Expanding Undergraduate Research in Mathematics: Making UR More Inclusive

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The CrowdMath Project

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15.1 Introduction

CrowdMath is an online program that was created to give math research opportunities to high school and college students around the world who do not have access to other math research projects [1]. Any student from any country can join CrowdMath just by signing up online. The projects are conducted in the CrowdMath section of the Art of Problem Solving website, where participants can post their questions, ideas, and proofs.

CrowdMath removes all barriers to entry for student research besides having Internet access. It is free for students to join CrowdMath. Students do not have to submit any application or request any letters of recommendation. Any student who is interested is welcome to participate.

The idea was inspired by the online Polymath projects for professional mathematicians led by Timothy Gowers and Terence Tao. The Polymath projects started in 2009 on Gowers's blog, where Gowers posted an open problem and invited others to collaborate on solving it in the blog's comment section. The Polymath projects produced multiple papers with substantial progress on major open problems.

In fall 2015, the first and third authors started discussing the possibility of running an online program with the Art of Problem Solving which would be like a Polymath project for high school students. The original Polymath projects were very successful in finding new mathematical results. With CrowdMath, it was initially unclear whether high school and undergraduate students would be able to solve a real research problem collectively in an online forum. In particular, what would motivate them to participate? It was also unclear whether the collaboration dynamics in CrowdMath would differ from those of Polymath.

Since 2010, the first and third authors have been organizers of the Program for Research in Mathematics, Engineering, and Science (PRIMES) at MIT. PRIMES is a selective year-long research program for high school students that requires a problem set, essays, and letters of recommendation for the application. Mentors at PRIMES, usually graduate students or postdocs in the MIT Mathematics Department, work on projects either one-on-one with students, or with a team of two or three students. The students in PRIMES perform very well in science competitions using their PRIMES projects. Since 2011, PRIMES students have won more than 340 awards at national and international science competitions, including 80 top awards in the Siemens competition, Intel/Regeneron Science Talent Search, ISEF, Davidson Fellowship, and S.-T. Yau High School Science Award. Students that participate in PRIMES usually go to the top colleges in the country, and they publish papers in reputable math journals such as *Representation Theory, Transactions of the AMS, Journal of Algebra, Journal of Algebraic Combinatorics, Electronic Journal of Combinatorics, European Journal of Combinatorics, Discrete Mathematics,* and *Journal of Combinatorics.*

Although the students in PRIMES usually have very high mathematical aptitude and a strong work ethic, most of them already go to highly ranked high schools with strong math programs, or their parents pay for them to take online math courses. Before CrowdMath, the first and third authors created two offshoots of PRIMES to provide less privileged students with an opportunity to do math research. Both programs focus on math enrichment rather than proving new results. One of the offshoot programs called PRIMES Circle was created to increase participation in math research by local high school students from underrepresented populations and underserved communities; it runs during the spring term. Another program is MathROOTS, a residential summer program at MIT open to the same category of students all over the United States.

CrowdMath serves a different purpose than MathROOTS and PRIMES Circle, because CrowdMath is focused on proving new results, and it is open to students all over the world, rather than only in the United States. Whereas MathROOTS and PRIMES Circle are in-person programs, CrowdMath is entirely online. All offshoots of PRIMES except CrowdMath require students to submit applications and be accepted into the programs. With CrowdMath, anyone is allowed to participate. The only thing that participants have to do is register on the Art of Problem Solving website.

In early November 2015, the first and third authors invited the second author, then a PRIMES mentor, to be the head mentor for CrowdMath, and the second author enthusiastically accepted. We recruited two other mentors and chose three topics for the first year: pattern avoidance in 0-1 matrices, incidence problems, and optimal codes. At first we restricted participation to only high school students, but later in the year we also opened it to college students since undergraduates were interested in participating. Now we also allow advanced middle school students.

CrowdMath is a good student research model for multiple reasons. For one, CrowdMath removes all barriers to entry for student research besides having Internet access. Even if they live somewhere with no one else who can help them do math research, a student who likes math can join the CrowdMath project and collaborate with the CrowdMath mentors and other participants.

A 2017 survey revealed that CrowdMath contributors often felt that they had been deeply transformed—from learners to researchers. We tried to design CrowdMath so that collaboration would be valued over competition and participants would exhibit empathy and comradery. Participants have commented on the "friendly environment" of CrowdMath and the opportunity for open, informal discussion and creative exploration [9].

In the following sections we discuss CrowdMath in detail. In Section 15.2, we discuss how CrowdMath works including the platform, how the papers are written, and the role of mentors. In Section 15.3, we compare CrowdMath to Polymath in terms of the target audience, the project leaders, the type of problems, and other factors. In Section 15.4, we discuss things that have worked and not worked in CrowdMath, including both successful and unsuccessful research topics, and ways that mentors have increased CrowdMath activity. In Section 15.5, we discuss the advantages of CrowdMath over in-person research groups including the cost, the ability to serve more students, the lack of pressure, and the ability to collaborate from anywhere in the world. In Section 15.6, we describe an example of a CrowdMath research thread from summer 2020 on the topic of metric dimension. Finally in Section 15.7, we discuss the benefits of CrowdMath for students, faculty, schools, and other math research programs.

15.2 How CrowdMath works

CrowdMath projects are conducted entirely on the CrowdMath section of the Art of Problem Solving website. The CrowdMath section includes a message board where mentors and students discuss the research project. The format of the CrowdMath message board is similar to other forums such as Reddit and Twitter, except the Art of Problem Solving website can compile LaTeX code, so it gives CrowdMath participants the ability to display math nicely in their posts. In the CrowdMath section, there is also a page with open problems and a page with references and exercises for each research topic.

Each year, we choose topics that are understandable for students with little to no mathematical background beyond the standard high school courses. In late December or early January, we post several references and exercises for each topic on the CrowdMath section of the Art of Problem Solving website. The exercises usually consist of known results that have short proofs. We also post a list of open problems for each topic.

Throughout the year, CrowdMath participants interact with CrowdMath mentors and each other by posting on the message board. Posts include questions, ideas for possible solutions, full solutions, and new open problems. The research direction goes wherever the participants want to take it. Sometimes a participant asks a new open problem on the message board and other participants start working on it, and this leads to a long research thread. At the end of the year, the mentors compile the results from the year's CrowdMath project and write papers for each topic that has interesting results. The papers appear under the collective authorship of P.A. CrowdMath.

In every year of CrowdMath, we have at least one topic in discrete math, but sometimes there are topics outside of discrete math. Part of this focus on discrete math is because of the second author's background, and another reason is because problems in discrete math usually do not require as much background for students to understand as many other areas of math. Most of the CrowdMath problems that we choose can be described in at most a paragraph. The problem descriptions for CrowdMath are entirely self-contained, so advanced high school students can understand them without having background in the research topic.

People with knowledge of the research topics or ideas for new topics are welcome to participate as mentors. Most of the mentors in CrowdMath are postdocs or graduate students, but recent math PhDs working in industry have also participated, using this as an opportunity to remain active in math research even after leaving academia. The mentoring workload varies, depending on the number of other mentors and student participation, and this provides flexibility for various degrees of mentor participation.

15.3 Comparing CrowdMath and Polymath

Some of the major differences between CrowdMath and Polymath are the platforms, the target audiences, the anonymity of the participants, the type of problems, and the educational support for participants. The target audience is professional mathematicians for Polymath but high school students for CrowdMath. While Polymath can choose open problems that require a sophisticated mathematical background to understand, we choose open problems for CrowdMath that advanced high school students can understand. Moreover, we offer online classes for newcomers on the Art of Problem Solving website to learn more about the CrowdMath topics. Another key difference is in the project leadership. While Polymath projects are led by high-profile mathematicians including Fields medalists, CrowdMath projects have been led by postdocs and graduate students.

The number of participants in Polymath 1 and CrowdMath 1 was similar, but 74% of Polymath 1 participants used their true identity, while only 3% of CrowdMath 1 participants used their true identity. When participants use an alias on the CrowdMath message board, neither the mentors nor the other students know their true identity. Participants in CrowdMath have noted that they like the anonymity of CrowdMath, since it results in less judgment based on demographic differences between participants [9]. CrowdMath 1 had a gender imbalance in participation, but it was not as bad as Polymath 1. While 14% of CrowdMath 1 participants identified as a girl, only 3% of Polymath 1 participants identified as women.

Collective credit was an issue in Polymath projects [10]. With CrowdMath, so far it has never been an issue because CrowdMath was designed to value collaboration over competition. In surveys, one thing that students have liked about CrowdMath is that the papers are written under the name P.A. CrowdMath. One student said that they were "really excited" to publish under the pseudonym, and that working together on the paper was their "best moment." Another student said, "I don't think it would have made much of a difference whether or not I was co-author, because I definitely gained more than having another paper I can list on a CV." The same survey found that making substantial progress toward the solution and publishing a collective paper are motivations for CrowdMath participants, even if the progress was not due to the student's personal contribution [9].

Participants of Polymath and other professional collaborative online forums felt that such forums lacked informal, exploratory aspects and experimentation [11]. On the other hand, most of the math done in CrowdMath has been exploratory. Although CrowdMath participants have solved some known open problems, most of the results in CrowdMath papers are solutions to problems that were suggested by other CrowdMath participants. Experimentation has frequently led to ideas and conjectures in CrowdMath. One survey respondent said that participating in CrowdMath showed them that research math is "about experimenting and playing with things over time" [9]. Some participants in CrowdMath are strong programmers and have been able to find the answers to some of the CrowdMath problems before the solutions had rigorous proofs. This experimentation was useful to other CrowdMath participants, since it

	CrowdMath 1	Polymath 1
Year	2016	2009
Platform for collaboration	message board on the Art of Problem Solving website	blog comment section
Target audience	High school students	Professional mathematicians
Educational support	Online classes, exercises, answers to questions	Answers to questions
Project leaders	MIT students and recent graduates	High-profile mathematicians
Number of participants	35	39
Identity of participants	97% anonymous	26% anonymous
Gender of participants	14% girls	3% women
Type of problems	Exploratory	Major open problems

Table 15.1. A comparison between CrowdMath 1 and Polymath 1

provided conjectures for them to attempt to prove instead of having no idea what the correct answer should be.

Another complaint about Polymath was that it lacked empathy and comradery [11]. By comparison, one respondent in a survey of CrowdMath participants noted that there is no criticism in CrowdMath, and another said that they "really appreciate the friendly environment on CrowdMath" [9]. Another respondent in the same survey noted that contributing to CrowdMath "made the idea of doing research less intimidating," and made them consider "a career in research." Other participants found that CrowdMath taught them that it was okay to post things that were "completely wrong," such as "ideas even if they were just ideas." They also learned that it was okay to ask questions and post if they did not understand something [9]. In comparison, Terence Tao noted that one of the cultural inhibitors to participation in projects like Polymath is that professional mathematicians are "reluctant to say anything on the public record which may end up being wrong, foolish, or naive" since this could damage their reputation [13].

CrowdMath has three rules which are designed to work against some of the toxic elements of the original Polymath: (1) be polite and constructive, (2) make your comments as easy to understand as possible, and (3) it's okay for a mathematical thought to be tentative, incomplete, or even incorrect. The first rule helps maintain CrowdMath's friendly environment, while the second rule helps make research less intimidating for participants. In addition to open problems, CrowdMath also has exercises which are more approachable for participants. These exercises allow students to participate, receive feedback, and learn new math even if they are unable to make progress on the open problems, which helps maintain a friendly environment and makes the idea of doing research less intimidating. The third rule, together with the anonymity of CrowdMath, helps students not worry about asking questions or writing things that are possibly wrong.

15.4 What has worked and what has not

The most successful topics so far have been metric dimension [8], pattern avoidance [3, 4], pursuit-evasion [12], broken sticks [5], and zero forcing [7]. So far, two CrowdMath papers were published in the *Electronic Journal of Combinatorics* [2] and in *Discrete Applied Mathematics* [8], some of the other papers are under review, and all of the CrowdMath papers have been posted on arXiv [6]. All of the topics are easy to describe, and the structures and processes in the problems are relatively easy to visualize. The proofs in these areas are usually pretty elementary, so students can read the reference papers without having to learn extra background. Students often post questions on the message board as they read the references, and the mentors and other students help answer their questions.

Most of the topics outside of discrete math have not been as successful. In the first year, we had some open problems about optimal codes and incidence geometry, but there was no progress on any of the problems for either of these topics. The only progress in the first year was on the problems about pattern avoidance in 0-1 matrices, and we ended up writing three different papers on that topic. Similarly in the third year, we had some open problems about neural codes, but this topic was also unsuccessful, and we ended up focusing more on powerful numbers. All of the unsuccessful topics required more background reading than the successful topics to understand the open problems, and the proofs in the reference papers for the unsuccessful topics were much less elementary, so it was more difficult to understand the references and the problems for the unsuccessful topics.

Another thing that has worked well for CrowdMath is having the mentors post new questions on the message board when activity slows down. In every year of CrowdMath, there are periods of time when no students work on the problems. When it seems like things are slowing down, we usually post some new related problems to try to increase CrowdMath activity. Sometimes when activity slows down, the mentors will also prove results on their own and lead the participants to the solutions by posting outlines on the message board with details to fill in and questions to answer.

15.5 Advantages over in-person and one-on-one or small groups

One of the clearest advantages of CrowdMath over a typical in-person research program with one-on-one or small groups is the ability to serve many more students. A mentor in PRIMES will mentor a research project with between one and three students. In CrowdMath, one mentor interacts with dozens of students. Other research programs usually have meetings between students and mentors, but there are no meetings in CrowdMath, so there is no need to worry about finding a meeting time that works for every participant in the project. Since students participate in CrowdMath over the Internet, there are no costs for transportation or the use of facilities, and the cost of mentors is much less per student.

As mentioned before, there is no barrier to entry for participation in CrowdMath, besides knowing math and having Internet access. Almost every other math research program requires students to apply and be accepted into the program, but CrowdMath will take anyone. There are many talented students who do not know anyone who can write them a reference letter, or who are not eligible for other math research programs, for example, because of their international student status. They can do CrowdMath and still get math research experience. Besides, while most math programs operate only in the summer, CrowdMath is active throughout the year.

Another benefit of CrowdMath is the lack of pressure on the students. With a one-on-one or small group project, there is an expectation that the students will get something done, and the project might be viewed as a failure if the students that were chosen to work on the project do not make any progress on it. With CrowdMath, there are no expectations that any specific student will prove anything, because no one besides the mentors is required to work on the projects. Even if an active participant leaves the forum for a while, someone else will usually fill in for them and continue making progress on the problems.

15.6 The story of a CrowdMath research thread

At any given time, many separate discussions of a variety of research problems are being conducted on the CrowdMath message board. To illustrate the dynamics of collective research in CrowdMath, we will describe one of the most productive research threads from summer 2020, which led to the paper by Geneson, Kaustav, and Labelle [8]. This research thread involved two students and one mentor. That number is typical of research threads in the CrowdMath projects, though many other students participate in CrowdMath by asking questions and working on the exercises.

The main topic for CrowdMath 2020 was metric dimension, a graph parameter that can be used in robot navigation. The robot moves from vertex to vertex in a graph, where some vertices are distinguished as landmarks. We can think of the graph as representing a surface, where the vertices in the graph correspond to a subset of the points on the surface, and the edges in the graph are pairs of vertices for which the robot can step between the corresponding points on the surface. The robot is able to measure its distance to the landmarks, and it uses only those distances to determine its location. The goal is to use as few landmarks as possible, and the *metric dimension* is the fewest number of landmarks that can be used. We also studied a variant of metric dimension called *edge metric dimension*, where the robot moves between edges instead of vertices.

For CrowdMath, we posted several open problems from publications about metric dimension. One problem was to find the maximum degree of a graph of metric dimension at most k. Before CrowdMath, this quantity was bounded between $3^k - k - 1$ and $3^k - 1$. Some similar problems were to find the maximum number of edges, the maximum chromatic number, the maximum degeneracy, and the maximum possible minimum degree of a graph of metric dimension at most k. Another problem was to determine if there exist any families of graphs of metric dimension k where the edge metric dimension is $\omega(2^k)$. A more general problem was to find new triples (a, b, c) of integers for which there exists a graph of metric dimension a, edge metric dimension b, and order c.

In early July, one of the students initiated the progress on the CrowdMath problems by solving the problem about maximum degree, improving the lower bound to $3^k - 1$ with an incredibly creative new construction and proof. Up to that point, there had been solutions to some of the exercises on the CrowdMath message board, but no progress on the open problems. Ten days later, the mentor realized that the same construction would solve one of the other problems: whether there exist any families of graphs of metric dimension k where the edge metric dimension is $\omega(2^k)$. The graphs that the student constructed had metric dimension k and edge metric dimension approximately 3^k, so the mentor posted some hints on how to show that the edge metric dimension was approximately 3^k using the pigeonhole principle, and the same student confirmed the details. CrowdMath mentors post hints when they see how to solve one of the open problems, since it makes the problems more approachable for students, while also allowing the students to generate the final solutions.

The mentor noted that the student's construction also gave new triples (a, b, c), and the same student and a second student found other constructions that gave more new triples (a, b, c). Then the mentor posted a new problem, to find the maximum possible size of a wheel subgraph in a graph of metric dimension at most k. CrowdMath mentors usually post new open problems as the project progresses, even if there are plenty of open problems remaining, since solutions to earlier problems often inspire ideas for new problems. The second student used a modification of the first student's construction to show that the answer for the wheel subgraph problem is 3^k.

Then the first student realized that their construction could be extended to a family of infinite graphs with vertices at lattice points. This family of infinite graphs was very useful for proving results about metric dimension, because all graphs of metric dimension k can be embedded in the k^{th} graph of the family. The result about maximum degree was immediate using this family of infinite graphs, along with several results about metric dimension and pattern avoidance. The family of infinite graphs that the first student discovered became the main tool for proving most of the results in the CrowdMath paper.

The first student used the family of infinite graphs to show that the maximum number of edges in a graph of metric dimension k and order n is approximately $\frac{n(3^k-1)}{2}$ and the maximum chromatic number is 2^k. The first student suggested that the family of infinite graphs could also give bounds for the maximum degeneracy, but wrote that they did not see how to get the lower bound. The mentor saw how to get $\Theta(3^k)$ bounds for the maximum degeneracy, so he first put up hints, and then a proof outline. The first student confirmed the details, and then later they proved a much stronger result. Using a simpler proof, they showed that the maximum possible degeneracy is $\frac{n(3^k-1)}{2}$.

After seeing how useful the family of infinite graphs was for solving problems about metric dimension, the second student asked if a similar family of infinite graphs could work for edge metric dimension. There was some discussion on this, and no similar family was found for edge metric dimension, but it is an interesting question that was included in the paper [8]. Then the mentor suggested that the family of infinite graphs could be used to find the maximum possible minimum degree of a graph of metric dimension at most k. The mentor suggested a construction using a spherical subgraph that seemed to give a lower bound of 3^{k-1}, and the second student used the pigeonhole principle to prove an upper bound of 3^{k-1}. However, the mentor's idea with the spherical construction was flawed, as observed by the first student.

The first student noted that the mentor's spherical construction did not work because it produced a graph of metric dimension greater than k, so the mentor suggested trying some polytopes. First he suggested that using a k-dimensional cross polytope achieved the lower bound of 3^{k-1}. A 2-dimensional cross polytope is a square, and that worked for $k = 2$. A 3-dimensional cross polytope is an octahedron. A few hours later, the mentor observed that using an octahedron was giving a graph with minimum degree too low for $k = 3$, so he changed it to a rhombic dodecahedron, and the students and mentor confirmed that this showed the answer was 3^{k-1} for $k = 3$. The first student suggested a way to generalize the construction to $k > 3$. The students and mentor conjectured that the maximum possible minimum degree is 3^{k-1}, and that it is obtained by a family of graphs formed by using k-dimensional generalizations of rhombic dodecahedrons, but this is still open for $k > 3$.

The entire discussion, from start to finish, took about one month. The students and the mentor took turns making conjectures and offering constructions and proofs. This research thread clearly displayed a collaborative, rather than hierarchical, model of research. The interaction in the research thread was much like a typical math research collaboration, except that the collaborators never met before they started working on the problems, and their work was visible online as they made progress on the problems. The resulting solutions of several open problems provided sufficient material for the research paper by Geneson, Kaustav, and Labelle [8].

15.7 Who benefits

CrowdMath benefits students, faculty, schools, and other math research programs. Students benefit from gaining math research experience and interacting with the mentors and other participants on the CrowdMath message board. Participants have noted that they perceive the CrowdMath message board not only as a place where they can learn "how math research is done" but also how to work with other students. Other respondents noted that CrowdMath transformed them from being a learner to a researcher. One student noted that transitioning from school math to CrowdMath is "like wading in the kiddy pool your whole life and then getting pushed into the deep end" [9]. Students that contribute solutions or other useful ideas to CrowdMath projects can also get letters of recommendation from the head mentor when they apply to other math research programs. For example, several CrowdMath participants have been accepted into PRIMES and other math programs using letters of recommendation from the head mentor.

There are also obvious benefits of CrowdMath for faculty. Experience with mentoring research projects is increasingly important for securing jobs in academia, so mentoring CrowdMath is one way to gain mentoring experience for the faculty mentors. Unlike other math research programs, CrowdMath has a very flexible schedule for the mentor with no meeting times, so faculty mentors can fit CrowdMath around their schedule, even if they would not have time to mentor a project that required regularly scheduled meetings. CrowdMath can also help faculty reflect on how to be a better mentor. For most research programs, it would be impossible for the mentor to remember every single interaction that they had with the student participants. In CrowdMath, every interaction happens on the message board, so it will be on the CrowdMath website as long as the website exists. This allows the mentors to review their posts and see which ones led to more student participation and new results, and the mentor can use this to improve their research supervision skills.

Another less obvious benefit of CrowdMath for mentors and students is the abundance of student participants. One might expect that it would be unmanageable for a mentor to work with many students, and that CrowdMath would become more difficult for the mentor as the number of students increased. However, the opposite has been true. CrowdMath becomes easier for the mentor with more student participants, since the students answer each other's questions and prove many new results. The first year of CrowdMath had the fewest student participants of all the years, and that year was the most work by far for the mentors. Later years had more participants, and those years were much easier for the mentors since the students proved most of the new results and often answered each other's questions. In a survey of CrowdMath participants, students said that other students' posts were a greater source of inspiration than mentors' comments [9].

In the same survey of CrowdMath participants, 79% said that they either definitely or probably want to become a research mathematician [9]. For schools, CrowdMath presents an opportunity to serve students who are interested in learning how to do math research, even if there is no teacher at the school who is willing to mentor a project that interests the student. A high school or college could have a CrowdMath course where students meet and work on the CrowdMath problems together and post on the message board when they have questions, ideas, or proofs. A teacher could oversee them and assign grades at the end of the course, and it would be straightforward for the teacher, since the students would be teaching themselves and interacting with the CrowdMath mentors and other participants. Similarly, schools could have an extracurricular club that works on CrowdMath problems, like a math team or computer team.

Another benefit of CrowdMath is helping other math research programs identify strong students. CrowdMath has benefited PRIMES by providing multiple students that performed well on their PRIMES projects. Some of these students would not have participated in PRIMES if they did not perform so well in CrowdMath. Other students that perform well in CrowdMath can refer to their CrowdMath participation whenever they apply for a math research program. In addition to a letter of recommendation from the CrowdMath mentor, participants can provide links to the

CrowdMath website to show exactly where and how they collaborated on the CrowdMath project. Another possible benefit of CrowdMath for other math research programs is as a template. Whenever other groups try to run online math research projects for college students and high school students, they can look at the CrowdMath website to see what has worked and what has not worked for CrowdMath.

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