Products of reflections in smooth Bruhat intervals

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Generators are elements such that we can write any other element as products of generators. Every element of the group can be written as a **word** or **string** in terms of generators.

Definition (Group Example)

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So we write $24135 = s_1 s_3 s_2$. (In particular, $s_1 s_3 s_2$ is a *word* in terms of the *letters* s_1, s_2, s_3, s_4.)

Coxeter Groups

Definition

Given some n^2 integers $m_{i,j}$, with

- $m_{i,i} = 1$ for all i, and
- $m_{i,j} \geq 2$ for all $i \neq j$,

the **Coxeter Group** for this (m_{ij}) is a group with generators $s_1, \ldots, s_n$ given by

$$(s_i s_j)^{m_{i,j}} = e$$

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The **Coxeter graph** for a Coxeter Group G has n vertices $s_1, \ldots, s_n$, and has an edge from i to j if and only if $m_{i,j} \geq 3$.

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- $(s_i s_{i+1})^3 = e$ for $i = 1, \ldots, n-1$. First swap $(i, i+1)$, then swap $(i+1, i+2)$:

$$i \to i \to i+1,$$
$$i+1 \to i+2 \to i+2,$$
$$i+2 \to i+1 \to i,$$

which is the *3-cycle* $(i, i+1, i+2)$, which cycles back to itself upon cubing.

These match up with the Coxeter Group rules! **Hence S_n is a Coxeter Group**, given by the following m-values:

- $m_{i,i} = 1$ for all i,
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Showing S_n is a Coxeter Group (cont.)

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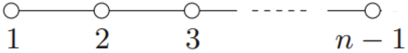
S_n's Coxeter graph is therefore a line graph:



where there are $n-1$ dots, each representing $s_1, \ldots, s_{n-1}$. This is called a Coxeter group of *type* A_{n-1} or *type* S_n.

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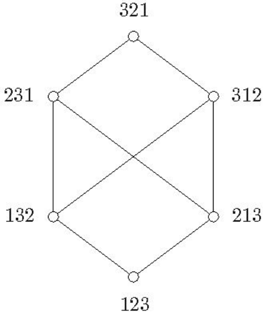

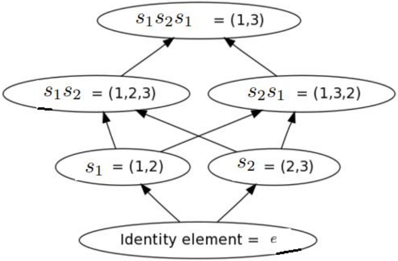

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Theorem (Gilboa and Lapid, 2020)

For any smooth $w \in S_n$, let $\{t_1, \ldots, t_k\}$ be the set of reflections less than or equal to w in Bruhat order. There exists a (compatible) order $t_1 \prec t_2 \prec \cdots \prec t_k$ for which $t_1 t_2 \cdots t_k = w$.

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Example

In S_3, everything is smooth. The set of reflections less than or equal to $w = s_1 s_2 s_1$ in Bruhat order is $\{s_1 s_2 s_1, s_1, s_2\}$. The claimed ordering exists:

$$s_2 \cdot (s_1 s_2 s_1) \cdot s_1 = s_2 s_1 s_2 = s_1 s_2 s_1.$$

Our Research

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Theorem (Gaetz and G., 2021)

For any smooth $w \in S_n$, let $\{t_1, \ldots, t_k\}$ be the set of reflections less than or equal to w in Bruhat order. There exists a (compatible) order $t_1 \prec t_2 \prec \cdots \prec t_k$ for which:

- $t_1 t_2 \cdots t_k = w$.
- $e \to t_1 \to t_1 t_2 \to \cdots \to t_1 \cdots t_k$ *is a saturated chain in Bruhat order.*
- $e \to t_k \to t_k t_{k-1} \to \cdots \to t_k \cdots t_1$ *is a saturated chain in Bruhat order.*

Our next goal is to generalize the above further to any *compatible order*, a kind of order used in the combinatorial constructions for these products of reflections.

Theorem Example

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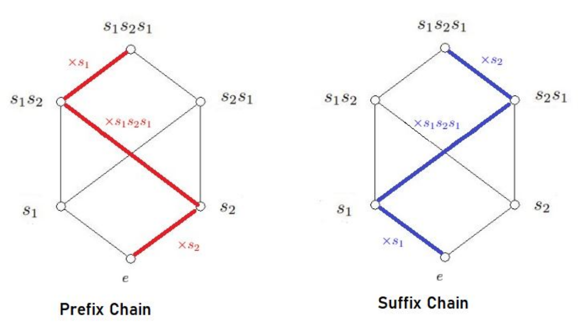
In S_3, consider $w = s_1 s_2 s_1 \in S_3$. We use the order $s_2 \prec s_1 s_2 s_1 \prec s_1$:



(Note $s_1 s_2 s_1 = s_2 s_1 s_2$.) Above on the left, the reflections s_2, then $s_1 s_2 s_1$, then s_1 are what we multiply in covering relations to make a saturated Bruhat chain. The right is a different chain, in the reverse (suffix products) order.

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