Representation Stability and Orthogonal Groups

Zifan Wang[1]
Mentor: Arun S. Kannan[2]

[1]Princeton International School of Mathematics and Science
[2]Department of Mathematics, Massachusetts Institute of Technology

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Group representation

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Fixing the standard basis on V, we can represent elements of S_3 as matrices:

$$\rho(123) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}; \quad \rho(321) = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}; \quad \rho(231) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}.$$

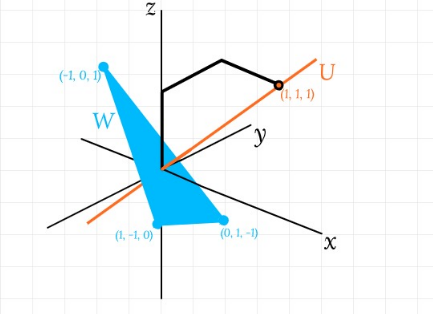
Subrepresentations

Example, continued Let

$$W = \{(x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_1 + x_2 + x_3 = 0\},$$

$$U = \{(t, t, t) \mid t \in \mathbb{R}\}$$

be subspaces of V, then $V = U \oplus W$. Furthermore, every $\sigma \in S_3$ preserves both components.



Representation theory of symmetric groups

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Theorem (Representation theory of S_n) Every rep of S_n over $\mathbb{Q}$ (or $\mathbb{R}, \mathbb{C} \ldots$) is a direct sum of irreps. Each distinct irrep of S_n corresponds naturally to an unordered integer partition of n.

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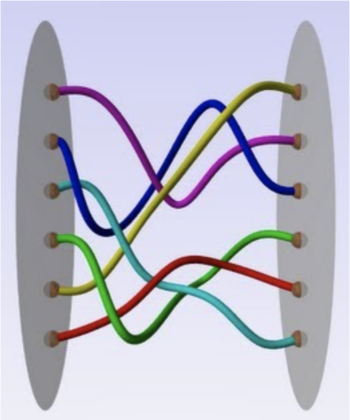
then $V = U \oplus W$ as representations. Here, U and W are both irreducible, and they correspond to the partitions $\{3\}$ and $\{2, 1\}$ respectively. Because of this, we denote $U = V_{\{3\}}$ and $W = V_{\{2,1\}}$, and $V = V_{\{3\}} \oplus V_{\{2,1\}}$.

Representation stability: Motivating example

Terminology Pure braid groups PB_n (picture credit [6])



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Examples

- $H^1(\mathrm{PB}_2; \mathbb{Q}) \cong V_{\{2\}}$;
- $H^1(\mathrm{PB}_3; \mathbb{Q}) \cong V_{\{3\}} \oplus V_{\{2,1\}}$;
- $H^1(\mathrm{PB}_4; \mathbb{Q}) \cong V_{\{4\}} \oplus V_{\{3,1\}} \oplus V_{\{2,2\}}$;
- $H^1(\mathrm{PB}_5; \mathbb{Q}) \cong V_{\{5\}} \oplus V_{\{4,1\}} \oplus V_{\{3,2\}}$;
- $H^1(\mathrm{PB}_6; \mathbb{Q}) \cong V_{\{6\}} \oplus V_{\{5,1\}} \oplus V_{\{4,2\}}$; etc ...

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Theorem (Church–Farb [1]) For each $i \geq 0$, the sequence of S_n–representations $H^i(\mathrm{PB}_n;\mathbb{Q})$ is multiplicity stable, stabilizing for $n \geq 4i$.

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Fix a sequence of groups with natural inclusions

$$G_0 \hookrightarrow G_1 \hookrightarrow G_2 \hookrightarrow G_3 \hookrightarrow \dots$$

Examples Symmetric groups S_n, braid groups B_n, general linear groups GL_n, symplectic groups Sp_{2n}, orthogonal groups $\mathrm{O}_n, \dots$

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Fix a ring K and consider a sequence of K-modules

$$A_0 \to A_1 \to A_2 \to A_3 \to \dots,$$

such that G_n has a K-linear action on A_n, and the maps between A_n are compatible with the action of G_n. This is a generalization of A_n being a G_n-rep.

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We often want to show that the sequence A_n stabilizes in some sense.

Orthogonal groups

Definition Let V be a finite-rank free R-module. A bilinear form $B : V \times V \to R$ is symmetric if $B(v, w) = B(w, v)$. It is nondegenerate if $B(v, w) = 0$ for all w implies $v = 0$. An orthogonal module is such a pair (V, B) where B is nondegenerate and symmetric.

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Definition An R-linear map between orthogonal modules $\varphi : (V, B_V) \to (W, B_W)$ is an isometry if $B_V(v, w) = B_W(\varphi(v), \varphi(w))$. It is necessarily injective. The orthogonal group $O_{V,B}$ is the group of isometries from (V, B) to itself.

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Theorem Let R be a finite local ring (where 2 is a unit), and let (V, B) be an orthogonal R-module. Then there exists a basis of V such that the matrix of B is either 1) the identity matrix, or 2) the diagonal matrix $\mathrm{diag}(1, \ldots, 1, x)$, where $x \in R$ is such that $\pi(x)$ is a nonsquare in $\mathbb{F}^\times$, and where different choices of x yield isometric forms.

Main result

Theorem (Stability with untwisted and twisted coefficients)

Let M be a finitely generated $\mathbf{OrI}(R)$-module over K. Then for a fixed $k \geq 0$, an isometry $(V, B_V) \to (W, B_W)$ induces maps

$$H_k(O_{V,B_V}(R); K) \to H_k(O_{W,B_W}(R); K)$$

and

$$H_k(O_{V,B_V}(R); M(V, B_V)) \to H_k(O_{W,B_W}(R); M(W, B_W)).$$

These are eventually isomorphisms for $\operatorname{rank} V \gg 0$.

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6. Picture credit: Ester Dalvit, "Braids. Chapter 1 - The group structure". `https://www.youtube.com/watch?v=u3Gt578803I`.