Regularities in the Lattice Homology of Seifert Homology Spheres

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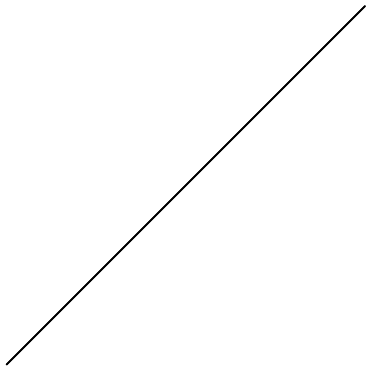

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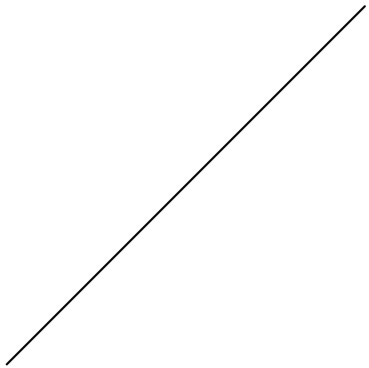

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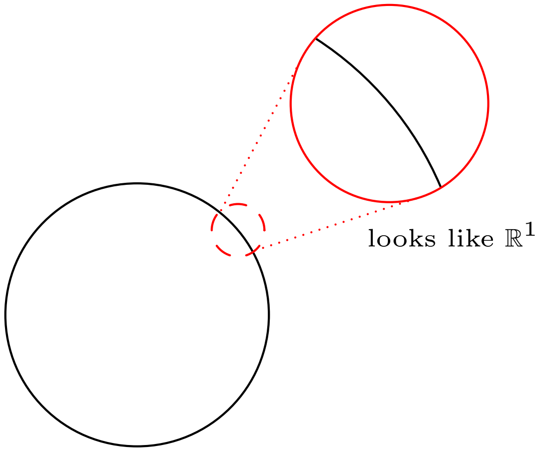

S^1. The 1-dimensional circle. A small ant sitting on a circle looks around and sees a line.

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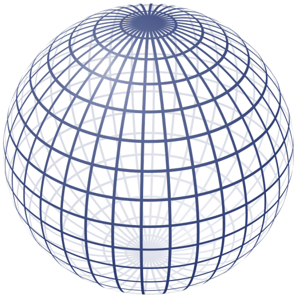

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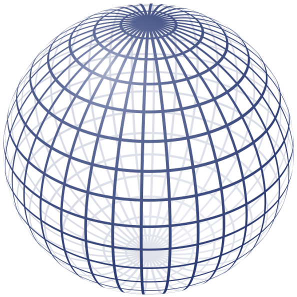

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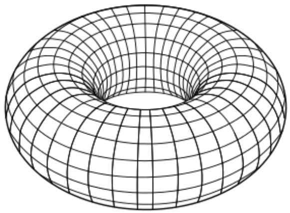

$T^2 = S^1 \times S^1$. The 2-dimensional surface of a torus. An ant sitting on a torus looks around and sees a plane.

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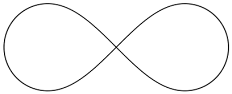

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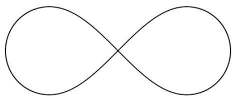

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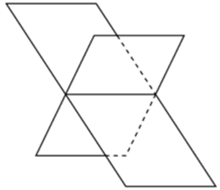

Two intersecting planes do not form a manifold, since an ant sitting on the line of intersection looks around and sees two intersecting planes, not $\mathbb{R}^2$.

Manifolds with Boundary

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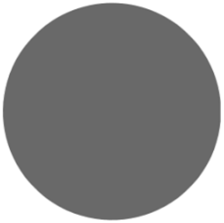

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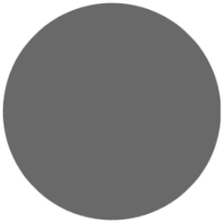

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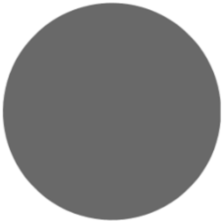

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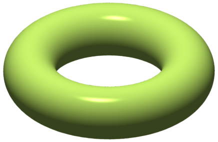

The filled-in torus $D^2 \times S^1$ has boundary $S^1 \times S^1$.

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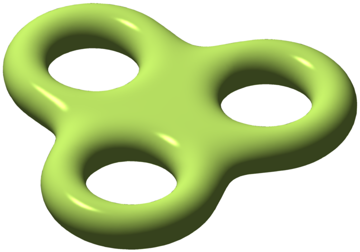

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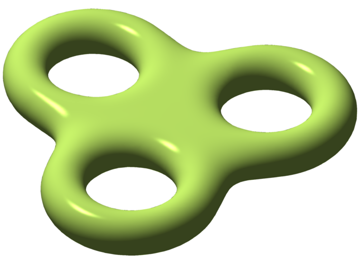

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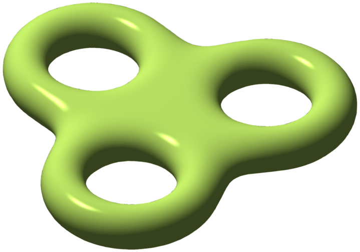

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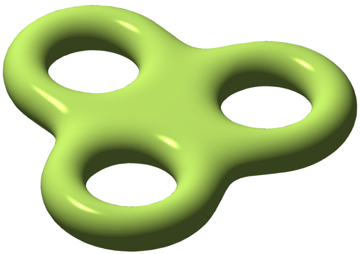

- Turns out, for $n = 1, 2$, the answer is indeed *all* manifolds.
- Very nontrivial fact: this holds true for $n = 3$ as well.
- $n = 4$ is when we get our first example of an n-dimensional manifold that *isn't* the boundary of some $(n+1)$-dimensional manifold, e.g. $\mathbb{C}P^2$.

Cobordisms

There's a way to reframe this question in a more generalized sense using the notion of *cobordisms*.

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This concept is best illustrated through some examples.

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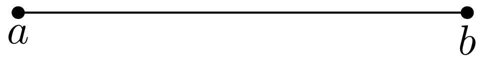

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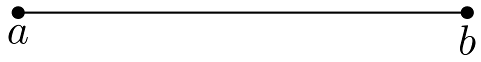

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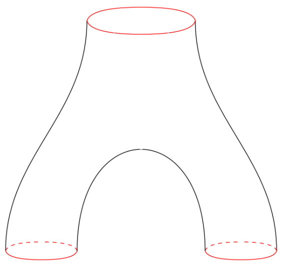
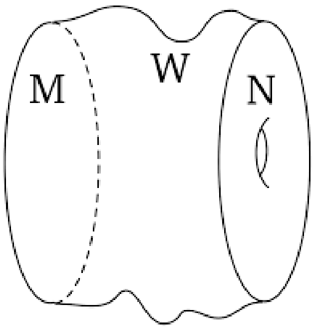


Let $M = S^1$ and $N = S^1 \sqcup S^1$. Then, the "pair of pants" manifold displays a cobordism between the M and N.

Cobordism

Cobordisms are hard to visualize and draw. In general, a cobordism looks something like the below, with two manifolds M and N connected by some manifold W. It is extremely hard to visualize what is going on in higher dimensions, so the figure is more of a schematic.



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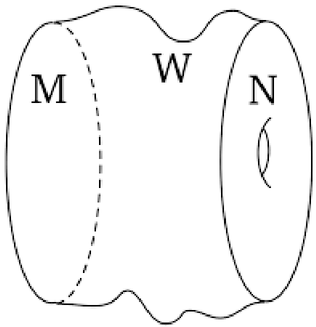

The study of cobordisms has been of intense interest the last few decades.

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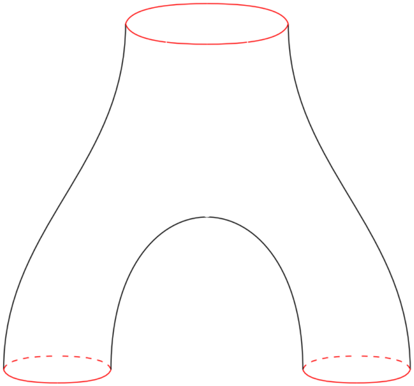
Bringing back the pair of pants analogy:



Note that the top circle S^1 bounds a disc D^2. Since the S^1 on top is cobordant to the $S^1 \sqcup S^1$ on the bottom through the pair of pants, $S^1 \sqcup S^1$ also bounds a 2-dimensional manifold, specifically the pair of pants with the top capped off with a disc.

Cobordism Classes of 3-manifolds

Note that cobordism is an equivalence relation (in particular, if X and Y are cobordant and Y and Z are cobordant, then we can see X and Z are cobordant). Therefore, it makes sense to talk about the *cobordism class* of a manifold X (it's simply the set of all manifolds cobordant to X).

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Because of this, we will actually study a slight specialization of cobordism called *homology cobordism* between 3-manifolds, which we will define later. In this case, there are infinitely many homology cobordism classes of 3-manifolds, and the classification problem is far from solved.

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- You can also think of S^3 as a cube except you grab all the points on the faces and fuse them together into a single point. Therefore, S^3 is roughly $\mathbb{R}^3$, just the outside points are wrapped around and fused together.

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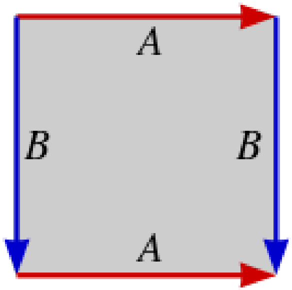

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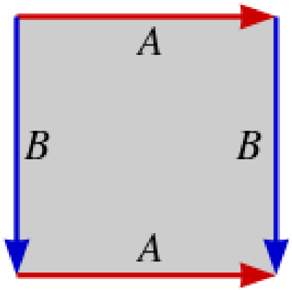

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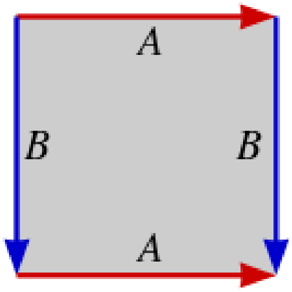

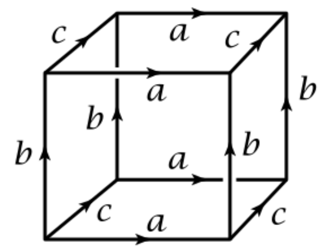

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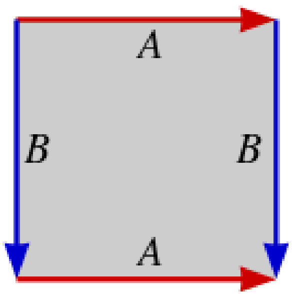

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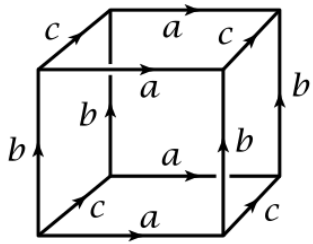

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Definition

1. Given some 3-manifold M with some link $\mathcal{L}$ (collection of knots) inside it, thicken the link to a group of tori and rip them out.

Surgery

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3. The resulting manifold is obtained from M via surgery along $\mathcal{L}$.

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Surgery is a very weird process that is practically impossible to visualize, but it is important since basically all 3-manifolds can be obtained via this process:

Theorem (Lickorish and Wallace)

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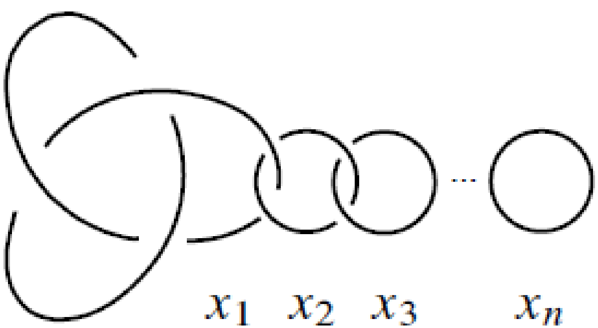
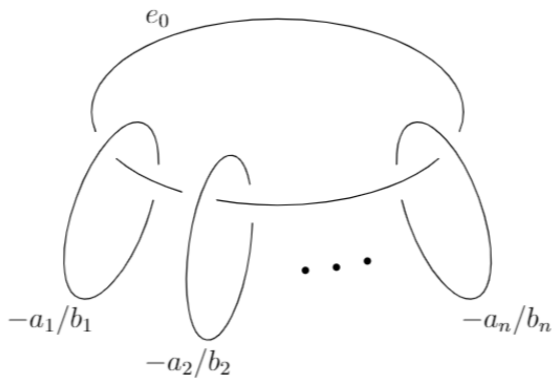
To represent the process in a better way, we can use *surgery diagrams*: we draw the link in S^3 (which is basically $\mathbb{R}^3$), and then label each with a number representing how we twist each solid torus (from thickening each knot in the link) when we glue it back in.



Seifert Homology Spheres

We are interested in Seifert homology spheres, a special class of 3-manifolds who have a specific surgery diagram:



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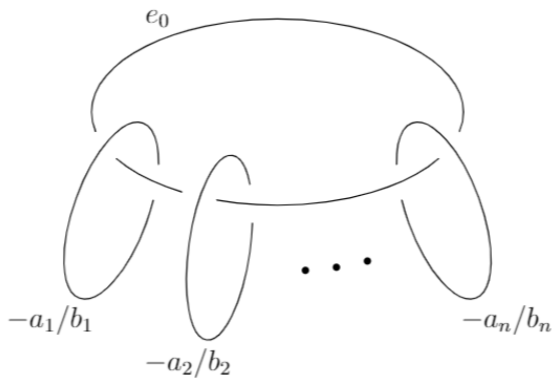

Definition

Seifert homology spheres are 3-manifolds with some special surgery diagram that can be parameterized by pairwise coprime integers $a_1, a_2, \ldots, a_n \geq 2$ for $n \geq 3$. We notate them as $\Sigma(a_1, a_2, \ldots, a_n)$.

Homology

Before, we noted that the problem of cobordisms between 3-manifolds is not very interesting, as they are all cobordant to each other. Therefore, we instead study a variant of cobordism called *homology cobordism*, which is much more interesting. To introduce this notion, we must first define *homology*.

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We say that a 4-manifold X is an *homology 4-cylinder* if X has the same homology groups H_i as $S^3 \times [0,1]$ for all $i \geq 0$.

Homology Cobordism

Now, we are interested in this specialization of cobordism:

Definition

Two homology spheres M and N are *homology cobordant* if there exists some homology cylinder W such that the disjoint union of M and N bounds W.

Homology Cobordism Invariants

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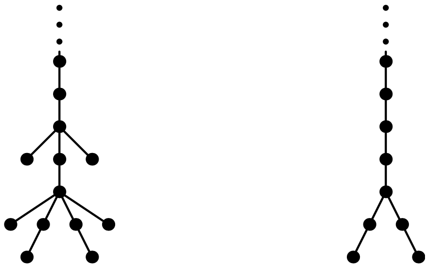

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$$d(\Sigma(a_1, a_2, \ldots, a_n)) = d(\Sigma(a_1, a_2, \ldots, a_{n-1}, a_n + \alpha)).$$

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Remark

In general, the maximal monotone subroots of the lattice homologies of $\Sigma(a_1, a_2, \ldots, a_n)$ *and* $\Sigma(a_1, a_2, \ldots, a_{n-1}, a_n + \alpha)$ *are not the same.*

Acknowledgements

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