Lower Central Series Quotients of Finitely Generated Algebras over the Integers

Katherine Cordwell and Teng Fei

Abstract

We study the lower central series that has elements $L_1 = A, L_2 = [L_1, A] \ldots, L_k = [L_{k-1}, A], \ldots$ for unital associative graded algebras A over $\mathbb{Z}$. Specifically, we consider the quotients $B_k = L_k/L_{k+1}$, each of which is graded and can be written as the direct sum of graded components. Each component is a finitely generated abelian group and may be further decomposed into a free part and a torsion part. The components of the B_i depend on the underlying algebra A in subtle ways; using *Magma*, we gather data, find patterns, prove that certain patterns continue, and formulate some conjectures for the B_i over various A. We mainly consider algebras $A \cong A_n(\mathbb{Z})/\langle f \rangle$ where f is a homogeneous relation and A_n is the free associative algebra on n generators. We completely describe B_i for free algebras modulo a relation of the form $f = xy - qyx$, where $q \in \mathbb{Z}$. We also outline a proof that shows that the ranks of $A_2/\langle x^d \rangle$ stabilize (for $d \in \mathbb{N}$) and present a result concerning a case in which the B_i are finite-dimensional.

1 Introduction

The algebraic approach to geometry is based on replacing geometric spaces by algebras of “nice” functions on them. For instance, the algebra of polynomials, $k[x_1, \ldots, x_n]$, corresponds to the n-dimensional space k^n. Similarly, the algebra $k[x_1, \ldots, x_n]/(x_1^2 + \cdots + x_n^2 - 1)$ corresponds to the sphere in the n-dimensional space defined by the equation $x_1^2 + \cdots + x_n^2 - 1 = 0$.

These algebras are commutative, since multiplication of functions is a commutative operation. Noncommutative geometry is a field where we replace these commutative algebras with similar noncommutative ones, pretending that they correspond to imaginary “noncommutative spaces”. For example, one replaces $k[x_1, \ldots, x_n]$ with the free algebra $A_n = k < x_1, \ldots, x_n >$, which corresponds to the nonexistent noncommutative n-dimensional space. Similarly, the algebra $A_n/(x_1^2 + \cdots + x_n^2 - 1)$ corresponds to the fictitious “noncommutative sphere”.

These noncommutative algebras A are much larger and more complicated than their commutative analogs. To understand their structure, one may study their lower central series $L_1 = A$, $L_2 = [A, L_1], L_3 = [A, L_2]$, ..., which samples their noncommutative nature “in steps”. The structure of this series for algebras of the above type, in particular, the structures of the quotients $B_i := L_i/L_{i+1}$, is therefore of interest. Specifically, it would be interesting to understand how this structure is related to the properties of the corresponding commutative algebras $A_{ab} := A/AL_2$, and the corresponding geometric spaces (in particular, their singularities). This direction has been explored in a number of previous papers, and indeed, the structure of B_i is intimately related to the geometry of the classical n-dimensional space. Based on [6] and later papers on this subject, we further explore this topic, focusing on algebras with relations and on algebras over the integers (in which case there is an additional interesting phenomenon of torsion).

Often, we will consider A to be the algebra with n generators modulo a relation f; we denote such an algebra by $A_n/\langle f\rangle$. We use $A_n(R)$ to denote the free algebra on n generators over a ring R.

The structures of the B_i have been fully or partially characterized for small i. For example, it is well-known that $B_1(A_n)$ has a basis consisting of all cyclic words. Feigin and Shoikhet [6] showed that $B_2(A_n(\mathbb{Q}))$ is isomorphic to the space of all even positive closed differential forms over $\mathbb{Q}^n$. Etingof [7] extended this result to more general algebras over $\mathbb{Q}$, and Balagovic and Balasubramanian [2] found the geometric description for $B_2(A_n/\langle f\rangle)$ where f is a generic homogeneous relation and $A_n(\mathbb{Q})$. Bhupatiraju, Etingof, Jordan, Kuszmaul, and Li [3] determined the structure of $\bar{B}_1(A_n)(\mathbb{Z})$, an object related to the B_i which we will define later, and formulated several conjectures, including one regarding the torsion structure of $B_2(A_n)$ over the integers, proving this conjecture for all but 2-torsion. The paper is organized as follows. In Section 2, we introduce background material. In Section 3, we present our results: a complete description of the B_i for $A/\langle f\rangle$ where f is of the form $xy - qyx$, the stabilization of the B_i for B_i, $i \geq 2$, in $A_2/\langle x^d\rangle$, and the finite-dimensionality of the B_i, $i \geq 2$, in algebras A for which the following properties hold: any two elements of A are algebraically dependent, and the radical of the abelianization of A, $Rad(A_{ab})$, is finite dimensional. In section 4, we conclude and providing suggestions for future work. The paper ends with an appendix in which we discuss our methods and present some of our data.

2 Preliminaries

Let A be a unital associative algebra. In this paper, unless otherwise, noted, we work with algebras over $\mathbb{Z}$; in this case, we think of A as a $\mathbb{Z}$-module, i.e., an abelian group. (In a few situations, we will want to consider A over $\mathbb{Q}$; then we think of A as a vector space over a field.)

For all associative algebras, we define a bilinear Lie bracket operation mapping $A \times A$ to A by $[a, b] = ab - ba$ for $a, b \in A$. This operation satisfies the following properties:

1. $[a, a] = 0$ and
2. $[a, [b, c]] + [b, [a, c]] + [c, [a, b]] = 0$.

An algebra that has such a bracket is called a *Lie algebra*.

If B and C are subspaces of A, we define $[B, C]$ as the set of all finite sums of $[b, c]$ where $b \in B$, $c \in C$.

We then construct the following series for A:

$$L_1 = A$$
$$L_2 = [A, L_1]$$
$$\vdots$$
$$L_{i+1} = [A, L_i].$$

This series is known as the *lower central series* of A. Note that L_{i+1} is a Lie algebra ideal of L_i. We define

$$B_i = L_i/L_{i+1}. \tag{2.1}$$

These are the objects that we study.

We also define $\bar{B}_1(A) = L_1/(L_2 + M_3)$ where $M_3 = A \cdot L_3$. This $\bar{B}_1(A)$ is interesting for a variety of reasons. It is obtained as the quotient of the graded Lie algebra, $\oplus_i B_i$, by part of its center. Also, $\bar{B}_1(A)$ exhibits a polynomial, rather than exponential growth with respect to degree, so it is interesting to compute combinatorially.

When A is an algebra generated by some elements $x_1, \ldots, x_n$ that do not satisfy any polynomial relation, we write $A = A_n$, which is said to be the *free algebra* on n generators. We always grade A_n by assigning each x_i to be of degree one.

2.1 Characterizing the B_i

Each of the algebras A we consider is naturally graded by degree, and this induces a grading on L_i. We may write L_i as a direct sum of its graded components:

$$L_i = \bigoplus_{j \geq 0} L_i[j].$$

Thus, each $B_i = L_i/L_{i+1} = \oplus_{j\geq 0} L_i[j]/L_{i+1}[j]$ is also graded. In all our cases, the components of the B_i are finitely generated.

Now, we can use the structure theorem, which states that every finitely generated abelian group G is isomorphic to the direct sum of a *free component* and a *torsion component*. The free component is isomorphic to $\mathbb{Z}^d$; we call d the *rank* of G and write $\text{rank}(G) = d$. The torsion part will be isomorphic to $\mathbb{Z}_{p_1^{d_1}} \oplus \mathbb{Z}_{p_2^{d_2}} \oplus \cdots \oplus \mathbb{Z}_{p_n^{d_n}}$ for some primes $p_1, \ldots, p_n$ and integers $d_1, \ldots, d_n$. So, to characterize a component of the B_i, we may find its rank and torsion structure.

3 Results

3.1 Patterns

Remark. There are patterns in the ranks of B_2, B_3, and B_4 in A_2 modulo $x^d + y^d$ for $d \in \mathbb{Z}$. The ranks of B_2 seem to form the arithmetic sequence $1, 2, 3, 4, \ldots$, and the ranks of B_3 seem to form the arithmetic sequence $2, 4, 6, \ldots$ (see Tables 1-5 in the appendix). The ranks of B_4 form only a quasi-arithmetic sequence: quotienting by $x^3 + y^3$ produces the rank sequence $3, 7, 3$ (see Table 2 in the appendix), quotienting by $x^4 + y^4$ produces the rank sequence $3, 8, 12, 8, 3$ (see Table 3 in the appendix), quotienting by $x^5 + y^5$ produces the rank sequence $3, 8, 13, 17, 13, 8, 3$ (see Table 4 in the appendix), and so on for polynomials of degree $d \leq 9$. This is an arithmetic sequence with a common difference of 5 except for the middle term, which has a difference of 4 on either side. We conjecture that this holds for all degrees d.

3.2 A Complete Description of $B_k(A_2/\langle xy - qyx\rangle)$

Let $A = A_2(\mathbb{Z})/\langle xy - qyx\rangle$ and k be an integer with $k \geq 2$. Then, we have the the following theorem:

Theorem 3.1.

1. *If* $q \neq \pm 1$, *then:*

$$B_k(A)[i,j] = \begin{cases} \mathbb{Z}_{|q-1|}, & i,j > 0;\ \ k < i+j \\ \mathbb{Z}, & i,j > 0;\ \ k = i+j \\ 0, & \textit{elsewhere} \end{cases} \tag{3.1}$$

2. *If* $q = 1$, *then* $B_k = 0$ *for all* k.

3. *If* $q = -1$, *then*

$$B_k(A)[i,j] = \begin{cases} \mathbb{Z}_2, & i+j > k;\ \ i,j > 0, \textit{not all even} \\ \mathbb{Z}, & i+j = k;\ \ i,j > 0, \textit{not all even} \\ 0, & \textit{elsewhere} \end{cases} \tag{3.2}$$

This may be proven by direct computation: we find a basis for L_k in degree (i,j) and then find a basis for $B_k = L_k/L_{k+1}$.

3.3 The Rank Stabilization Theorem

An interesting pattern arises in the ranks of the $B_i(A_2/\langle f\rangle)$ for generic f, and for certain special f. In the generic case, where f is a homogeneous relation f in degree d, the ranks of the B_i increase monotonically until a certain point, after which they decrease (see Tables 1-5).

However, for certain special f, particularly $f = x^d$, the ranks of $B_i(A_2/\langle f\rangle)$ increase monotonically and then stabilize, as may be seen in tables 6-9 of the appendix. Indeed, we have:

Theorem 3.2 (The Rank Stabilization Theorem)**.** *In $A_2/\langle x^d\rangle$, for each $i \geq 2$ there exists $k \in \mathbb{Z}$ such that rank$(B_i[k]) =$ rank$(B_i[j])$ for any $j \in \mathbb{Z}$ with $j \geq k$. Furthermore, for $l < k$, rank$(B_i[l]) <$ rank$(B_i[k])$. If $i \geq 3$, then $k \geq 2i + d - 5$, and if $i = 2$, then $k \geq d$.*

3.3.1 Proof Outline

Consider W_2, the space of polynomial vector fields in two variables with elements $f(x, y)(\partial/\partial x) + g(x, y)(\partial/\partial y)$ where $f, g \in \mathbb{C}[x, y]$. This space is a Lie algebra. Furthermore, W_2 has a Lie subalgebra, W_1, which is the space of polynomial vector fields in one variable, y. Elements of W_1 are of the form $h(y)(\partial/\partial(y))$ for polynomials $h(y) \in \mathbb{C}[y]$.

It has already been established by Feigin and Shoikhet that $B_i(A_2)$ is a W_2-module for $i \geq 2$ [6]. Furthermore, Arbesfeld and Jordan demonstrated that this $B_i(A_2)$ is a finite length module and that its structure involves tensor field modules, $F_{p,q}$ with $p+q \leq 2i-3$ for $i \geq 3$ [1] (this will become important later).

Because $B_i(A_2)$ is a module over W_2, it is also a module over W_1. If the algebra A we work with is no longer a free algebra, then B_i will no longer be a W_2-module; however, W_1 will still act on $B_i(A_2/\langle x^d \rangle)$, and thus $B_i(A_2/\langle x^d \rangle)$ is still a W_1-module. We claim that $B_i(A_2/\langle x^d \rangle)$ is of finite length as a W_1-module.

We may consider the canonical quotient mapping π that sends $B_i(A_2)$ to $B_i(A_2/\langle x^d \rangle)$. Note that π is a mapping of modules over W_1. We can directly compute that $[W_1, x^d W_2] \subset x^d W_1$, and thus $(x^d W_2) B_i(A_2)$ is a W_1-submodule of $B_i(A_2)$.

Moreover, $\pi((x^d W_2) B_i(A_2)) = 0$. This means that there exists a surjective map $\pi' : B_i(A_2)/((x^d W_2) B_i(A_2)) \to B_i(A_2/\langle x^d \rangle)$, and thus $B_i(A_2/\langle x^d \rangle)$ is a quotient module of $B_i(A_2)/((x^d W_2) B_i(A_2))$. So, to show that $B_i(A_2/\langle x^d \rangle)$ is of finite length, we may show that $B_i(A_2)/((x^d W_2) B_i(A_2))$ is finite.

Because we know that $B_i(A_2)$ is a finite length module [6], we have the following structure: $0 = M_n \subset M_{n-1} \subset \cdots \subset M_1 \subset M_0 = B_i(A_2)$ where each $M_j/M_{j+1} = P_j$ is an irreducible W_2 module. We see that we have the following commutative diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & (x^d W_2) M_{j+1} & \longrightarrow & (x^d W_2) M_j & \longrightarrow & (x^d W_2) M_j / (x^d W_2) M_{j+1} & \longrightarrow & 0\,. \\ & & \downarrow & & \downarrow & & \downarrow f & & \\ 0 & \longrightarrow & M_{j+1} & \longrightarrow & M_j & \longrightarrow & P_j & \longrightarrow & 0 \end{array}$$

The rows are exact sequences and the vertical maps are inclusions (canonical mappings that send an element of one algebraic object to itself as an element of another algebraic object), which is significant because it means their kernels are zero. We apply the snake lemma to obtain the following long exact sequence:

$0 \to \ker(f) \to M_{j+1}/((x^d W_2)M_{j+1}) \to M_j/((x^d W_2)M_j) \to \text{coker}(f) \to 0.$

The cokernel of a mapping $q : X \to Y$ is defined to be $Y/\text{im}(q)$. Thus, $\text{coker}(f) = P_j/(\text{im}(f)) = P_j/((x^d W_2)P_j)$. We see that if we show that each cokernel is of finite length, we will have shown that each $M_j/((x^d W_2)M_j)$ is finite-length, and thus we will have our result.

Now,

$$F_{p,q} = \mathbb{C}[x,y] \otimes Sym^{p-q}(dx, dy) \otimes (dx \wedge dy)^{\otimes q}.$$

We directly compute that $(x^d W_2)F_{p,q} = x^{d-1}F_{p,q}$, and so

$$F_{p,q}/((x^d W_2)F_{p,q}) \cong \bigoplus_{l=0}^{d-2} \bigoplus_{k=q}^{p} x^l (dx)^{p+q-k} F_k,$$

and each $F_k = \mathbb{C}[y](dy)^k$ is irreducible. The result that $F_{p,q}/((x^d W_2)F_{p,q})$ is of finite length follows.

Furthermore, F_k has dimension 1 in each degree greater than or equal to k. So, $x^l(dx)^{p+q-k}F_k$ has dimension 1 after degree $p + q + l$. It follows from the result of Arbesfeld and Jordan [1] that $p + q + l \leq 2i + d - 5$, thus, the rank of $B_i(A_2/\langle x^d \rangle)$, $i \geq 3$ stabilizes from this degree onward. This shows the patterns in B_3, B_4, and B_5 in tables 6-9. The result for $i = 2$ can be obtained in like manner.

The Finite Dimensionality Theorem

We present the following theorem:

Theorem 3.3 (The Finite Dimensionality Theorem). *If any two elements $x, y \in A_{ab}$ are algebraically dependent over k, and $Rad(A_{ab})$ is finite-dimensional, then $B_i(A)$ is finite-dimensional for $i \geq 2$.*

3.3.2 Motivation

This result is based on the work of Jordan and Orem [8] (and has since been independently proven by them). We provide several examples as motivation:

Example 1. Suppose A has n generators and $n-1$ generic relations. In this case, any two generators are algebraically dependent in the abelianization A_{ab}. Furthermore, we may check that $\text{Rad}(A_{ab}) = \{0\}$. So, the conditions of the theorem hold, and we expect that in this case the B_i are finite dimensional.

Example 2. Consider the data for $x^d + y^d$ in $B_i(A_2(\mathbb{Z})/\langle x^d + y^d \rangle)$, $2 \leq d \leq 9$. The ranks of $B_i[m]$ increase and then decrease, indicating that each B_i has nonzero rank in only finitely many gradings. These ranks over $\mathbb{Z}$ are identical to the dimensions of $B_i(A_2(\mathbb{Q})/\langle x^d + y^d \rangle)$. As such, we wish to check if the conditions of the theorem hold for $A_2(\mathbb{Q})/\langle x^d + y^d \rangle$. Since we have two variables, x and y, and one relation in x and y, by definition x and y are algebraically dependent. Now, we wish to consider the radical of the abelizanization of A. In general, to abelianize an algebra of the form $A_n(k)/\langle f \rangle$, we consider the polynomial ring $k[x_1, \ldots, x_n]$ modulo the abelian polynomial that corresponds to f. So, in this case, $A_{ab} = \mathbb{Q}[x, y]/(x^d + y^d)$. Because $x^d + y^d$ has no multiple factors, $\mathbb{Q}[x, y]/(x^d + y^d)$ has no nilpotent elements, and so $\text{Rad}(\mathbb{Q}[x, y]/(x^d + y^d))$ is simply the zero set, which is finite-dimensional. So, the conditions of the theorem hold for $B_i(A_2(\mathbb{Q})/\langle x^d + y^d \rangle)$, just as we wanted.

Example 3. Now, let us consider $B_2(A_2/\langle x^d \rangle)$. Dobrovolska, Kim, and Ma [7] showed that, for all i and j, the brackets $[x^i, y^j]$ form a basis of $B_2(A_2(\mathbb{Q}))$. Because $B_2(A_2(\mathbb{Q})/\langle x^d \rangle)$ is a quotient space of $B_2(A_2(\mathbb{Q}))$, these brackets must span $B_2(A_2(\mathbb{Q})/\langle x^d \rangle)$. Now, the relation $x^d = 0$ will only affect brackets $[x^i, y^j]$ where $i > d$. Thus, the brackets $[x^i, y^j]$, $1 \leq i \leq d-1$, $1 \leq j$ remain linearly independent in the quotient space, and because j is arbitrary, $B_2(A_2(\mathbb{Q})/\langle x^d \rangle)$ is infinite dimensional. Because $A_{ab} = k[x, y]/(x^d)$ has radical (x), which is infinite-dimensional, this does not contradict our theorem.
Furthermore, this shows that $[x^i, y^j]$, $1 \leq i \leq d-1$, $1 \leq j$ is a basis of $B_2(A_2(\mathbb{Q}/\langle x^d \rangle))$, which proves the pattern we see in the ranks of B_2 in tables 6-9.

Example 4. The conditions of the theorem are not satisfied for free algebras with at least two generators, because these generators are not algebraically dependent. So, it is not surprising that the ranks of the B_i increase arbitrarily in free algebras.

Proof Outline

Let $M_i = AL_i$ be the ideal generated by L_i.

By Bapat and Jordan [4], $[M_j, L_k] \subset L_{k+j}$ if j is odd. Let $k = 1$ and $j = 2r + 1$; then we have $[L_1, M_{2r+1}] \subset L_{2r+2}$ which implies that $\sum [x_i, M_{2r+1}] \subset L_{2r+2}$ for all generators x_i of L_1. However, by definition, if $z \in L_j$, then $z = [a, b]$ for $a \in L_1$ and $b \in L_{j-1}$. Now, a is some polynomial of the x_i's. If $a = a_1 a_2$, for some polynomials $a_1, a_2 \in L_1$, we may apply the following identity: $[a_1 a_2, b] = [a_1, a_2 b] + [b a_1, a_2]$. By definition, $a_2 b \in M_{j-1}$ and $b a_1 \in M_{j-1}$. Because of this, we see that $[a, b]$ will be the sum of elements of the form $[x_i, M_{j-1}]$; thus, $L_j \subset \sum [x_i, M_{j-1}]$. In particular, if $j = 2r$, then $L_{2r} \subset \sum [x_i, M_{2r-1}]$.

Now, we see (to be understood in each graded component):

$$\dim L_{2r} - \dim L_{2r+2} \leq \dim \sum [x_i, M_{2r-1}] - \dim \sum [x_i, M_{2r+1}]. \tag{3.3}$$

Now, let V be the n-dimensional vector space spanned by the x_i. Then, we have the following surjections:

$$f : V \otimes M_{2r-1} \to \sum [x_i, M_{2r-1}]$$

$$g : V \otimes M_{2r+1} \to \sum [x_i, M_{2r+1}]$$

We have the following commutative diagram:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \ker f & \longrightarrow & V \otimes M_{2r-1} & \longrightarrow & \sum [x_i, M_{2r-1}] & \longrightarrow & 0 \\
 & & \downarrow a & & \downarrow b & & \downarrow c & & \\
0 & \longrightarrow & \ker g & \longrightarrow & V \otimes M_{2r+1} & \longrightarrow & \sum [x_i, M_{2r+1}] & \longrightarrow & 0.
\end{array}$$

Because $M_{2r-1} \subset M_{2r+1}$, ker $b = 0$ and ker $c = 0$.

Now, by the snake lemma, we get a short exact sequence:

$$0 \to \text{coker } a \to \text{coker } b \to \text{coker } c \to 0,$$

which gives us that $\dim\ \text{coker}\ c \leq \dim\ \text{coker}\ b - \dim\ \text{coker}\ a \leq \dim\ \text{coker}\ b$.

By our previous equations, this means that

$\dim\ \sum[x_i, M_{2r-1}] - \dim\ \sum[x_i, M_{2r+1}] \leq \dim\ V \otimes M_{2r-1} - \dim\ V \otimes M_{2r+1} =$
$\dim\ V(\dim\ M_{2r-1} - \dim\ M_{2r+1}) = n(\dim\ M_{2r-1} - \dim\ M_{2r+1})$.

Thus, we have the following:

$$\dim\ B_{2r} + \dim\ B_{2r+1} \leq n(\dim\ N_{2r-1} + \dim\ N_{2r}). \qquad (3.4)$$

Now, recall that $L_j \subset \sum[x_i, M_{j-1}]$; if $j = 2r+1$, then $L_{2r+1} \subset \sum[x_i, M_{2r}]$.

Thus,

$$\dim L_{2r+1} - \dim L_{2r+2} \leq \dim \sum[x_i, M_{2r}] - \dim \sum[x_i, M_{2r+1}]. \qquad (3.5)$$

If we define maps:

$$\hat{f} : V \otimes M_{2r-1} \to \sum[x_i, M_{2r}]$$

$$\hat{g} : V \otimes M_{2r+1} \to \sum[x_i, M_{2r+1}]$$

then, similarly to the above, we find that

$\dim\ \sum[x_i, M_{2r}] - \dim\ \sum[x_i, M_{2r+1}] \leq n(\dim\ M_{2r} - \dim\ M_{2r+1})$ Thus,

$$\dim B_{2r+1} \leq n \dim N_{2r}. \qquad (3.6)$$

Thus, if $k \geq 3$, showing that the N_i are finite-dimensional will show that the B_i are finite-dimensional.

It is known that $A/M_3 = R$ is a commutative ring with product operation $a \cdot b = \frac{1}{2}(ab + ba)$. Now, N_i is a module over R. We define the operation as follows: Let a be the lift of $a \in A$ and let m be the lift of $m \in N_i$. Then, take $\frac{1}{2}(am + ma)$. Since $m \in M_i$, $\frac{1}{2}(am + ma) \in M_i$ and thus we may consider $a \cdot m$ in N_i to be the equivalence class of $\frac{1}{2}(am + ma)$.

The first question is whether this product is well-defined. To show that it is, consider m' to be an alternative lift of $m \in N_i$. By definition, $m - m' \in M_{i+1}$, and thus $\frac{1}{2}(a(m - m') + (m - m')a) \in M_{i+1}$, which means that our product does not depend on the choice of the lift of m. Now, let a' be an alternative lift of a. By definition, $a - a' \in M_3$; thus $(a - a')m \in M_3M_i$ and $m(a - a') \in M_iM_3$, and by Corollary 1.4 in [4], $(a - a')m$ and $m(a - a')$ are both elements of M_{i+2}, and so $\frac{1}{2}((a - a')m - m(a - a')) \in M_{i+1}$. This shows that the product does not depend on the choice of the lift of a. Thus, the product operation is well-defined. To check that N_i is a module with this product is straightforward.

It is also known that if R is a finitely generated algebra over $\mathbb{Q}$, and M is a finitely generated module over R, then $\dim_{\mathbb{Q}} M$ is finite iff M has finite support. In this case, if supp(N_i) is finite, then the N_i are finite. By definition, supp(N_i) is the set of all prime ideals p of R such that $(N_i)_p \neq 0$, where $(N_i)_p$ is the fraction module $S^{-1}M$ and $S = R \setminus p$.

Now, let $A_{ab} = A/M_2 = X$. We have posited that if X is at most one-dimensional as ring and that if Rad(X) is finite-dimensional as a vector space, then the B_i are finite-dimensional.

We have a short exact sequence:

$$0 \to M_2/M_3 \to A/M_3 = R \to A/M_2 = X \to 0.$$

Let $I = M_2/M_3$, and note that I is an ideal of R. Because A has n generators, then by Lemma 2.5 of [8], $I^n = 0$; thus, the prime ideals of R will correspond to the prime ideals of $R/I = X$.

Our goal is still to show that there are only finitely many prime ideals p s.t. $(N_i)_p \neq 0$. If we can show that, if $(N_i)_p \neq 0$, then p corresponds to a singular ideal of X, then we will be done. But this is equivalent to showing that $(N_i)_p = 0$ for smooth prime ideals p, and to accomplish this, we only need show that the completion of $N_i(A)$, or $(N_i(A))_{(m)}$, is 0 when m is a maximal smooth ideal of X. By [8], $(N_i(A))_{(m)} = N_i(A_{(m)})$. By our assumption that X is at most 1-dimensional, $A_{(m)} = k[[t]]$, the (commutative) one-variable power series ring over k. Thus, $N_i(A_{(m)}) = 0$, and we are done.

4 Conclusion

Thus, we have given the structure of B_i for an infinite class of algebras, and we have presented an outline of a proof for a result that partially characterize the B_i for another infinite class of algebras. We have also shown that the B_i will be finite-dimensional under certain conditions. Finally, we have gathered data which should be useful in future explorations of this problem and similar problems. Topics for future investigation include:

- Working with an algebra modulo more than one relation; in particular, generalizing the stabilization theorem to $A_n/\langle x_1^{d_1}, \ldots, x_{n-1}^{d_r}\rangle$, where $r < n$. More generally, we may consider $A_n/\langle g_1, \ldots, g_{n-1}\rangle$, where the g_i are homogeneous noncommutative polynomials of $x_1, \ldots, x_{n-1}$ whose images form a regular sequence in $\mathbb{C}[x_1, \ldots, x_n]$. Still more generally, we may consider $A_n/\langle f_1 \ldots f_{n-k}\rangle$ with $f_1, \ldots, f_{n-k}$ homogeneous polynomials of $x_1, \ldots, x_{n-k}$, which are a regular sequence in $\mathbb{C}[x_1, \ldots, x_n]$ (for example, perhaps $f_i = x_i^{d_i}$). We expect that the dimensions of the homogeneous parts of $B_i[m]$ are polynomials in m of degree $k-1$ for $m > 0$. The proof may be similar to that of $A_2/\langle x^d\rangle$, involving the representation theory of W_k, which is the Lie algebra of polynomial vector fields.
- Further investigating the structure of $\bar{B}_1$.

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6 Appendix

6.1 Methods

To determine the free and torsion components of the B_i, we use the computer program *Magma* [5]. Code exists to compute the B_i over $\mathbb{Q}$, $\mathbb{Z}$, and $\mathbb{F}_p$ for algebras $A = A_n$ or $A = A_n/\langle f \rangle$ for a relation f. The codes work by sequentially computing each integer grading for every L_i and B_i. As the gradings get larger, the computational complexity increases. Thus, it is usually only feasible to compute small gradings, especially when working with several variables. Because of the computational complexity, we compute up to at most 12 gradings in two variables, 8 gradings in three variables, and 6 gradings in 4 variables. Note that this also limits the number of B_i we can compute, because B_i is zero in all degrees less than i.

We compute gradings of B_i for many different relations over the integers. Our primary examples are quotients of A_n by $x_1^d + x_2^d + \cdots + x_n^d$ for $d \in \mathbb{Z}$. We also consider relations $xy - qyx$ and x^d in A_2.

We present a portion of our data for the B_i over $\mathbb{Z}$, formatted formatted as follows: gradings are designated in the first row of the table, and the B_i are designated in the first column. Nonparenthetical terms correspond to rank, and parenthetical terms indicate torsion. Tables 1-5 are for algebras of the form $A = A_n/\langle x^d + y^d \rangle$, and Tables 6-9 are for algebras of the form $A = A_2/\langle x^d \rangle$.

$A_2/\langle x^2+y^2\rangle$	0	1	2	3	4	5	6	7	8	9	10	11	12
B_2	0	0	1	(2^2)	(2)	(2^2)	(2)	(2^2)	(2)	(2^2)	(2)	(2^2)	(2)
B_3	0	0	0	2	(2^2)	(2^4)	(2^2)	(2^4)	(2^2)	(2^4)	(2^2)	(2^4)	(2^2)
B_4	0	0	0	0	2	(2^4)	(2^3)	(2^6)	(2^3)	(2^6)	(2^3)	(2^6)	(2^3)
B_5	0	0	0	0	0	4	(2^3)	(2^6)	(2^4)	(2^8)	(2^4)	(2^8)	(2^4)
B_6	0	0	0	0	0	0	3	(2^6)	(2^4)	(2^8)	(2^5)	(2^{10})	(2^5)
$\bar{B}_1$	1	2	2	2	2(2)	2	2(2)	2	2(2)	2	2(2)	2	2(2)

Table 1: $A = A_2/\langle x^2+y^2\rangle$

$A_2/\langle x^3+y^3\rangle$	0	1	2	3	4	5	6	7	8	9	10	11
B_2	0	0	1	2	$1(3^2)$	(3^3)	(3^2)	(3^3)	(3^3)	(3^2)	(3^3)	(3^3)
B_3	0	0	0	2	4	$2(3^4)$	(3^6)	(3^6)	(3^6)	(3^6)	(3^6)	(3^6)
B_4	0	0	0	0	3	7(2)	$3(2^2\cdot 3^7)$	$(2^2\cdot 3^{14})$	(3^{14})	(3^{12})	(3^{14})	(3^{14})
B_5	0	0	0	0	0	6	$13(2^2)$	$6(2^6\cdot 3^{14})$	$(2^5\cdot 3^{25})$	$(2^2\cdot 3^{25})$	(3^{27})	(3^{27})
B_6	0	0	0	0	0	0	9	$22(2^5)$	$10(2^{12}\cdot 3^{22})$	$(2^{12}\cdot 3^{40})$	$(2^5\cdot 3^{48})$	$(2^2\cdot 3^{50})$
$\bar{B}_1$	1	2	3	3	3(2)	3	3(3)	3	3	3(3)	3	3

Table 2: $A = A_2/\langle x^3+y^3\rangle$

$A_2/\langle x^4+y^4\rangle$	0	1	2	3	4	5	6	7	8	9	10
B_2	0	0	1	2	3	$2(4^2)$	$1(2^2\cdot 4)$	(4^4)	$(2\cdot 4^4)$	(4^4)	$(2^2\cdot 4^4)$
B_3	0	0	0	2	4	6	$4(2\cdot 4^3)$	$2(4^6)$	$(2^4\cdot 4^4)$	(4^8)	$(2^2\cdot 4^6)$
B_4	0	0	0	0	3	8	$12(2\cdot 5)$	$8(2^4\cdot 3^2\cdot 4^6\cdot 5^2)$	$3(2^8\cdot 3^4\cdot 4^9\cdot 5^2)$	$(2^8\cdot 3^2\cdot 4^{12})$	$(2^{14}\cdot 4^6)$
B_5	0	0	0	0	0	6	15	$24(2^2\cdot 5^2)$	$17(2^{13}\cdot 3^4\cdot 4^8\cdot 5^4)$	$6(2^{18}\cdot 3^6\cdot 4^{18}\cdot 5^4)$	$(2^{32}\cdot 3^7\cdot 4^{14})$
B_6	0	0	0	0	0	0	9	30	$46(2^6\cdot 4\cdot 5^5)$	$34(2^{26}\cdot 3^{10}\cdot 4^{18}\cdot 5^{10})$	$12(2^{61}\cdot 3^{18}\cdot 4^{22}\cdot 5^9)$
$\bar{B}_1$	1	2	3	4	4(2)	4	$4(2^2\cdot 3)$	4	$4(2\cdot 4)$	4	$4\cdot(2^2)$

Table 3: $A = A_2/\langle x^4+y^4\rangle$

$A_2/\langle x^5+y^5\rangle$	0	1	2	3	4	5	6	7	8	9	10
B_2	0	0	1	2	3	4	$3(5^2)$	$2(5^3)$	$1(5^4)$	(5^5)	(5^4)
B_3	0	0	0	2	4	6	8	$6(5^4)$	$4(5^6)$	$2(5^8)$	(5^{10})
B_4	0	0	0	0	3	8	13	$17(2\cdot 5)$	$13(4^2\cdot 5^{10})$	$8(2^2\cdot 4^2\cdot 5^{17})$	$3(2^2\cdot 4^2\cdot 5^{22})$
B_5	0	0	0	0	0	6	15	26	$35(2^3\cdot 5^2)$	$28(2^6\cdot 4^2\cdot 5^{20})$	$17(2^6\cdot 3^2\cdot 4^6\cdot 5^{36})$
B_6	0	0	0	0	0	0	9	30	54	$72(2^7\cdot 5^7\cdot 7)$	$60(2^{18}\cdot 3^2\cdot 4^4\cdot 5^{43})$
$\bar{B}_1$	1	2	3	4	5(2)	5	$5(2^2\cdot 3)$	5	5(4)	5	5(5)

Table 4: $A = A_2/\langle x^5+y^5\rangle$

$A_2/\langle x^6+y^6\rangle$	0	1	2	3	4	5	6	7	8	9	10
B_2	0	0	1	2	3	4	5	$4(2^2\cdot 3^2)$	$3(2\cdot 3^3)$	$2(2^4\cdot 3^2)$	$(2^2\cdot 3^5)$
B_3	0	0	0	2	4	6	8	10	$8(2^3\cdot 3^4)$	$6(2^6\cdot 3^6)$	$4(2^5\cdot 3^8)$
B_4	0	0	0	0	3	8	13	18	$22(5\cdot 7)$	$18(2^6\cdot 3^8\cdot 5^4\cdot 7^2)$	$13(2^9\cdot 3^{16}\cdot 5^4\cdot 7^2)$
B_5	0	0	0	0	0	6	15	26	$37(2)$	$46(5^2\cdot 7^2)$	$39(2^{11}\cdot 3^{16}\cdot 5^8\cdot 7^4)$
B_6	0	0	0	0	0	0	9	30	54	80	$98(2^2\cdot 5^7\cdot 7^6\cdot 8)$
$\bar{B}_1$	1	2	3	4	$5(2)$	6	$6(2^2\cdot 3)$	6	$6(2^2\cdot 4)$	$6(3^2)$	$6(2^3\cdot 5)$

Table 5: $A = A_2/\langle x^6+y^6\rangle$

$A_2/\langle x^2\rangle$	0	1	2	3	4	5	6	7	8	9	10
B_2	0	0	1	$1(2)$	1	$1(2)$	1	$1(2)$	1	$1(2)$	1
B_3	0	0	0	2	$2(2)$	$2(2^2)$	$2(2)$	$2(2^2)$	$2(2)$	$2(2^2)$	$2(2)$
B_4	0	0	0	0	2	$3(2^2)$	$3(2^3)$	$3(2^5)$	$3(2^3\cdot 5)$	$3(2^4)$	$3(2^3\cdot 7)$
B_5	0	0	0	0	0	4	$5(2^3)$	$5(2^6\cdot 3)$	$5(2^7\cdot 3)$	$5(2^8)$	$5(2^8\cdot 3)$
B_6	0	0	0	0	0	0	5	$9(2^5)$	$9(2^{10}\cdot 3)$	$9(2^{14}\cdot 3)$	$9(2^{14}\cdot 3)$
B_7	0	0	0	0	0	0	0	9	$15(2^7)$	$15(2^{18}\cdot 3^2\cdot 5)$	$15(2^{24}\cdot 3^2\cdot 5)$
B_8	0	0	0	0	0	0	0	0	12	$24(2^{12})$	$25(2^{29}\cdot 3^2\cdot 5)$
B_9	0	0	0	0	0	0	0	0	0	20	$40(2^{18})$
B_{10}	0	0	0	0	0	0	0	0	0	0	29
$\bar{B}_1$	1	2	2	2	$2(2)$	2	$2(2)$	2	$2(2)$	2	$2(2)$

Table 6: $A = A_2/\langle x^2\rangle$

$A_2/\langle x^3\rangle$	0	1	2	3	4	5	6	7	8	9	10
B_2	0	0	1	2	$2(3)$	$2(3)$	2	$2(3)$	$2(3)$	2	$2(3)$
B_3	0	0	0	2	4	$4(3^2)$	$4(3^2)$	$4(3^2)$	$4(3^2)$	$4(3^2)$	$4(3^2)$
B_4	0	0	0	0	3	$7(2)$	$8(2\cdot 3^3)$	$8(2^2\cdot 3^6)$	$8(2\cdot 3^6)$	$8(2^2\cdot 3^5\cdot 5)$	$8(2\cdot 3^6)$
B_5	0	0	0	0	0	6	$13(2)$	$16(2^3\cdot 3^6)$	$16(2^5\cdot 3^{11})$	$16(2^5\cdot 3^{12})$	$16(2^5\cdot 3^{13})$
B_6	0	0	0	0	0	0	8	$24(2^2)$	$31(2^5\cdot 3^{10})$	$32(2^{11}\cdot 3^{22})$	$32(2^{10}\cdot 3^{28})$
B_7	0	0	0	0	0	0	0	16	$44(2^3)$	$59(2^{10}\cdot 3^{19})$	$60(2^{22}\cdot 3^{21}\cdot 5\cdot 9)$
B_8	0	0	0	0	0	0	0	0	25	$79(2^5)$	$112(2^{19}\cdot 3^{34})$
B_9	0	0	0	0	0	0	0	0	0	45	$146(2^8)$
B_{10}	0	0	0	0	0	0	0	0	0	0	76
$\bar{B}_1$	1	2	3	3	$3(2)$	3	$3(2\cdot 3)$	3	$3(2)$	$3(3)$	$3(2)$

Table 7: $A = A_2/\langle x^3\rangle$

$A_2/\langle x^4\rangle$	0	1	2	3	4	5	6	7	8	9	10
B_2	0	0	1	2	3	3(4)	3(2)	3(4)	3	3(4)	3(2)
B_3	0	0	0	2	4	6	$6(2\cdot 4)$	$6(4^2)$	$6(2\cdot 4)$	$6(4^2)$	$6(2\cdot 4)$
B_4	0	0	0	0	3	8	$12(2\cdot 5)$	$13(2^3\cdot 3\cdot 4^2\cdot 5)$	$13(2^3\cdot 3^2\cdot 4^3\cdot 5)$	$13(2^2\cdot 3\cdot 4^5\cdot 5)$	$13(2^4\cdot 3\cdot 4^3\cdot 5)$
B_5	0	0	0	0	0	6	15	$24(2\cdot 5^2)$	$27(2^7\cdot 3^2\cdot 4^3\cdot 5^2)$	$27(2^8\cdot 3^3\cdot 4^6\cdot 5^2)$	$27(2^{11}\cdot 3^4\cdot 4^7\cdot 5^2)$
B_6	0	0	0	0	0	0	9	29(2)	$48(2^2\cdot 4\cdot 5^3)$	$57(2^{12}\cdot 3^5\cdot 4^6\cdot 5^4)$	$58(2^{21}\cdot 3^9\cdot 4^{12}\cdot 5^4)$
B_7	0	0	0	0	0	0	0	18	55(2)	$95(2^8\cdot 4\cdot 5^6)$	$113(2^{29}\cdot 3^9\cdot 4^{11}\cdot 5^9)$
B_8	0	0	0	0	0	0	0	0	29	$103(2^3)$	$186(2^{18}\cdot 3\cdot 4^2\cdot 5^{11})$
B_9	0	0	0	0	0	0	0	0	0	54	$198(2^6)$
B_{10}	0	0	0	0	0	0	0	0	0	0	94
$\bar{B}_1$	1	2	3	4	4(2)	4	$4(2^2\cdot 3)$	4	$4(2\cdot 4)$	4(3)	$4(2^2)$

Table 8: $A = A_2/\langle x^4\rangle$

$A_2/\langle x^5\rangle$	0	1	2	3	4	5	6	7	8	9	10
B_2	0	0	1	2	3	4	4(5)	4(5)	4(5)	4(5)	4
B_3	0	0	0	2	4	6	8	$8(5^2)$	$8(5^2)$	$8(5^2)$	$8(5^2)$
B_4	0	0	0	0	3	8	13	$17(2\cdot 5)$	$18(4\cdot 5^5)$	$18(2\cdot 4\cdot 5^7)$	$18(2\cdot 4\cdot 5^7)$
B_5	0	0	0	0	0	6	15	26	$35(2^2\cdot 5^2)$	$38(2^3\cdot 4\cdot 5^{10})$	$38(2^4\cdot 3\cdot 4^3\cdot 5^{15})$
B_6	0	0	0	0	0	0	9	30	53(7)	$74(2^3\cdot 5^4\cdot 7)$	$83(2^8\cdot 4^2\cdot 5^{20}\cdot 7)$
B_7	0	0	0	0	0	0	0	18	57	$106(7^2)$	$149(2^8\cdot 5^9\cdot 7^2)$
B_8	0	0	0	0	0	0	0	0	30	109(2)	$212(2\cdot 7^3)$
B_9	0	0	0	0	0	0	0	0	0	56	211(2)
B_{10}	0	0	0	0	0	0	0	0	0	0	98
$\bar{B}_1$	1	2	3	4	5(2)	5	$5(2^2\cdot 3)$	5	$5(2\cdot 4)$	5(3)	$5(2^2\cdot 5)$

Table 9: $A = A_2/\langle x^5\rangle$

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