PRIMES Math Problem Set

PRIMES 2020

Due December 1, 2019

Dear PRIMES applicant:

This is the PRIMES 2020 Math Problem Set. Please send us your solutions as part of your PRIMES application by **December 1, 2019**. For complete rules, see `http://math.mit.edu/research/highschool/primes/apply.php`

- Note that this set contains two parts: "General Math problems" and "Advanced Math." Please solve as many problems as you can in both parts.
- You can type the solutions or write them up by hand and then scan them. Please attach your solutions to the application as a PDF file. The name of the attached file must start with your last name, for example, "`smith-solutions`". Include your full name in the heading of the file.
- Please write not only answers, but also proofs (and partial solutions/results/ideas if you cannot completely solve the problem). Besides the admission process, your solutions will be used to decide which projects would be most suitable for you if you are accepted to PRIMES.
- Submissions in LaTeX are preferred, but handwritten submissions are also accepted.
- You are allowed to use any resources to solve these problems, *except other people's help*. This means that you can use calculators, computers, books, and the Internet. However, if you consult books or Internet sites, please give us a reference.
- **Note that posting these problems on problem-solving websites before the application deadline is strictly forbidden!** Applicants who do so will be disqualified, and their parents and recommenders will be notified.

Note that some of these problems are tricky. We recommend that you do not leave them for the last day. Instead, think about them, on and off, over some time, perhaps several days. We encourage you to apply if you can solve at least 50% of the problems.

Enjoy!

Why it makes no sense to cheat

PRIMES expects its participants to adhere to MIT rules and standards for honesty and integrity in academic studies. As a result, **any cases of plagiarism, unauthorized collaboration, cheating, or facilitating academic dishonesty during the application process or during the work at PRIMES may result in immediate disqualification from the program, at the sole discretion of PRIMES.** In addition, PRIMES reserves the right to notify a participant's parents, schools, and/or recommenders in the event it determines that a participant did not adhere to these expectations. For explanation of these expectations, see *What is Academic Integrity?*, `integrity.mit.edu`.

Moreover, even if someone gets into PRIMES by cheating, it would immediately become apparent that their background is weaker than expected, and they are not ready for research. This would prompt an additional investigation with serious consequences. By trying to get into PRIMES by cheating, students run very serious risks of exposing their weak background and damaging their college admissions prospects.

General Math Problems

Problem G1. Let $n \geq 4$ be an integer. We wish to arrange the numbers $1, \ldots, n$ in a circle so that any two consecutive numbers sum to a prime number. For example, $(1, 2, 3, 4)$ would be a valid arrangement when $n = 4$.

(a) Is there an odd $n \geq 5$ for which this is possible?

(b) For each of $n = 6$, $n = 8$, $n = 10$, determine whether this is possible.

Problem G2. Consider pairs (a, b) of positive integers where b is not a perfect square and $a^2 > b$. Let

$$q = \sqrt{a + \sqrt{b}} + \sqrt{a - \sqrt{b}}.$$

(a) Assume $(a, b) = (17943, 321185624)$. Show that q is rational and determine its value.

(b) Find another pair (a, b) as above for which q is rational.

(c) Determine whether there are infinitely many pairs (a, b) for which q is rational.

(d) Is it possible to find (a, b) such that q is rational but not an integer?

Problem G3. Three distinct points are chosen on the parabola $y = x^2$, determining a triangle Δ.

(a) Express the area of Δ as a function of the three slopes of the sides of Δ.

(b) Determine the largest possible area of Δ, given that all slopes of sides of Δ have absolute value at most m.

Problem G4. Let k be a positive integer. A 3×3 matrix M with integer entries is given. It turns out that each of the four continuous 2×2 submatrices has determinant 1. (These are the four minors obtained by deleting either the first row or last row, and either the first column or last column.) Moreover, the center entry is equal to k.

Given this information, find all possible values of $\det M$, in terms of k.

Problem G5. Let n be a positive integer and let $S = \{1, \ldots, n\}$. We choose three subsets A, B, C of S uniformly at random (from the 2^n possible subsets), with replacement.

(a) Find the expected value of $|A \cap B \cap C|$.

(b) Find the expected value of $|A \cap B| \cdot |B \cap C|$.

Problem G6. A robot starts at the point 0 on a number line. Thereafter, if it is at the number n, then

- it goes to $n+1$ with probability $1/2$,
- it goes to $n-1$ with probability $1/3$,
- it goes to $n-2$ with probability $1/6$.

Determine the probability the robot ever reaches 1. You may take for granted the probability this occurs is not 1.

Problem G7. Let n be a positive integer. An $n \times n$ board is given, and some rooks are placed on the board. In a move, any rook may capture another rook in the same row or column (i.e. if rook R is in the same row or column as rook R', then R' is removed and R takes its place). This continues until no more captures are possible; the resulting configuration is called peaceful.

(a) Describe an algorithm to compute, in $O(n^2)$ time, the maximum number of captures that can be made before reaching a peaceful configuration.

(b) Describe an algorithm to compute, in $O(n^3)$ time, the minimum number of captures that can be made before reaching a peaceful configuration.

If you use a standard or well-known algorithm as part of your solution, you do not need to describe the exact steps of the algorithm itself, but you should reference what the algorithm does, give either a name or citation, and state on the runtime.

Advanced Math Problems

Problem M1. There are six towers which are each 24 blocks tall. Bob the Builder wishes to merge them into a tower which is 144 blocks tall.

In a move, Bob may take a block from any tower (say with a blocks) and move it on top of any other tower (say with b blocks), as long as $a \le b$. Due to gravity, the effort of doing so takes Bob $b - a + 2$ hours. Determine the minimum amount of time it takes Bob to construct a tower of height 144.

Problem M2. Let $f\colon [0,1] \to \mathbb{R}$ be a strictly increasing function which is differentiable in $(0,1)$. Suppose that $f(0) = 0$ and for every $x \in (0,1)$ we have

$$\frac{f'(x)}{x} \ge f(x)^2 + 1.$$

How small can $f(1)$ be?

Problem M3. Let n be a fixed positive integer and consider the vector space V of real polynomials of degree at most n. We define the map $T\colon V \to V$ by

$$f(x) \mapsto \frac{d}{dx}\left[(x+1)^{n+1} \cdot f\left(\frac{1}{x+1}\right)\right].$$

Is T a linear map? If so, compute its determinant.

Problem M4. Let G be a finite group and let $A = \operatorname{Aut}(G)$ denote its automorphism group.

(a) If $|G| = 1048576$, can it happen that $|A|$ is divisible by 1048575?

(b) If $|G| = 1048572$, can it happen that $|A|$ is divisible by 1048571?

Problem M5. We say an integer $n \geq 2$ is *chaotic* if for any monic nonconstant polynomial $f(x)$ with positive integer coefficients, the set

$$\{f(1), f(2), \ldots, f(n)\}$$

contains fewer than $10^{\deg f} \cdot \frac{n}{\log n}$ prime numbers. Are there finitely many chaotic integers?

(Possible hint: use the prime number theorem for arithmetic progressions.)

Problem M6. Fix an additive abelian group G. Say a family $\mathcal{I}$ of *finite* subsets of G is *G-admissible* if the following properties hold:

- if $A \in \mathcal{I}$ and $g \in G$, then $g + A = \{g + a \mid a \in A\}$ is also in $\mathcal{I}$;
- whenever $A, B \in \mathcal{I}$, the symmetric difference

  $$A \triangle B = \{x \mid (x \in A \text{ and } x \notin B) \text{ or } (x \in B \text{ and } x \notin A)\}$$

  is also in $\mathcal{I}$.

If $A \subseteq G$ is finite, let $\overline{A}$ denote the smallest admissible family containing A.

(a) Is every $\mathbb{Z}$-admissible set of the form $\overline{A}$ for some A?

(b) Describe all $\mathbb{Z}$-admissible sets.

(c) For each $n = 2, 3, \ldots, 9$ compute the number of $\mathbb{Z}/n\mathbb{Z}$-admissible sets.

(d) What can you say about $\mathbb{Z}/n\mathbb{Z}$-admissible sets for general n?