section 7, we bound the column extremal function of a new pattern Q for $k = 4, 5$. Finally, in section 8, we discuss the implications of our results and propose questions for future inquiry.

3 Bounds on Operations for $ex_k(m, P)$

3.1 Operations on Patterns for $ex(n, P)$

Earlier research has yielded several notable examples of operations that induce bounded changes in weight extremal functions. Tardos [9] proved that $ex(n, P') \leq ex(n, P) + n$, where P' is a matrix formed by appending a column with a single 1-entry to the right of a 1-entry in the final column of P. Keszegh [7] showed that when we join patterns P and Q at opposite, 1-entry containing corners, $ex(n, R) \leq ex(n, P) + ex(n, Q)$, where R is the resultant matrix. Here, we extend the concept of operations to $ex_k(m, P)$.

Lemma 3.1. *Let the final column of P contain a 1-entry in row r. If P' is the pattern formed by appending a column with a single 1-entry in row r to the right of P, as shown in Figure 4, $ex_k(m, P') \leq ex_k(m, P) + m$.*

Proof. In pattern P, call the 1-entry in row r of the last column x. Suppose we have a matrix A with dimensions $m \times ex_k(m, P')$ and k 1-entries per column which avoids the pattern P'. Create

[1] Here, we use Bachmann-Landau notation, which will be implemented throughout the paper.

$$\begin{pmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix}$$

Figure 4: Operation appending 1-entry

a new matrix A' by deleting the columns containing the rightmost 1-entry in each row. Note that no more than m columns could have been deleted in this process.

Assume for contradiction that A' contains P. Then by construction, there must exist some 1-entry y to the right of x in the copy of P which was deleted in creating A'. Thus, A must contain P', a contradiction. Hence, $ex_k(m, P') \leq ex_k(m, P) + m$. □

Lemma 3.2. *Let P and Q be pattern matrices with height j where the last column of P and the first column of Q are both completely filled with j 1-entries. Then we claim that $ex_j(m, P) + ex_j(m, Q) \geq ex_j(m, R)$, where R is the matrix created by overlapping P and Q at their last and first columns respectively, as shown in Figure 5.*

$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 \end{pmatrix}$$

Figure 5: Operation overlapping P and Q at filled columns

Proof. Suppose A has m rows and $ex_j(m, P) + ex_j(m, Q) + 1$ columns, each with j 1-entries. Since $ex_j(m, P) + ex_j(m, Q) + 1 > ex_j(m, P)$, we can find a copy of P in A. Delete the rightmost column of the copy and repeat this process $ex_j(m, Q) + 1$ times, obtaining a submatrix A' of $ex_k(m, Q) + 1$ deleted columns. By definition, we can find a copy Q within A'. Note that the column in A' which contains the leftmost column of the copy of Q also contains the rightmost column of some copy of P. All 1-entries in this column will align and overlap exactly, so A contains R and

$$ex_j(m, P) + ex_j(m, Q) \geq ex_j(m, R).$$

□

Lemma 3.3. *Let P and Q be matrices such that P has a 1 in its bottom-right corner and no other 1-entries in its bottommost row, and Q has a 1 in its top-left corner and no other 1-entries in* *its topmost row. Overlap P and Q at their aforementioned corners, as shown in Figure 6. For sufficiently large m, $ex_{j+k-1}(m, R) \leq ex_j(m, P) + ex_k(m, Q)$, where R is the resultant matrix.*

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

Figure 6: Operation joining P and Q at corners

Proof. Let matrix A have m rows, $j + k - 1$ minimum 1-entries per column and $ex_j(m, P) + ex_k(m, Q) + 1$ columns. Only considering the top j 1's in each column, we can find $ex_k(m, Q) + 1$ copies of P, each time deleting the rightmost column of the copy in A. We form a submatrix A' from the $ex_k(m, Q) + 1$ deleted columns. Next, we find a copy of Q using only the bottom k 1-entries in each column. Delete all rows lying exclusively between the second row of the copy of Q and the bottom row of the corresponding copy of P. This forms a copy of R. Hence, $ex_{j+k-1}(m, R) \leq ex_j(m, P) + ex_k(m, Q)$. □

4 Bounds on the extremal functions of L_i and $T_{r,s}$

4.1 Bar s-visibility hypergraphs and $0 - 1$ matrices

Define $T_{r,s}$ to be the collection of matrices M with $r+s+2$ rows and $r+2s+2$ columns such that M restricted to the first $s+1$ columns and rows $2, \ldots, s+2$ is an $(s+1) \times (s+1)$ permutation matrix, M restricted to the last $s+1$ columns and rows $2, \ldots, s+2$ is an $(s+1) \times (s+1)$ permutation matrix, M restricted to the middle r columns and the last r rows is an $r \times r$ permutation matrix, and M has 1-entries in the middle r columns in row 1. For example $T_{1,0}$ contains a single 3×3 matrix with four 1-entries in a diamond formation. Figure 7 represents a pattern in $T_{4,1}$. Black squares are cells with 1's; white squares are cells with 0's.

We extend Fulek's method to show $exs(n, T_{r,s}) = O(n)$ for all $r \geq 1$ and $s \geq 0$. First, we prove a linear bound on the number of edges in a bar s-visibility hypergraph with n vertices. This proof is similar to the proof of the maximum number of edges in a bar s-visibility graph with n vertices

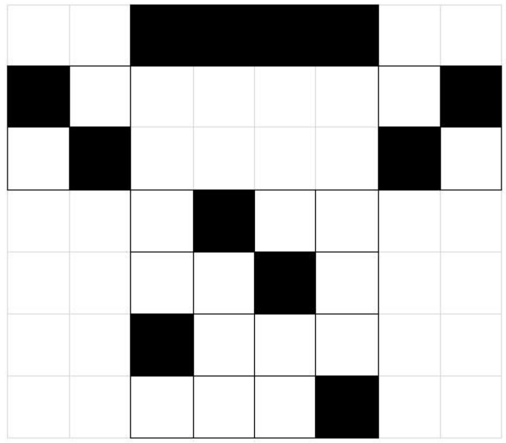

Figure 7: element of $T_{4,1}$

in [12] by Dean et al. We assume all bar endpoints have distinct coordinates since this does not decrease the maximum number of edges.

Lemma 4.1. *All bar s-visibility hypergraphs with n vertices have at most $(2s+3)n$ edges.*

Proof. Scan any representation of the given bar s-visibility hypergraph in the plane from left to right, listing distinct edges. List an edge when, for the first time, a vertical segment can be drawn which intersects only the vertices in the edge. Then edges will be listed only when the scan passes the left or right end of some bar.

For each bar B, the maximum possible number of edges added to the list when the scan passes the left end of B is $s+2$ since there are at most $s+2$ vertical segments representing different edges which pass through the left end of B and through $s+1$ other bars. The maximum possible number of edges added to the list when the scan passes the right endpoint of B is $s+1$ since there are at most $s+1$ vertical segments representing different edges which pass just right of B and through $s+2$ other bars, at least one of which is below B and at least one of which is above B. With n bars, we have at most $(2s+3)n$ edges. □

In the following theorem, we will change $0-1$ matrices avoiding $T_{r,s}$ into bar s-visibility hypergraphs, and then show that the resulting hypergraphs have edge multiplicity at most $r-1$.

Theorem 4.2. *For all $r \geq 1$ and $s \geq 0$, $exs(n, T_{r,s}) = O(n)$.*

Proof. Let M be an $n \times n$ matrix which avoids $T_{r,s}$. Define M' to be the matrix obtained from M by deleting the first $s+1$ and last $s+1$ 1-entries in every row, and the last r 1-entries in every column. Construct a representation of a bar s-visibility hypergraph H from M' by drawing a bar in each row with left end at the first 1-entry of M' in the row and right end at the last 1-entry of M' in the row. For each 1-entry which is not among the bottommost $s+1$ 1-entries in its column in M', draw a vertical line starting from the 1-entry and extending through s bars until reaching the $(s+1)^{st}$ bar below the 1-entry.

Suppose for contradiction that H contains some edge e with multiplicity at least r. Let $u_1, \dots, u_{s+2}$ be the rows of M' which contain the vertices in the edge e, and let $c_1, \dots, c_r$ be columns of M' which contain r vertical segments representing the copies of e. Let $v_1, \dots, v_r$ be distinct rows of M such that v_i contains one of the bottommost r 1-entries of c_i in M for each $i = 1, \dots, r$; let $d_1, \dots, d_{s+1}$ be distinct columns of M such that d_i contains one of the $s+1$ leftmost 1-entries of u_i in M for each $i = 1, \dots, s+1$; and let $e_1, \dots, e_{s+1}$ be distinct columns of M such that e_i contains one of the $s+1$ rightmost 1-entries of u_i in M for each $i = 1, \dots, s+1$. Then the submatrix of M consisting of rows $u_1, \dots, u_{s+2}, v_1, \dots, v_r$ and columns $c_1, \dots, c_r, d_1, \dots, d_{s+1}, e_1, \dots, e_{s+1}$ contains an element of the collection $T_{r,s}$, a contradiction.

Then every edge of H has multiplicity less than r, so the number of 1-entries in M is at most $(2s+2+r)n + (r-1)(2s+3)(n-r)$. □

Observe that every element of $T_{r,s}$ contains the pattern L_3, shown in Figure 8, for $r \geq 3$ and $s \geq 1$. Hence we obtain the following corollary.

$$\begin{pmatrix} 0 & 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix}$$

Figure 8: Pattern L_3

Corollary 4.3. $ex(n, L_3) = O(n)$

Let L_1 and L_2 be the patterns depicted in Figure 1. Visibility representations can also be used to derive linear bounds on the column extremal functions of L_1, L_2, and the collections $T_{r,s}$. The first proof is much like Fulek's [6] proof that $ex(n, L_1) = O(n)$.

Lemma 4.4. *For $k \geq 3$, $ex_k(m, L_1) \leq \frac{5m-13}{k-2}$.*

Proof. We follow Fulek's proof for $ex(n, L_1)$ with some slight modifications. Begin with matrix A which contains k 1's in each column. Delete columns containing the first and last 1's in each row. We note that at most $2m$ columns are deleted in this process. Next, let us denote the cell in row r and column c of A as (r, c). Construct a visibility representation by creating a bar from the first 1-entry to last 1-entry of every nonempty row excluding the bottommost nonempty row. We call these bars vertices of our representation of A. We say that (r, c) is directly above (r', c) if there exists a direct, vertical path along column c between the two cells such that the path intercepts no other bar lying between vertices r and r', and (r, c) and (r', c) lie on bars r and r' respectively. For every 1-entry (r_i, c) that is not the bottommost or second bottommost 1-entry in its column and lies directly above a cell (r_j, c), drop an edge from vertex r_i to r_j in column c.

We claim that A cannot avoid L_1 if it contains multiple edges between the same pair of vertices. Suppose for contradiction that the visibility representation of A contains edges projecting down from 1-entries (r_1, c_1) and (r_1, c_2) on vertex r_1 to vertex r_2. Then by our construction, we can find 1-entries on row r_2 to the left of column c_1 and to the right of c_2, namely, the endpoints of the r_2. Call these (r_2, c_{left}) and (r_2, c_{right}). Also by construction, we must have a 1-entry below row r_2 in column c_1, namely, the bottommost 1-entry in c_1. Call this (r_{bottom}, c_1). We see that the configuration created by 1-entries (r_1, c_1), (r_1, c_2), (r_2, c_{left}), (r_2, c_{right}), and (r_{bottom}, c_1) contains pattern L_1, as shown in Figure 9, a contradiction.

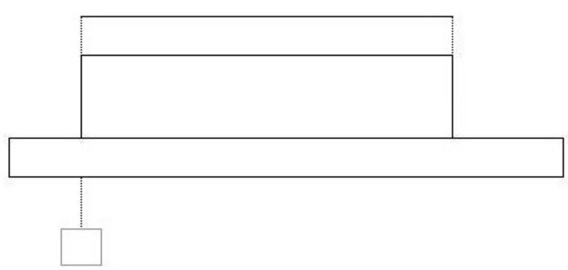

Figure 9: Portion of visibility representation that yields L_1

We use the well-known fact that a simple, connected planar graph with $x \geq 3$ vertices can have at most $3x - 6$ edges. Our visibility representation has at most $m - 1$ vertices and hence at most $3m - 9$ edges, which correspond to 1-entries that are not the bottommost or second bottommost 1's in their column or the leftmost and rightmost 1-entries in their row. To account for these, we add back a possible $2(ex_k(m, L_1) - 2)$ and $2m$ 1-entries respectively and obtain $k(ex_k(m, L_1)) \leq 2m + 3m - 9 + 2(ex_k(m, L_1)) - 4$, which becomes $ex_k(m, L_1) \leq \frac{5m-13}{k-2}$ for $k \geq 3$.

By a similar argument, if we have a matrix A which avoids L_2, the visibility representation of A cannot have any edges of multiplicity 3. Then $ex_k(m, L_2) \leq \frac{8m-22}{k-2}$ for $k \geq 3$. □

We extend this technique to obtain upper bounds on the column extremal functions of $T_{r,s}$ which are tight up to constant gaps.

Lemma 4.5. $exs_k(m, T_{r,s}) = O(\frac{m}{k})$

Proof. Fix $k \geq r + s + 2$. Let M be a matrix with m rows and k 1's per column which avoids $T_{r,s}$. Define M' to be the matrix obtained from M by deleting the columns which contain the first $s+1$ and last $s+1$ 1-entries in every row. At most $2(s+1)m$ columns are deleted. Construct a representation of a bar s-visibility hypergraph H from M' by drawing a bar in every row besides the bottommost r rows with left end at the first 1-entry of M' in the row and right end at the last 1-entry of M' in the row. For each 1-entry which is not among the bottommost $s + r + 1$ 1-entries in its column in M', draw a vertical line starting from the 1-entry and extending through s bars until reaching the $(s + 1)^{st}$ bar below the 1-entry.

Suppose for contradiction that H contains some edge e with multiplicity at least r. Let $u_1, \ldots, u_{s+2}$ be the rows of M' which contain the vertices in the edge e, and let $c_1, \ldots, c_r$ be columns of M' which contain r vertical segments representing the copies of e. Let $v_1, \ldots, v_r$ be distinct rows of M such that v_i contains one of the bottommost r 1-entries of c_i for each $i = 1, \ldots, r$; let $d_1, \ldots, d_{s+1}$ be distinct columns of M such that d_i contains one of the $s+1$ leftmost 1-entries of u_i in M for each $i = 1, \ldots, s + 1$. Let $e_1, \ldots, e_{s+1}$ be distinct columns of M such that e_i contains one of the $s + 1$ rightmost 1-entries of u_i in M for each $i = 1, \ldots, s + 1$. Then the submatrix of M consisting of rows $u_1, \ldots, u_{s+2}, v_1, \ldots, v_r$ and columns $c_1, \ldots, c_r, d_1, \ldots, d_{s+1}, e_1, \ldots, e_{s+1}$ contains an element of the collection $T_{r,s}$, a contradiction.

A bar s-visibility hypergraph with x vertices can have at most $(2s+3)x$ distinct edges, so the s-visibility representation of M has at most $(r-1)(2s+3)(m-r)$ edges which correspond to 1-entries that are not among the $s + r + 1$ bottommost 1's in their column or the $s + 1$ leftmost and $s + 1$ rightmost 1-entries in their row. To account for these, add back a possible $(s+r+1)(exs_k(m, T_{r,s}) - 2(s + 1))$ and $2(s + 1)m$ 1-entries respectively. Summing all components, we obtain the inequality

$k(exs_k(m, T_{r,s})) \leq 2(s+1)m + (r-1)(2s+3)(m-r) + (s+r+1)(exs_k(m, T_{r,s}) - 2(s+1))$. Rearranging confirms the lemma. □

This result gives an upper bound on the column extremal function of L_3 that is tight up to a constant.

Corollary 4.6. $ex_k(m, L_3) = O(\frac{m}{k})$

5 Column Extremal Functions of Rectangular Patterns

5.1 $k = r$

In the case that k is equal to the number of rows in our rectangular pattern, we can find the exact value of $ex_k(m, P_{r,c})$ using a simple counting argument.

Lemma 5.1. $ex_k(m, P_{k,c}) = (c-1)\binom{m}{k}$.

Proof. There exist $\binom{m}{k}$ configurations of k 1-entries in a column with m rows. By the pigeonhole principle, a matrix with $(c-1)\binom{m}{k} + 1$ columns must have one configuration that occurs at least c times. A submatrix consisting of c identical columns and their k nonempty rows forms a copy of $P_{k,c}$. Hence, $ex_k(m, P_{k,c}) \leq (c-1)\binom{m}{k}$. We obtain the same lower bound by constructing a matrix with $c-1$ copies of each configuration; this avoids $P_{k,c}$. Therefore, $ex_k(m, P_{k,c}) = (c-1)\binom{m}{k}$. □

5.2 General bounds for $k \geq r$

Theorem 5.2. *For $k \geq r$,*

$$ex_k(m, P_{r,c}) = \theta(m^r).$$

Proof. We proceed by induction on k to prove a lower bound of order m^r. For our base case, we see that $ex_r(m, P_{r,2}) = \theta(m^r)$, according to Lemma 5.1. Since $ex_k(m, P)$ is decreasing in k, this also serves as an upper bound.

For any 0-1 matrix M, let G_M be the graph obtained from M by letting every column of M be a vertex and adding an edge between two vertices if and only if their corresponding columns have 1-entries in exactly $r-1$ common rows. Note that $r-1$ is the maximum number of rows a pair of columns may share without containing $P_{r,2}$. If M is a matrix with m rows and r 1-entries per column which avoids $P_{r,2}$, then the maximum degree of any vertex of G_M, $\Delta(G_M)$, is at most $r(m-r)$.

Fix $k > r$. Let M be the matrix obtained in the last inductive step avoiding $P_{r,2}$ with xm rows, $\binom{m}{r}$ columns, and $k-1$ 1's per column, such that $\Delta(G_M) \leq ym$ for some constants x and y.

Using a greedy algorithm, color the vertices in G_M using $\Delta(G_M)+1$ colors so that no two vertices with a common edge share a color. Let $C : V_{G_M} \to \{1, \ldots, \Delta(G_M)+1\}$ be the coloring function. Place a 1-entry in row $xm + r$ of column b if and only if $C(b) = r$. Since $\Delta(G_M) \leq ym$, the resulting matrix M' has at most $xm + ym + 1$ rows.

We bound the maximum degree of $G_{M'}$ using the fact that G_M is a subgraph of $G_{M'}$ and bounding the increase from $\Delta(G_M)$ to $\Delta(G_{M'})$. Fix a column b of M'. A column c is a neighbor of b in $G_{M'}$ but not in G_M only if there are exactly $r-2$ rows in M which have 1s in both b and c, and b and c were assigned the same color in G_M. There are $\binom{k-1}{r-2}$ ways to choose which $r-2$ rows contain 1's in both columns b and c. For every set of $r-2$ rows with shared 1-entries there are at most $\frac{xm-(k-1)}{(k-1)-(r-2)}$ possible new neighbors c of b in M', since every pair of new neighbors of b in $G_{M'}$ get the same color in G_M and thus have no common rows in M containing 1-entries besides the $r-2$ rows each neighbor shares with b. Then $\Delta(G_{M'}) \leq \frac{xm-(k-1)}{(k-1)-(r-2)}\binom{k-1}{r-2} + ym$, completing the induction.

We proved that for each $k \geq r$, there exists a constant a_k such that $ex_k(a_k m, P_{r,2}) \geq \binom{m}{r}$. To generalize the bounds for all integral values of m, find n which satisfies $a_k n \leq m \leq a_k(n+1)$. We see that

$$ex_k(a_k n, P_{r,2}) \geq \binom{n}{r} \geq \binom{\frac{m}{a_k} - 1}{r}.$$

Hence, $ex_k(m, P_{r,2}) = \theta(m^r)$.

For $c \geq 2$, we can say that $ex_k(m, P_{r,c}) \geq ex_k(m, P_{r,2})$ since $P_{r,c}$ contains $P_{r,2}$. Therefore, $ex_k(m, P_{r,c}) = \Omega(m^r)$ for all $k \geq r$. Since $ex_r(m, P_{r,c}) = (c-1)\binom{m}{r}$, then $ex_k(m, P_{r,c}) = O(m^r)$ for all $k \geq r$. This gives $ex_k(m, P_{r,c}) = \theta(m^r)$ for all $k \geq r$ and $c \geq 2$, completing the proof.

□

6 Relating $ex_k(m, P)$ to $ex(m, n, P)$

Nivasch [8] proved the correspondence between blocked and unblocked sequences that avoid symbol alternations of a given length. We show that such a relation also exists between the weight and column extremal functions of certain patterns.

Theorem 6.1. *For any pattern P that is range-overlapping,*

$$ex(m, n, P) \leq k(ex_k(m, P) + n).$$

Proof. Create a matrix A with m rows and n columns which avoids a range-overlapping pattern P. For each column c in A, from top to bottom, section off clusters of k 1-entries. Delete all 1-entries remaining outside of a cluster. Starting from the second cluster, move each cluster to a new column. Call the newly formed matrix A', and let there be n' columns in A'.

Suppose for contradiction that A' contains P. Consider two cases.

Case 1: The copy of P in A' contains columns that originated from the same original column c. In this case, P could not be range-overlapping because our construction separated columns of A into clusters with disjoint row indices. This contradicts our assumption.

Case 2: The copy of P in A' only contains columns derived from different original columns. Here, we see that our original matrix A must also contain P, a contradiction.

Therefore, A' cannot contain P. In our construction, we deleted a maximum of $n(k-1)$ 1-entries. Each column of A' contains k 1-entries, so we see that $ex(m, n, P) \leq k(n' + n) \leq k(ex_k(m, P) + n)$, and $ex(m, n, P) \leq k(ex_k(m, P) + n)$. □

7 Column Extremal Function for Patterns with Alternating 1-entries

$$\underset{Q}{\begin{pmatrix} 0 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}} \qquad \underset{P_4}{\begin{pmatrix} 0 & 1 \\ 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{pmatrix}}$$

Figure 10: Patterns Q and P_4

Lemma 7.1. $ex_4(m, P_4) \geq \binom{m-2}{2}$.

Proof. Let M be a matrix with m rows and $\binom{m-2}{2}$ columns such that every column is indexed with a unique (i, j), where $i + 1 < j < m$. The column with index (i, j) contains 1-entries in rows

$i, i+1, j, j+1$. Order these columns lexicographically. If columns c_1 and c_2 have indices (i_1, j_1) and (i_2, j_2) respectively, order c_1 to the left of c_2 if and only if $i_1 < i_2$, or $i_1 = i_2$ and $j_1 < j_2$. We claim that M cannot contain P_4, shown in Figure 10. From top to bottom, let the 1-entries in P_4 be $e_1 \dots e_4$. Assume for contradiction that columns c_x and c_y, with indices (i_x, j_x) and (i_y, j_y) respectively, contain P_4. Suppose c_x is to the left of c_y. Then e_1 and e_3 must lie in c_y. Consider two cases.

Case 1: $i_x = i_y$. By construction, there exist no 1-entries in column c_x below row j_y in M. Therefore e_4 cannot be in column c_x, a contradiction.

Case 2: $i_x < i_y$. Since $i_x < i_y$, then e_2 must lie in either row j_x or row $j_x + 1$. In either case, e_4 cannot be in column c_x, a contradiction.

Therefore, M does not contain P_4, so $ex_4(m, P_4) \geq \binom{m-2}{2}$.

□

Proposition 7.2. $ex_4(m, Q) = \theta(m^2)$.

Proof. We find quadratic upper and lower bounds on $ex_4(m, Q)$.

Because pattern Q, shown in Figure 10, contains P_4, a matrix which contains Q must also contain P_4, so $ex_4(m, Q) \geq ex_4(m, P_4)$ and $ex_4(m, Q) = \Omega(m^2)$. Let M be a matrix with 4 1-entries per column and m rows containing $2\binom{m-2}{2} + 1$ columns. Beginning from the top of each column, call the 1-entries $e_1 \dots e_4$. Define e_2 and e_4 to be even 1-entries. To find an upper bound, we consider the $\binom{m-2}{2}$ distinct arrangements of even 1-entries. We claim that $ex_4(m, Q) \leq 2\binom{m-2}{2}$.

By the pigeonhole principle, we have 3 columns $c_x < c_y < c_z$ which have even 1-entries in identical rows. Take e_2 and e_4 from c_x, e_2 from c_y, and e_1 and e_3 from c_z; these form a copy of Q. Hence,

$$\binom{m-2}{2} \leq ex_4(m, Q) \leq 2\binom{m-2}{2}$$

so, $ex_4(m, Q) = \theta(m^2)$. □

Proposition 7.3. $ex_5(m, Q) = \theta(m \log(m))$.

Proof. We use a lower bound of order $m \log(m)$ on $ex_5(m, P_4)$ to bound $ex_5(m, Q)$ as well. The proof of this bound can be found in section A.1 of the Appendix.

For an upper bound, we count columns by category. Divide matrix M with $2m$ rows into a top m rows and a bottom m rows. Consider columns in which all 5 1-entries are either in the top or bottom section. There are at most a total of $2ex_5(m, Q)$ of these columns. Next, we consider columns which have at least 3 but fewer than 5 1-entries in the top section. There are at most $ex_3(m, Q_2)$ of these because if configuration Q_2, shown in Figure 11, occurs in the top section, then adding the 1-entry that must occur in the bottom section in the leftmost column of this copy of Q_2 would create Q. Similarly, there are at most $ex_3(m, Q_1)$ columns with 3 or 4 1-entries in the bottom section, where Q_1 is also shown in Figure 11. We note that these categories cover all cases.

$$\underset{Q_1}{\begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}} \qquad \underset{Q_2}{\begin{pmatrix} 0 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}}$$

Figure 11: Patterns Q_1 and Q_2

Hence, we obtain the inequality

$$ex_5(2m, Q) \leq 2ex_5(m, Q) + ex_3(m, Q_1) + ex_3(m, Q_2).$$

In general, $ex_k(m, Q_1)$ and $ex_k(m, Q_2)$ are linear. We see that Q_1 is a subpattern of L_1, so Q_1 also has linear extremal function. Furthermore, Q_2 is contained within a pattern which can be derived from applying elementary operations to a trivial pattern, so Q_2 is linear as well. We then rewrite our inequality as $ex_5(2m, Q) \leq 2ex_5(m, Q) + lm$ for some constant l. Using an induction argument found in section A.2 of the Appendix, we show $ex_5(m, Q) \leq cm \log(m)$ for some constant c, which ultimately yields $ex_5(m, Q) = \theta(m \log(m))$. □

Corollary 7.4. $ex(n, Q) = O(n \log n)$

Proof. This follows from Theorem 6.1 and the previous proposition. □

8 Conclusion

8.1 Discussion

With analogous methods, we extended bounds on several elementary operations to $ex_k(m, P)$. It remains unknown which other operations can be applied successfully in the context of column extremal functions. A basic example is the insertion of a column with a single 1-entry between columns containing 1-entries in the same row. While weight extremal functions remain unchanged after rotating patterns, column extremal functions present the challenge that they are not preserved through rotation. For instance, $ex_k(m, P_{2,3})$ clearly differs from $ex_k(m, P_{3,2})$. Furthermore, rotated operations are also unequivalent in column extremal functions. Inserting a 1-entry between columns in the manner mentioned earlier may not change the function in the same way as inserting a 1-entry between two rows containing 1-entries in the same column.

We examined $ex_k(m, P)$ for patterns that had been studied exclusively in $ex(n, P)$, including L_1, L_2, P_4, and the new pattern Q. Examining the extremal functions of these specific patterns will help develop methods for bounding the extremal functions of more general classes of patterns. In particular, we extended the use of visibility representations to find linear bounds on the set extremal functions of collections $T_{r,s}$ and, in turn, linearly bounded the extremal functions of L_3. We would like to investigate $ex_k(m, Q)$ for $k > 5$ as well and examine the growth of the order of the function as k increases. Using related techniques, perhaps we can also derive bounds for other pattern matrices created by adding a column with a single 1-entry to an alternating pattern P_n.

As a case study, we then determined $\theta(m^r)$ bounds on the column extremal functions of $r \times c$ rectangular patterns. However, we have yet to bound the coefficients of these functions in terms of k. With our present construction, we attain an exponentially decreasing lower bound, but these coefficients are not necessarily optimal.

We were also able to relate modified extremal functions to the original extremal functions for range-overlapping patterns. We used the derived inequality to find an analogous upper bound of order $n \log(n)$ on $ex(n, Q)$. Our relation may aid in future attempts to find bounds on column and weight extremal functions that are difficult or impractical to analyze directly.

8.2 Future Work

Additionally, we hope to consider the following open problems: Are there patterns P where $ex_k(m, P)$ is finite and nonlinear in m, but $ex_{k+1}(m, P)$ is linear? Is $ex_k(m, P) = O(m)$ for all permutation matrices P with at most k rows? Is there any pattern P for which $ex(n, P)$ is linear but $ex_k(m, P)$ is nonlinear, or vice versa? Similarly, can certain collections of patterns have linear set weight extremal functions but nonlinear set column extremal functions, or vice versa? For what patterns P is $ex_k(m, P)$ nonlinear in m (or infinite) for every k, but $ex_k(m, P')$ is linear in m (or infinite) for every k, for every pattern P' properly contained in P? How many minimal nonlinear patterns, which become linear upon deletion of any 1-entry, are there, e.g., are there infinitely many? Answering these questions will provide fundamental insight on the nature of column extremal functions and their relation to weight extremal functions.

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A Appendix

A.1 Lower bound on $ex_k(m, P_4)$

Proof. We modify Pettie's [10] matrix-sequence transformations to show bounds on $ex_k(m, P)$ for alternating patterns P. Let a_s be an alternation $aba\dots$ of length s and let P_s be the 0-1 matrix with s rows $0, \dots, s-1$ and 2 columns $0, 1$ such that the number in each entry is the sum of its row and column mod 2. A ***block*** of a sequence is a contiguous, possibly empty, subsequence with no repeated letters. Define $Ex_k(m, s)$ to be the maximum number of letters in a sequence on m blocks which avoids a_s such that every letter occurs at least k times. Let $Ex(m, n, s)$ be the maximum length of a sequence with n letters on m blocks which avoids a_s.

Pettie's sequence to matrix transformation starts with a sequence Q with n letters on m blocks which avoids alternations of length $s+2$, and results in a 0-1 matrix A with n columns and m rows which avoids the matrix P_{s+1}. The letters of Q are named $0, \dots, n-1$ by first occurrence and the blocks are named $0, \dots, m-1$ from left to right. We place a 1-entry in column i and row j of A if and only if the letter i occurs in block j of Q.

Suppose for contradiction that A contains P_{s+1}. Then there is a submatrix of A with 2 columns $c_0 < c_1$ corresponding to an alternation $c_1 c_0 \dots$ of length $s+1$ in Q. However, the first occurrence of c_0 is before the first occurrence of c_1 in Q, so Q contains an alternation of length $s+2$, a contradiction. This implies $Ex(m,n,s+2) \leq ex(m,n,P_{s+1})$ for $m,n,s \geq 1$ and $Ex_k(m,s+2) \leq ex_k(m,P_{s+1})$ for $s,k,m \geq 1$.

Pettie then uses a matrix to sequence transformation to show that $ex(m,n,P_{s+1}) \leq Ex(m,n,s+2)+n$. He converts a 0-1 matrix A with m rows $0,\ldots,m-1$ and n columns $0,\ldots,n-1$ to a sequence Q with m blocks and n letters, named likewise. Letter i occurs in block j of Q if and only if there is a 1-entry in column i and row j.

Let C_j be the letters in block j of Q which occur in no block before j and let D_j be the letters in block j of Q which occur in a block before j. All letters in D_j occur before all letters in C_j in block j, and letters in D_j occur in block j in reverse order of their last appearance before block j. In Pettie's paper the letters in C_j were ordered arbitrarily, but here letter x in C_j occurs before letter y in C_j if and only if $x<y$.

Suppose that Q contains an alternation of length $s+2$ on letters x and y such that $x<y$. List all alternations of length $s+2$ on the letters x and y in Q lexicographically, so that alternation f appears before alternation g if there exists some $i \geq 1$ such that the first $i-1$ elements of f and g are the same, but the i^{th} element of f appears before the i^{th} element of g in Q.

Let f_0 be the first alternation on the list and let π_i be the number of the block which contains the i^{th} element of f_0 for $1 \leq i \leq s+2$. Suppose for contradiction that for some $2 \leq i \leq s+1$, $\pi_i = \pi_{i+1}$. Let a be the i^{th} element of f_0 and let b be the $(i+1)^{st}$ element of f_0.

Since the b in π_i is not the first occurrence of b, the a in π_i cannot be the first occurrence of a, as D_{π_i} precedes C_{π_i}. Hence, a and b in π are both in D_{π_i}, and the last occurrence of a before π_i follows the last occurrence of b before π_i. Let f_1 be the subsequence obtained by replacing the letter a in π_i from f_0 with the last occurrence of a before π_i. Then f_1 is an alternation of length $s+2$ and f_1 occurs before f_0 on the list. This contradicts the definition of f_0, so for $2 \leq i \leq s+1$, $\pi_i < \pi_{i+1}$. We now consider two cases.

Case 1: The first element of f_0 is x. Here, the submatrix of A with 2 columns x,y and $s+1$ rows $\pi_2,\ldots,\pi_{s+2}$ contains P_{s+1} since $x<y$.

Case 2: The first element of f_0 is y. Suppose for contradiction $\pi_1 = \pi_2$. The occurrences of x and y in block π_1 are both first occurrences of x and y in Q. Otherwise, f_0 would not be the first alternation on the list. Since x and y are in C_{π_1}, then x appears before y in block π_1 because $x<y$, a contradiction. Then the submatrix of A with 2 columns x,y and $s+1$ rows $\pi_1,\ldots,\pi_{s+1}$

contains P_{s+1}.

This implies $Ex(m, n, s+2) \geq ex(m, n, P_{s+1})$ for $m, n, s \geq 1$ and $Ex_k(m, s+2) \geq ex_k(m, P_{s+1})$ for $s, k, m \geq 1$. We proved $ex_5(m, P_4) = ex_5(m, a_5)$, and Nivasch [9] showed that $ex_5(m, a_5) = \Omega(m \log(m))$. Hence, $ex_5(m, P_4) = \Omega(m \log(m))$. $\square$

A.2 Induction

Proof. In Proposition 7.3, we proved $ex_5(2m, Q) \leq 2ex_5(m, Q) + lm$ for some constant l.

Inductive hypothesis and base case: Assume $ex_5(m, Q) \leq cm \log(m)$ for all $m \leq i$ for some constant c. This can be done because there are finitely many such m. Additionally, pick $c \geq \frac{2l}{\log(2)}$.

Inductive step: By substituting values into our original inequality, for all m such that $i < m \leq 2i$, we see that $ex_5(2m, Q) \leq 2cm \log(m) + lm$, which we rewrite as $ex_5(2m, Q) \leq 2cm \log(2m) - 2cm \log(2) + lm$. Because we chose $c > \frac{2l}{\log(2)}$,

$$ex_5(2m, Q) \leq 2cm \log(2m) - \frac{2lm}{\log(2)} \log(2),$$

which simplifies to $ex_5(2m, Q) \leq cm \log(2m)$, completing our induction. $\square$

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