PRIMES Math Problem Set

PRIMES 2021

Due December 1, 2020

Dear PRIMES applicant:

This is the PRIMES 2021 Math Problem Set. Please send us your solutions as part of your PRIMES application by **December 1, 2020**. For complete rules, see `http://math.mit.edu/research/highschool/primes/apply.php`

- Note that this set contains two parts: "General Math problems" and "Advanced Math." Please solve as many problems as you can in both parts.
- You can type the solutions or write them up by hand and then scan them. Please attach your solutions to the application as a PDF file. The name of the attached file must start with your last name, for example, "`etingof-solutions.pdf`" or similar. Include your full name in the heading of the file.
- Please write not only answers, but also proofs (and partial solutions/results/ideas if you cannot completely solve the problem). Besides the admission process, your solutions will be used to decide which projects would be most suitable for you if you are accepted to PRIMES.
- Submissions in LaTeX are preferred, but handwritten submissions are also accepted.
- You are allowed to use any resources to solve these problems, *except other people's help*. This means that you can use calculators, computers, books, and the Internet. However, if you consult books or Internet sites, please give us a reference.
- **Note that posting these problems on problem-solving websites before the application deadline is strictly forbidden!** Applicants who do so will be disqualified, and their parents and recommenders will be notified.

Note that some of these problems are tricky. We recommend that you do not leave them for the last day. Instead, think about them, on and off, over some time, perhaps several days. We encourage you to apply if you can solve at least 50% of the problems.

Enjoy!

Why it makes no sense to cheat

PRIMES expects its participants to adhere to MIT rules and standards for honesty and integrity in academic studies. As a result, **any cases of plagiarism, unauthorized collaboration, cheating, or facilitating academic dishonesty during the application process or during the work at PRIMES may result in immediate disqualification from the program, at the sole discretion of PRIMES.** In addition, PRIMES reserves the right to notify a participant's parents, schools, and/or recommenders in the event it determines that a participant did not adhere to these expectations. For explanation of these expectations, see *What is Academic Integrity?*, `integrity.mit.edu`.

Moreover, even if someone gets into PRIMES by cheating, it would immediately become apparent that their background is weaker than expected, and they are not ready for research. This would prompt an additional investigation with serious consequences. By trying to get into PRIMES by cheating, students run very serious risks of exposing their weak background and damaging their college admissions prospects.

General Math Problems

Problem G1. A polynomial $f(x)$ has complex coefficients. It turns out that $f(x) \cdot f'(x)$ is a degree five polynomial whose x^5, x^4, x^1, x^0 coefficients are respectively 3, 10, 25, 12. Determine the polynomial f.

Problem G2. Scientists have found a vaccine that produces undesirable side effects with probability p. Initially, the number p is distributed uniformly across the interval $[0, 0.1]$. To test the vaccine, the scientists test the vaccine on 148374 volunteers and find that no one experiences adverse side effects.

Find the smallest real number λ such that the scientists can assert $p < \lambda$ with probability at least 95%. Round your answer to four significant figures.

Problem G3. Let p be an odd prime number. Calculate the number of triples $(a, b, c) \in \mathbb{F}_p \times \mathbb{F}_p \times \mathbb{F}_p$ for which $a + b + c = a^3 + b^3 + c^3 = 1$.

Problem G4. We roll a fair six-sided die and let s_1 be the result of the roll. Then, we roll s_1 fair six-sided dice and let s_2 be the sum of the rolls. Then, we roll s_2 fair six-sided dice and let s_3 be the sum of the rolls. The process continues to generate an infinite sequence $(s_1, s_2, \dots)$.

(a) Find the probability that 3 appears in the sequence.

(b) Find the expected value of s_n, for each integer n.

(c) We say the sequence grows exponentially if there exists a constant $c > 1$ such that $s_n > c^n$ for all sufficiently large integers n. Does the sequence grow exponentially almost surely?

Problem G5. For each positive integer $n \geq 4$, find all positive real numbers a_1, a_2, $\dots$, a_n such that

$$a_i^2 = 19a_{i+1} + 20a_{i+2} + 21a_{i+3}$$

holds for all $i = 1, \dots, n$ with indices taken modulo n.

Problem G6. If s is a finite binary string, then we denote by $f(s)$ the sum of the squares of the lengths of the *consecutive runs* of f. For example, $f(10110001111100) = 1^2 + 1^2 + 2^2 + 3^2 + 5^2 + 2^2 = 44$.

Suppose that a binary string s of length n is specified by letting the ith bit be 1 with probability p_i and 0 with probability $1 - p_i$, all independent. We wish to calculate the expected value of $f(s)$ given the values of n, p_1, p_2, $\dots$, p_n.

(a) Exhibit an algorithm with the best runtime you can find, in terms of n.

(b) Give the best lower bounds you can on the runtime of such an algorithm.

Advanced Math Problems

Problem M1. For positive integers n, find a closed form for

$$\sum_{\substack{a+b+c+d=n \\ a,b,c,d\geq 0}} 2^{a+2b+3c+4d}$$

in terms of n.

Possible hint: use generating functions.

Problem M2. We say a real number α is *good* if there exist nonzero integers m and n such that $e^{\alpha m}$ is an integer divisor of 2020^n.

(a) Let V denote the set of real numbers which are the sum of two good numbers. Show that V is a $\mathbb{Q}$-vector space under addition.

(b) Calculate $\dim V$ and give an example of a basis of V.

Problem M3. Let T be a finite tournament. For any vertex v, the indegree and outdegree of v is denoted by $\operatorname{indeg} v$ and $\operatorname{outdeg} v$, respectively. For each positive integer d we then define

$$A_d = \sum_v (\operatorname{indeg} v)^d \qquad B_d = \sum_v (\operatorname{outdeg} v)^d.$$

(a) Find all d such that $A_d = B_d$ holds for any tournament T.

(b) Prove or disprove: if $A_3 \geq B_3$ then $A_4 \geq B_4$.

(c) Prove or disprove: if $A_4 \geq B_4$ then $A_5 \geq B_5$.

Problem M4. A particle is initially on the number line at a position of 0. Every second, if it is at position x, it chooses a real number $t \in [-1, 1]$ uniformly and at random, and moves from x to $x + t$.

Find the expected value of the number of seconds it takes for the particle to exit the interval $(-1, 1)$.

Possible hint: for each $0 < x < 1$, let $E(x)$ denote the expected value of the amount of time until the particle exits the interval. You may assume without proof that $E(x)$ is a well-defined and analytic function on the interval $(0, 1)$.

Problem M5. Suppose G is a finite group and $\varphi\colon G \to G$ a homomorphism. Denote by $0 \leq k \leq 1$ the fraction of elements $g \in G$ which satisfy

$$\varphi(g) = g^2.$$

(a) Give an example where $k = 0.03$.

(b) If $k \neq 1$, how large can you get k to be?

Problem M6. A unit regular tetrahedron is a tetrahedron whose edge lengths are all equal to 1. Two unit regular tetrahedrons $ABCD$ and $WXYZ$ lie in Euclidean space. The labelings of $ABCD$ and $WXYZ$ are *oppositely* oriented.

(a) How small can $\max(AW, BX, CY, DZ)$ be?

(b) Generalize from 3 dimensions to n dimensions.

Problem M7. Let G be a finite simple graph with n vertices. Say that two Hamiltonian paths P_1 and P_2 of G are *neighbors* if they have exactly $n-2$ edges in common; also say a Hamiltonian path P and a Hamiltonian cycle C of G are *neighbors* if every edge of P is also an edge of C. Finally, we say that two Hamiltonian cycles C_1 and C_2 of G are *equivalent* if there exist some number of Hamiltonian paths $P_1, P_2, \ldots, P_k$ of G such that every pair of consecutive terms in the sequence $C_1, P_1, P_2, \ldots, P_k, C_2$ are neighbors.

(a) Give an example of a graph G with at least two inequivalent Hamiltonian cycles.

(b) Give an example of a graph G with at least 2020 inequivalent Hamiltonian cycles or prove that no such graph exists.