Pulses of Flow-firing Processes

Rupert Li

Abstract

Flow-firing is a natural generalization of chip-firing, or the abelian sandpile model, to higher dimensions, operating on infinite planar graphs. The edges of the graph have flow, which is rerouted through the faces of the graph. We investigate initial flow configurations which display terminating behavior and global confluence, meaning the terminating configuration is unique. The pulse configuration over a hole, or a configuration of flow going around a face that cannot redirect flow, is known to display global confluence, and we expand this result to initial configurations that have multiple pulses, identifying which terminating configurations are possible. We also generalize the analysis of the global confluence of pulses to configurations with flow outside of the hole, especially to the configuration of a pulse with radius, and prove under what conditions this displays global confluence. We conclude with a conjecture on the global confluence of a generalization of a pulse with radius, a uniform conservative configuration, or contour.

1 Introduction

Chip-firing has been convergently discovered numerous times by researchers operating independently of each other, such as in [7] as an example of self-organized criticality, which was introduced in [2]. The automaton model was invented to model stacked items at sites, which would topple if a critical height was reached, and send the items to adjacent sites; for example, using a lattice graph to model a plane, the abelian sandpile model could mimic a pile of sand on a flat surface as it collapses under gravity to reach a certain stable state, and the destabilizing behavior of adding additional sand analyzed. The analysis of these models have led to the generalization of it for arbitrary locations and compositions of sites.

The **chip-firing game**, also referred to as the **abelian sandpile model**, on a directed graph G consists of a collection of an integral number of chips at each vertex of G. If a vertex v has at least as many chips as its outdegree, then it can fire, sending one chip along each outgoing edge to its neighboring vertices. This continues until no vertex can fire. Articles on chip-firing in the literature include [3–6, 10, 13, 14].

Chip-firing can be thought of as acting on a simplicial complex, where chips are on 0-dimensional cells (vertices) and fired through 1-dimensional cells (edges). This naturally leads to the generalization of chip-firing into higher dimensions, as done in [8]. **Ridge-firing**, as explained in [9], works over a polytopal decomposition of n-dimensional space, where values are assigned to $(n-1)$-dimensional cells (**ridges**) and rerouted through n-dimensional cells (**facets**). The first generalization of chip-firing into ridge-firing is the two-dimensional form of ridge-firing, or **flow-firing**, which works on a planar graph with **flow** on 1-dimensional cells (edges), which are rerouted through 2-dimensional cells (faces).

Chip-firing displays local confluence, meaning the order of the firing sequence does not impact the overall firing process, and terminating behavior, which implies that it displays global confluence as well, meaning there exists a unique terminating configuration. Flow-firing, on the other hand, does not display local confluence, and only under certain conditions displays terminating behavior. Hence, while global confluence is well-understood in chip-firing, which configurations in flow-firing display global confluence is not well-understood.

In Section 2, we establish the basic theory surrounding flow-firing, and define the primary object of study, the **pulse configuration**, along with its Aztec pyramid. We provide a simpler proof of the result in [9, Theorem 9] which states that the pulse configuration over a hole, or a distinguished face that cannot redirect flow, displays global confluence and identifies the terminating configuration as the Aztec pyramid. In Section 3, we begin stating our own results, investigating unidirectional flow-firing and when it displays local confluence. In Section 4, we generalize the results of [9] to allow for multiple pulses and holes in the flow-firing process, analyzing which initial configurations terminate and what configurations they may terminate at, especially noting which configurations display global confluence. In Section 5, we expand the study of pulses to include initial configurations that contain a pulse but are not necessarily composed solely of a pulse, studying which configurations do or do not display global confluence. In Section 6, we look at a particular type of initial configuration that contains a pulse, the generalization of a pulse to a pulse of radius r, finding which configurations do or do not display global confluence. We conclude in Section 7 with a conjecture surrounding the behavior of a contour, or a uniform conservative flow containing a pulse, a generalization of the pulse of radius r studied in Section 6 and a special case of the initial conservative configurations containing a pulse studied in Section 5.

2 Preliminaries

We refer readers to [9] for detailed background on flow-firing. Let G be an infinite planar graph, where each edge has a specified orientation. A **flow configuration** C is an integer assignment to the edges of G, representing flow through each edge. A negative flow corresponds to the flow going in the direction opposite of the specified orientation. Under the flow-firing process, if an edge e has a flow of magnitude at least 2, it may fire, rerouting one unit around each of the two faces containing e. See Fig. 1 for an example of the flow-firing process on the $\mathbb{Z}^2$ lattice, the infinite grid graph embedded as $\mathbb{Z}^2$, composed of square faces each with a coordinate (x, y) for $x, y \in \mathbb{Z}$ where faces $(x_1, y_1), (x_2, y_2)$ share an edge when $|x_1 - x_2| + |y_1 - y_2| = 1$.

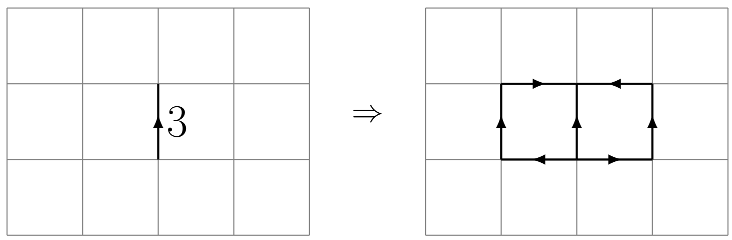


Figure 1: Flow-firing process on the $\mathbb{Z}^2$ lattice. Unlabeled flow is of magnitude 1.

Chip-firing displays **local confluence**, or equivalently it satisfies the diamond lemma, which states that if a state C can reach states C_1 and C_2 after a single step, then there exists a state D that both C_1 and C_2 can reach after a single step. It has been previously shown in [11] that local confluence in terminating systems implies **global confluence**, meaning regardless of the choices made during the process, an initial configuration will always terminate at a unique final configuration. However, as shown in [9], flow-firing does not in general display local confluence nor does the process generally terminate, because of the ability to cancel flow (in chip-firing, the non-firing vertices can only gain chips, but this is not the case in flow-firing); to help resolve this, the definition of a conservative configuration was created in [9].

Definition 2.1. A flow configuration is **conservative** if for all vertices v,

$$\text{inflow}(v) - \text{outflow}(v) = 0,$$

where inflow(v) denotes the sum of the magnitudes of the flows going into vertex v and outflow(v) denotes the sum of the magnitudes of the flows going into vertex v.

It has been demonstrated in [9] that conservative configurations display terminating behavior. Under the flow-firing process the quantity inflow(v) − outflow(v) is conserved at each vertex. Because of this, conservativity is preserved under the flow-firing process.

It has been previously shown in [9] that a conservative configuration C allows for a face representation of the flow configuration, where each face is given a flow, with positive flow interpreted as in the clockwise direction and negative in the counterclockwise direction. The flow over a face F in C is denoted $C(F)$. The flow at each edge e is determined by the addition of the two flows induced on the edge by the two faces containing e, where flows in opposite directions cancel each other.

The flow-firing process can then be redefined in terms of the face flow representation. An edge with flow of magnitude at least 2 corresponds to the two faces that share that edge having flows that differ by at least 2. Hence, if faces that share an edge have flows that differ by at least 2, the common edge may fire, resulting in the face with higher flow losing one unit flow and the face with lower flow gaining one unit of flow.

In chip-firing, the pulse configuration consists of n chips at a single vertex and no chips elsewhere. The pulse, as a fundamental configuration in chip-firing, is of interest to study, and the properties of a pulse on the $\mathbb{Z}^1$ lattice, or the infinite path graph, have been studied extensively in [1] and [12]. This motivates the definition of a pulse configuration for conservative configurations of the flow-firing process.

Definition 2.2. A pulse of force p over a face f is the conservative configuration whose face flow representation has p flow at face f and no flow elsewhere.

The Aztec pyramid of a pulse will be of significant use throughout this paper.

Definition 2.3. The **Aztec pyramid** of a nonnegative pulse of force p at face f is the configuration C where $C(F) = \max(p - d + 1, 0)$ flow for all F except f, where d is the distance of F from the pulse, or the minimal number of edges that must be crossed (one may not cross through vertices) to reach F from the distinguished face. The face f still has p flow. The flow through each face F in the Aztec pyramid configuration of pulse p is denoted by $\text{Aztec}_{p,f}(F)$ or simply $\text{Aztec}(F)$ if which pulse is being referred to is obvious. The Aztec pyramid of a negative pulse of force $-p$ at face f is the configuration C where $C(F) = -\text{Aztec}_p(F)$.

See Fig. 2 for an example of an Aztec pyramid on the $\mathbb{Z}^2$ lattice.

In some configurations, some faces, called holes, are not allowed to redirect flow, as seen in the following definition.

Definition 2.4. A **hole** is a distinguished face of the infinite planar graph that the flow-firing process operates on that acts as a topological hole, meaning the edges cannot redirect flow through the hole. An edge e bordering a hole only needs flow of magnitude at least 1 to fire, and it only redirects over the undistinguished face.

In terms of face flow representations for conservative configurations, the hole can still have flow, but the hole and a face that shares an edge with it only need to have flow differing by at least 1; moreover, under the flow-firing process the flow on the hole is unchanged. The modified flow-firing process that results from the presence of a hole displays desirable properties in regards to pulses. This leads to the following result from [9, Theorem 9].

			1			
		1	2	1		
	1	2	3	2	1	
1	2	3	3	3	2	1
	1	2	3	2	1	
		1	2	1		
			1			

Figure 2: The Aztec pyramid for a pulse of force 3 over the $\mathbb{Z}^2$ lattice. The pulse is on the face colored gray.

Theorem 2.1 (Felzenszwalb, Klivans)**.** *A pulse of force p over a hole f displays global confluence, with its terminating configuration being its Aztec pyramid.*

We provide a simpler proof of this result.

Proof. As the pulse is a conservative configuration, it must terminate. Say it terminates at a stable configuration D. We will show that for the Aztec pyramid is both an lower bound and upper bound for D at all faces, thus showing D is the Aztec pyramid.

The initial configuration only contains faces with flow f and 0, so all faces in D must have flow between 0 and f, inclusive. Notice that starting from the faces sharing an edge with the hole, which must have flow f to be stable, and proceeding outwards, the most the flow can differ between faces that share an edge is 1, so each undistinguished face F is bounded from below by $\max(p - d + 1, 0)$, where d is the distance from F to the pulse.

On the other hand, as all of the undistinguished faces start with no flow, the only source of flow is the hole, and the maximum amount of flow they may acquire is also $\max(p-d+1, 0)$.

Hence, D is the Aztec pyramid. ■

Because of this highly desirable property, throughout this paper, when referring to a pulse we will assume that the pulse is over a hole, unless explicitly stated otherwise. Hence, the hole is somewhat analogous to the sink in chip-firing, which is a distinguished vertex that cannot fire and whose chips are not tracked, as the sink gives chip-firing terminating behavior and thus global confluence. Furthermore, the sum of the flows over all faces is conserved under the flow-firing process except when interacting with the hole, just as in chip-firing where the sum of the chips at all vertices is conserved under the chip-firing process except when chips interact with the sink. A pulse's terminating configuration is referred to as the Aztec pyramid, and is remarkably simple to define for all infinite planar graphs and all choices of pulse locations and forces.

3 Unidirectional flow-firing

Definition 3.1. A **unidirectional flow-firing configuration** C is a conservative configuration on a planar graph isomorphic to the $\mathbb{N}$ lattice (a 1-dimensional lattice of square faces enumerated uniquely with the nonnegative integers such that two faces share an edge if their corresponding numbers are consecutive).

The **sequence of flow** S_C of a unidirectional flow-firing configuration C is the sequence formed by the flows over the faces of C, with the ith term (0 indexed) corresponding to the flow of face $i \in \mathbb{N}$. In other words, the ith term of S_C is $C(i)$.

See Fig. 3 for an example of a unidirectional flow-firing configuration and its sequence of flow.

4	4	3	2	2	1	0	0	$\cdots$

Figure 3: A unidirectional flow-firing configuration with sequence of flow 4, 4, 3, 2, 2, 1, 0, 0, $\ldots$.

When the sequence of flow is weakly decreasing, as is the case for the example in Fig. 3, the following lemma states how this unidirectional flow-firing configuration has the desirable property of local confluence.

Lemma 3.1. *A unidirectional flow-firing configuration C with $C(F) \geq 0$ and with S_C weakly decreasing displays local confluence.*

Proof. First we notice that if S_C is weakly decreasing, no face k can fire towards the face $k-1$, so if it can fire it can only fire to $k+1$. Furthermore, if a face k fires towards face $k+1$, the sequence of flow remains weakly decreasing as the flow at face k decreases and the flow at face $k+1$ increases, while the flow at face k was initially at least 2 greater than the flow at face $k+1$ so that boundary remains weakly decreasing. Hence, the faces can only fire in the positive direction.

Say we have a configuration C that has two distinct successors C_1 and C_2 after one step, meaning that C has at least 2 options to fire. That means for two distinct values of k,

$C(k+1) - C(k) \geq 2$. If these two values of k, k_1 and k_2, are at least 2 apart, then their firings do not affect each other, so if k_1 fires, then k_2 can fire afterwards, thus demonstrating local confluence. If k_1 and k_2 are consecutive, then firing one will increase the difference of the other, meaning that the other still can fire, hence demonstrating local confluence. Thus a unidirectional flow-firing configuration C with $C(F) \geq 0$ and with S_C weakly decreasing displays local confluence. ■

Furthermore, notice that a unidirectional flow-firing configuration C with $C(F) \geq 0$ and S_C weakly decreasing, during its stabilization process, will always have S_C weakly decreasing.

This allows us to prove the following proposition about the terminating configuration of a particular type of unidirectional flow-firing configuration with a weakly decreasing sequence of flow.

Proposition 3.2. *A unidirectional flow-firing configuration C with $C(F) = f$ for $F \leq n \in \mathbb{N}$ and $C(F) = 0$ otherwise will stabilize to the configuration D where S_D is weakly decreasing, with $C(k) - C(k+1) \leq 1$ for all k and at most one value k where $0 < C(k) = C(k+1) < f$.*

See Fig. 4 and Fig. 5 for examples of the stabilization of a unidirectional flow-firing configuration described in Proposition 3.2.

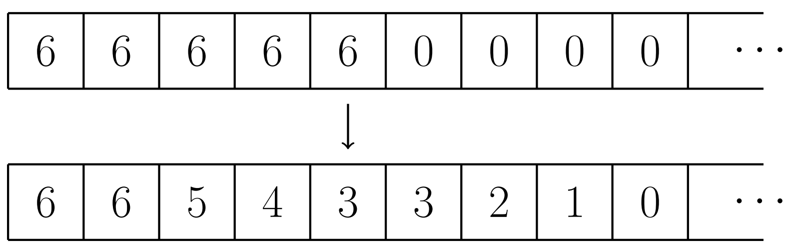


Figure 4: The unidirectional flow-firing configuration with sequence of flow 6, 6, 6, 6, 6, 0, ... stabilizes to 6, 6, 5, 4, 3, 3, 2, 1, 0, ...

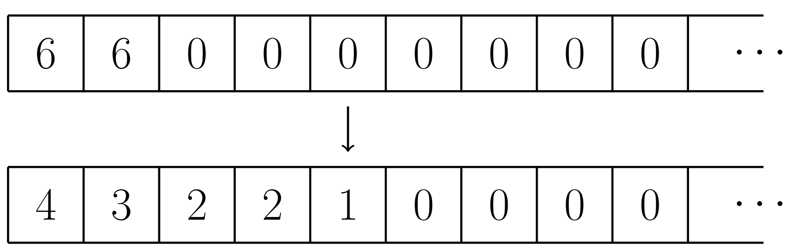


Figure 5: The unidirectional flow-firing configuration with sequence of flow 6, 6, 0, ... stabilizes to 4, 3, 2, 2, 1, 0, ...

Proof. Notice that there is only one such configuration with the properties specified for a particular f and n.

We merely need to show such a configuration is reachable, as the initial configuration has local confluence and thus global confluence by Lemma 3.1, meaning this stable configuration is the unique stabilization. We demonstrate this by imposing the restriction for the firing process that for each step, among all the possible pairs of consecutive faces that may fire, fire the pair furthest from face 0.

During the flow-firing process, we will divide the configuration into three sections: the source, the pyramid, and the zeros. The zeros section is the section composed of all faces with flow 0. Notice that the zeros section is contiguous and is the furthest section in regards to the positive direction, which we will call the rightwards direction. The pyramid section is the contiguous section consisting of, starting from the rightmost face and moving left, the face to the left of the zeros section, continuing leftward until there is a difference in flow of at least 2 or until the next face has flow f. The source section is the section composed of all faces to the left of the pyramid section.

The initial configuration has an empty pyramid section. The rightmost face in the source section fires, creating a pyramid section starting at the face to its right. As the process continues, the process consists of the rightmost face in the source section firing, followed by that unit of flow cascading down the pyramid section. We can each of these occurrences a roll. At the end of each roll, notice that the rightmost face of the pyramid section has flow 1. Furthermore, the pyramid section can only contain one pair of consecutive faces that have the same flow at the end of each turn, with each roll either stopping when it hits this pair and moving this pair one face leftwards or stopping at the leftmost face of the zeros section and thus expanding the pyramid section and creating that pair at the bottom of the pyramid.

Eventually, we reach the stable configuration at the end of some roll. As the pyramid section only contains one pair of consecutive faces that have the same flow at the end of each turn, and the rightmost face of the source section and the leftmost face of the pyramid section cannot have flow differing by more than 1 but cannot have the same flow (or else both faces would be part of the pyramid section), the source section consists entirely of faces with flow f, and the terminating configuration has only one pair of consecutive faces that have the same flow. ■

4 Multiple pulses

We investigate which configurations with multiple pulses, or conservative configurations with multiple holes with flow only on the holes, display terminating behavior, and among those identify the possible terminating configurations.

For a planar graph with n holes $h_1, \ldots, h_n$ each with a pulse of force $p_1, \ldots, p_n$, respectively, the flow-firing process proceeds as if each hole was the only hole in the planar graph, with specific interactions at overlaps of the supports of each pulse.

If the holes of two pulses share an edge and are of differing force, then there is a flow through the edge between them, but the edge has no way to fire. This edge thus will not fire, and it need not be taken into consideration for the stability of a flow configuration, but is critical to maintain the conservativity of a configuration.

In order to understand the interactions between multiple pulses, we define the support of a pulse.

Definition 4.1. The **support** of a pulse of nonzero force is the union of the faces that have a nonzero flow in the Aztec pyramid of that pulse, including the hole of the pulse itself. The support of a pulse of zero force is the union of the faces that share an edge with the hole of the pulse in addition to the hole itself.

The support of any configuration is the set of faces that have nonzero flow through them. There are three **signs** of pulses: positive, negative, and zero, which directly corresponds to the sign of the force of the pulse. We use the convention that a positive sign is opposite to a negative sign, but a sign of 0 is not opposite to any sign. Note that if exactly one of two signs is zero, then the signs are differing but not opposite.

If none of the supports overlap or share an edge, the flow-firing process will terminate, with the final configuration in each support being the final configuration if the location of the pulse was the only hole in the planar graph, and 0 flow on all the other faces.

If any supports share an edge or overlap, the interactions of the flow-firing process are as follows:

- The supports of multiple pulses share a face
  - If the supports of any two pulses of differing signs overlap, the flow-firing process will not terminate as the final configurations of the two individual pulses do not agree at at least one face, and each pulse will continue the flow-firing process towards its final configuration.
  - If the overlapping supports' pulses are of the same sign, then the flow-firing process will terminate unless other supports' interactions cause it to fail to terminate. If it terminates, the final flow for any face that is part of multiple supports, say of pulses of force p_1, p_2, $\ldots$, p_n at faces f_1, f_2, $\ldots$, f_n, respectively, is the $\text{Aztec}_{p_i,f_i}(F)$ for which $|\text{Aztec}_{p_i,f_i}(F)|$ is maximized among all $1 \leq i \leq n$.
- The supports of multiple pulses share an edge, but not a face

- If the supports of any two pulses of opposite signs share an edge, the flow-firing process will not terminate as one face containing that edge will have a positive flow and the other a negative flow, meaning that the flow-firing would not be able to terminate, as it would attempt to balance the two faces, but then the pulses would return the face to its previous value.
- If the supports of any two pulses of non-opposite signs share an edge, the flow-firing process will terminate unless other supports' interactions cause it to fail to terminate. The two pulses do not impact each other's supports in any way.

5 Conservative flows and a single hole

We now consider initial conservative configurations that contain a pulse, meaning that the initial configuration consists of a pulse but some of the other undistinguished faces may have flow through them as well.

Lemma 5.1. *If an initial conservative configuration C containing a single pulse has $0 \leq C(F) \leq \mathrm{Aztec}(F)$ for all faces F, then the initial configuration exhibits global confluence.*

Proof. At no face can the configuration ever exceed the Aztec pyramid configuration. Because of this, as the pulse must terminate at a configuration no less than the Aztec pyramid at any face, regardless of the order of firing the configuration will terminate at the Aztec pyramid. ■

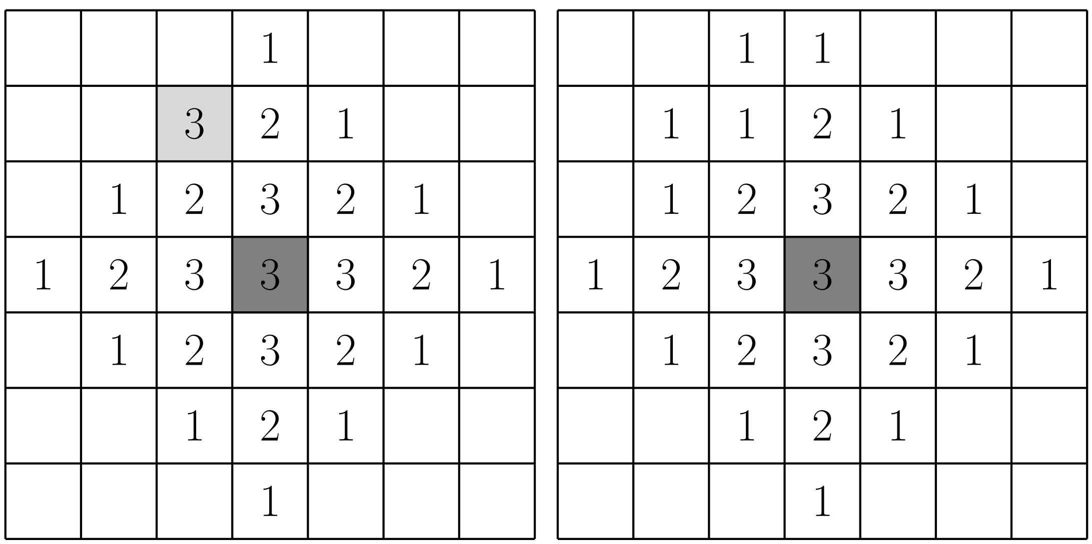
Proposition 5.2. *Over the $\mathbb{Z}^2$ lattice, if an initial conservative configuration C containing a pulse has $C(F) \geq 0$ for all faces F and either contains exactly one face F' in the support of the pulse with $C(F') = \mathrm{Aztec}(F') + 1$, or outside the support with $C(F') \geq 2$, with $C(F) \leq \mathrm{Aztec}(F)$ for all faces F except F', then C does not exhibit global confluence.*

Proof. In the case that flow is outside the support, it has at least 2 places that it can flow to, thus able to violate global confluence.

In the case that the flow is inside the support, either the face is in the same row or column of the hole, or it is not. If it is not, create the Aztec pyramid at all faces except that face by using the hole, and then let that face's flow run down the Aztec pyramid, where it has at least 2 places that it can flow to, thus able to violate global confluence.

If the flow is inside the support and in the same row or column of the hole, then without loss of generality it is to the right of the hole (so in the same row). Then, the Aztec pyramid in that row, to the right of the face that is greater than the Aztec pyramid configuration, cannot be formed without firing that face. However, at all other faces the Aztec pyramid can be made, and thus the flow in that face can travel up or down the Aztec pyramid. At that point, the Aztec pyramid can be made at the faces to the right of the original location of the additional flow. The resulting configuration is a configuration with an additional flow outside the support or inside the support but not in the same row or column of the hole, and thus by the previous cases does not exhibit global confluence. ■

Notice that the condition in Proposition 5.2 that there exists a face F' such that $C(F') = \text{Aztec}(F') + 1$ cannot be generalized to say that there exists a face F' in the support of the Aztec pyramid such that $C(F') > \text{Aztec}(F')$; for example, the initial configuration shown in Fig. 6a where the hole is in the center and colored gray, with flow 3, has one face, the face with flow 3, colored light gray, with flow greater than it would in the Aztec pyramid configuration. This face can only fire to the left or up, and once it does one of these the only remaining option is to fire in the other of the two options, resulting in the terminating configuration shown in Fig. 6b, and hence demonstrating that this initial configuration displays global confluence.



(a) Initial configuration (b) Terminating configuration

Figure 6: A configuration with a single face F' where $C(F') \geq \text{Aztec}(F')$ that displays global confluence.

6 A single pulse of radius r

Define a pulse of radius $r \in \mathbb{N}$ and force p to be the initial configuration with flow p through all faces that are a distance at most r away from the hole. Notice that when $r = 0$ we recover the original definition of a pulse of force p.

See Fig. 7 for an example of a pulse of radius 2.

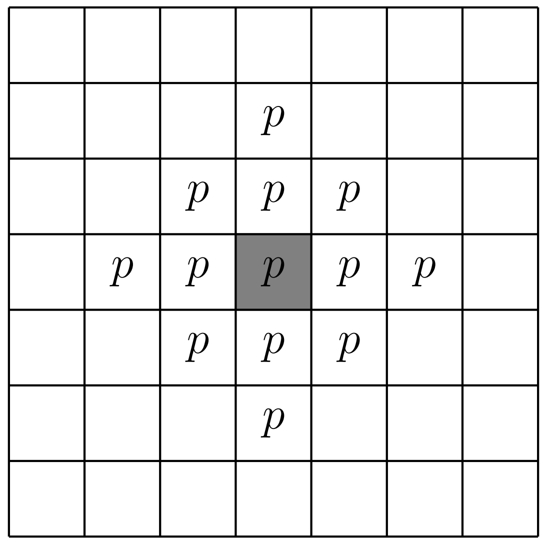


Figure 7: A pulse of radius 2 and force p. The hole is in gray.

Proposition 6.1. *On the $\mathbb{Z}^2$ lattice, a pulse of radius r and force p exhibits global confluence if and only if $|p| \leq 1$ or $r \leq 1$.*

Proof. We assign coordinates to each face of the $\mathbb{Z}^2$ lattice, with the hole being at the origin $(0, 0)$. For example, the four faces that the hole shares an edge with are at $(1, 0)$, $(0, 1)$, $(-1, 0)$, and $(0, -1)$.

If $|p| \leq 1$, then the pulse is already stable, and thus exhibits global confluence.

If $r \leq 1$, then the initial configuration is less than or equal to the Aztec pyramid of the pulse, and thus by Lemma 5.1, the final configuration must be the Aztec pyramid, implying global confluence.

If $|p| \geq 2, r \geq 2$, without loss of generality let $p \geq 2$. If $p = 2$, we see the result as the $4r$ outermost faces (r away from the hole) have multiple options for firing, and once all of them have stabilized (which they will as there can be at most $4r$ firings), the configuration is stable. Hence as there were multiple options for the ending stable configuration, it does not display global confluence. We now assume $p \geq 3$. We proceed by casework, based on whether $r \leq \frac{p+1}{2}$ or not.

Case 1: $r \leq \frac{p+1}{2}$. We will show that for $r \leq \frac{p+1}{2}$, the pulse does not display global confluence.

We will first demonstrate that there exists a firing sequence that turns the pulse of radius r into a configuration less than or equal to the Aztec pyramid, and thus the Aztec pyramid is a possible terminating configuration. We do this by first proving that there exists a way to have the squares directly to the right of the hole be less than the Aztec pyramid without changing the flow through any other of the faces. We note that starting from the square immediately to the right of the hole and continuing right, in order for the final configuration to be stable, the flows through those faces must form a weakly decreasing sequence starting at p and ending at 0, where each decreasing step can only go down by 1. Hence, any stable configuration on those faces must have at least $p+(p-1)+\cdots+2+1+0 = \frac{p(p+1)}{2}$ total flow. We fire the faces directly to the right of the hole towards the right until they, by themselves, are stable relative to each other. As the flow is conserved, there is a total of rp flow. But if $rp \leq \frac{p(p+1)}{2}$, or equivalently $r \leq \frac{p+1}{2}$, then once it stabilizes it must be less than or equal to the Aztec pyramid. We apply this firing sequence similarly to the faces directly above, below, and to the left of the hole as well. Furthermore, using the same logic for the faces with flow off the axis, for example quadrant I could all fire to the right, the entire pulse of radius r can be stabilized. In that case, at a y-coordinate of n, the total flow would be $rp - np$, where the total flow in the Aztec pyramid would be

$$\frac{p(p+1)}{2} - p - (p-1) - \cdots - (p-n+1) \geq \frac{p(p+1)}{2} - np,$$

so for those cases it results in a configuration less than or equal to the Aztec pyramid. Hence, there exists a firing sequence that takes the pulse of radius r into a configuration less than or equal to the Aztec pyramid, and thus the Aztec pyramid is a possible terminating configuration.

Now we will show that there exists a successor of the pulse of radius r that has flow outside of the support of the Aztec pyramid. Stabilize the pulse of radius r, except do not fire the face at $(1,1)$. The resulting configuration for all of the faces except the one at $(1,1)$ must be greater than or equal to the Aztec pyramid. If it is greater than the Aztec pyramid, then there must be flow outside of the support of the Aztec pyramid. If it is equal to the Aztec pyramid, then by Proposition 5.2, there exist terminating configurations that have a support that is a strict superset of the support of the Aztec pyramid.

Case 2: $r > \frac{p+1}{2}$. We will now prove the $r > \frac{p+1}{2}$ cases by considering the parity of p.

Case 2a: p is odd.

For odd flow p, fire the faces with $x = \pm 1$ in the positive y half of the lattice upwards. By Proposition 3.2, this will use the uppermost $\frac{p-1}{2}$ squares of flow p and create a pyramid going from a flow of $p-1$ at $\left(\pm 1, r-\frac{p-1}{2}\right)$ to a flow of 1 at $\left(\pm 1, r+\frac{p-1}{2}-1\right)$. The two faces at $(\pm 1, r)$ have flow $\frac{p-1}{2}$. If $p \geq 5$, then the face at $(0, r)$ with p flow may fire to the left and right once, allowing those flows to cascade upwards and reach the faces at $\left(\pm 1, r+\frac{p-1}{2}\right)$. The faces from $\left(0, r-\frac{p-1}{2}\right)$ to $(0, r-1)$, still with flow p, along with the face at $(0, r)$ with flow $p-2$ by Proposition 3.2 will form a pyramid from $\left(0, r-\frac{p-1}{2}\right)$ to $\left(0, r+\frac{p-1}{2}\right)$ starting with $p-1$ flow and going down to 1 flow, with the double flow at flow $p-2$. Hence, this configuration will have positive flow at the three faces from $\left(-1, r+\frac{p-1}{2}\right)$ to $\left(1, r+\frac{p-1}{2}\right)$. However, if we merely fire those three columns purely upwards and then outwards to the left and right as necessary, the two faces at $\left(\pm 1, r+\frac{p-1}{2}\right)$ will have 0 flow through them as the vertical firing by Proposition 3.2 will yield a pyramidal flow from flow $p-1$ at $\left(\pm 1, r-\frac{p-1}{2}\right)$ to flow 1 at $\left(\pm 1, r+\frac{p-1}{2}-1\right)$. From this configuration it is possible for flow to never reach the faces from $\left(-1, r+\frac{p-1}{2}\right)$ to $\left(1, r+\frac{p-1}{2}\right)$; if the other columns are fired upwards initially, then each column at $x = \pm a$ for $a \leq r - \frac{p-3}{4}$ will be able to reach $\left(\pm a, r+\frac{p-1}{2}-a\right)$, but no further, as afterwards any firing is only to the left or right, meaning it is not possible to have flow go through $\left(\pm 2, r+\frac{p-1}{2}\right)$ and then inwards to $\left(\pm 1, r\frac{p-1}{2}\right)$. Alternatively, another way to look at this is that the flow configuration now is less than or equal to the Aztec pyramid with $r+\frac{p-1}{2}$ flow at the hole at the origin if the same firing process is done on the lower half of the lattice firing downwards, meaning that by the proof of Lemma 5.1, flow cannot reach the squares $\left(\pm 1, r+\frac{p-1}{2}\right)$, which are outside of the Aztec pyramid's support.

See Fig. 8 for a visual example of these two firing processes.

Hence for odd flow $p \geq 5$ with $r \geq \frac{p+1}{2}$, the pulse of flow p and radius r does not have global confluence. For flow $p = 3$, there is not sufficient flow for the square $(0, r)$ to fire both to the left and right, but there is sufficient to fire to one of the two, and so firing to the right, the proof still holds as that one flow will reach $(1, r)$, which is not possible when all three columns fired upwards first. This proof works as long as the hole does not interfere with the aforementioned process, or when $r \geq \frac{p+1}{2}$. This completes the proof for odd p and $r > \frac{p+1}{2}$, completing Case 2a.

Case 2b: p is even.

For even flow $p \geq 4$, fire the faces with $x = \pm 1$ in the positive y half of the lattice upwards. By Proposition 3.2, this will use the uppermost $\frac{p}{2}$ squares of flow p and create a pyramid going from flow $p-1$ at $\left(\pm 1, r-\frac{p}{2}\right)$ to flow 1 at $\left(\pm 1, r+\frac{p}{2}-1\right)$, with the double flow being of flow $\frac{p}{2}$ at $(\pm 1, r-1)$ and $(\pm 1, r)$. The two squares $(\pm 1, r)$ have flow $\frac{p}{2}$, with the pyramid lacking any double flows onward from that point. If $p \geq 6$, the square $(0, r)$ can fire to the left and then the right, and the proof follows similar to the odd flow $p \geq 5$ case aforementioned. If $p = 4$, there is only sufficient flow at $(0, r)$ to fire either to the left or right but not both, and the proof follows similar to the odd flow $p = 3$ case. The proof works as long as the hole does not interfere with the aforementioned process, or when $r > \frac{p}{2}$.

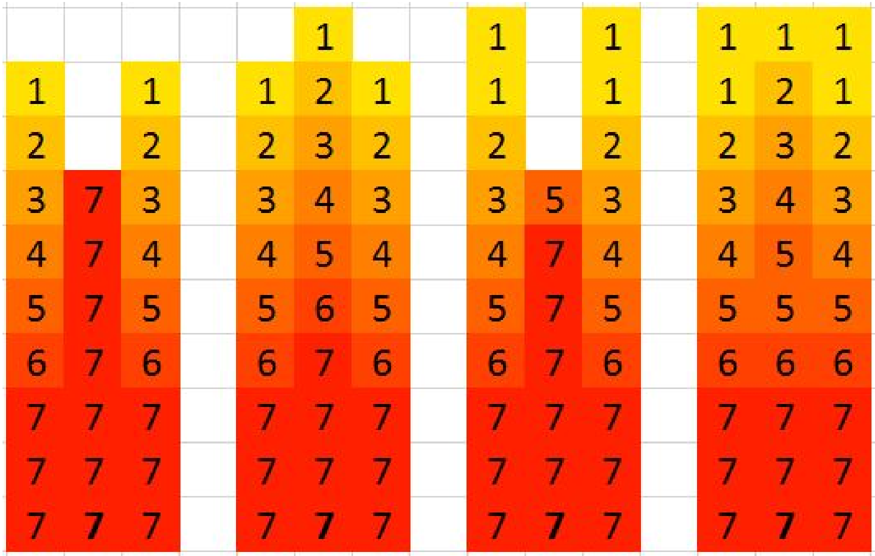


Figure 8: Example diagram for $p = 7, r = 6$. The hole is on the face with the bolded 7 at the bottom center. From left to right: (1) firing the faces with $x = \pm 1$ upwards; (2) the final configuration after firing the central column downwards from (1); (3) firing the face at $(0, r)$ to the left and right, with the flow cascading upwards from (1); (4) the final configuration after firing central column downwards from (3). The faces are colored according to a gradient, with red being the maximum flow of 7 to yellow being the minimum flow of 1, with faces with 0 flow being unlabeled and white.

This completes the proof for even p and $r > \frac{p+1}{2}$, completing Case 2b.

The proof is complete. ■

7 A conjecture on contours

Define a **contour** to be the initial configuration defined by a set of faces in a planar graph with a single topological hole and a positive integer p, where the configuration has a flow of p through each of the faces in the set and through the topological hole. Notice that a contour is a uniform conservative flow containing a pulse, and thus is a generalization of the pulses of radius r studied in Section 6 while being a special case of the conservative flows containing a pulse studied in Section 5.

Notice that if $|p| \leq 1$, the contour trivially has global confluence as all the non-hole faces are stable and any face adjacent to the hole will have a flow of p because of the hole.

As contours are a generalization of pulses of radius r, we wish to generalize Proposition 6.1

to contours. Empirically, the only contours that display global confluence are essentially of the same nature as the pulses of radius r that have global confluence, leading to the following conjecture on the global confluence of contours.

Conjecture 7.1. *On the $\mathbb{Z}^2$ lattice, a contour C has global confluence if and only if either $|p| \leq 1$ or the contour only contains faces that share an edge with the hole.*

The if case can be easily proven, with the $|p| \leq 1$ case being trivially true as the initial configuration will immediately stabilize once the faces sharing an edge with the hole have flow p, and with the other case displaying global confluence due to Lemma 5.1 as the configuration bounded between the all-zero configuration and the Aztec pyramid.

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References

[1] Richard Anderson, László Lovász, Peter Shor, Joel Spencer, Éva Tardos, and Shmuel Winograd. Disks, balls, and walls: analysis of a combinatorial game. *The American Mathematical Monthly*, 96(6):481–493, 1989.

[2] Per Bak, Chao Tang, and Kurt Wiesenfeld. Self-organized criticality: An explanation of 1/f noise. *Phys. Rev. Lett.*, 59:381–384, 1987.

[3] N. L. Biggs. Chip-firing and the critical group of a graph. *J. Algebraic Combin.*, 9(1):25–45, 1999.

[4] Norman L. Biggs. Algebraic potential theory on graphs. *Bull. London Math. Soc.*, 29(6):641–682, 1997.

[5] Anders Björner and László Lovász. Chip-firing games on directed graphs. *J. Algebraic Combin.*, 1(4):305–328, 1992.

[6] Anders Björner, László Lovász, and Peter W. Shor. Chip-firing games on graphs. *European J. Combin.*, 12(4):283–291, 1991.

[7] Deepak Dhar. Self-organized critical state of sandpile automation models. *Phys. Rev. Lett.*, 64(23):2837, 1990.

[8] Art M. Duval, Caroline J. Klivans, and Jeremy L. Martin. Critical groups of simplicial complexes. *Ann. Comb.*, 17(1):53–70, 2013.

[9] Pedro Felzenszwalb and Caroline Klivans. Flow-firing processes. *arXiv preprint arXiv:1902.02404*, 2019.

[10] Caroline J Klivans. *The mathematics of chip-firing*. Chapman and Hall/CRC, 2018.

[11] Maxwell Herman Alexander Newman. On theories with a combinatorial definition of" equivalence". *Annals of mathematics*, pages 223–243, 1942.

[12] Joel Spencer. Balancing vectors in the max norm. *Combinatorica*, 6(1):55–65, 1986.

[13] Gábor Tardos. Polynomial bound for a chip firing game on graphs. *SIAM J. Discrete Math.*, 1(3):397–398, 1988.

[14] Jan van den Heuvel. Algorithmic aspects of a chip-firing game. *Combin. Probab. Comput.*, 10(6):505–529, 2001.