UPPER BOUND ON THE DISTORTION OF CABLED KNOTS

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Abstract. The torus knots are a class of knots generated by ordered pairs (p, q) of relatively prime integers, where the (p, q)-torus knot is the curve defined by a ray of slope $\frac{p}{q}$ emanating from the origin in the representation of the torus as a square with opposing sides identified. Furthermore, given a curve K, we can define the (p, q)-cabling of K to be the (p, q)-torus knot living on an embedding of the torus which follows K, as opposed to the standard embedding of the torus which follows S^1 in $\mathbb{R}^3$.

We show that for all p and $q \gg p$, there exists a curve in the isotopy class of the (p, q)-torus knot whose supremal ratio of arc length to Euclidean distance, called the distortion of the curve, is bounded above by $\frac{7q}{\log(q)}$, and additionally show that this bound holds for the (p, q)-cabling of any knot. This extends a result of Studer establishing sublinear upper bounds for the distortion of the $(2, q)-$torus knots.

1. Introduction

We will be studying the distortion of cabled knots. The *distortion* $\delta(\gamma)$ of a rectifiable simple closed curve $\gamma \subset \mathbb{R}^n$ is defined as

$$\delta(\gamma) = \sup_{\substack{u,v\in\gamma \\ u\neq v}} \left(\frac{d_\gamma(u, v)}{|u - v|} \right)$$

where $d_\gamma(u, v)$ is the shortest arc length between u and v along γ. The distortion is the supremal arc length to distance ratio over γ. To extend this to a knot K, we define $\delta(K)$ to be the infimal distortion over all representatives of the isotopy class of K.

This distortion is difficult to compute exactly, and not known for any knot except for the unknot. In [4], Gromov shows that the distortion of any knot is at least $\frac{\pi}{2}$, realized by the standard embedding of the unknot. In [2], Denne and Sullivan showed that all other knots have distortion at least $\frac{5\pi}{3}$, but even for the trefoil knot, perhaps the simplest non-trivial knot, computational evidence suggests that the distortion is around 10.7. In [3], Gromov asked whether there exist knots with arbitrarily large distortion; for example, does every knot have a representative γ with $\delta(\gamma) < 100$? This problem was open for 30 years, until in [6], John Pardon demonstrated the existence of highly distorted knots, showing that for the (p, q)-torus knot $T_{p,q}$

$$\delta(T_{p,q}) > \frac{1}{160}\min(p, q).$$

As such,

$$\lim_{n\to\infty} \delta(T_{n,n+1}) \geq \lim_{n\to\infty} \frac{1}{160}\min(n, n+1) = \lim_{n\to\infty} \frac{n}{160} = \infty.$$

Pardon's bound also applies to the $(p, q)-$cabling of any knot. A *cable knot* is constructed for some knot K by taking an embedding of the torus knotted in the shape of K, for instance, the boundary of a tubular neighborhood of K, and drawing the torus knot $T_{p,q}$ on this torus. We denote the resulting knot $K_{p,q}$, called the (p, q)*-cabling of* K. Observe that if K is the unknot, $T_{p,q} = K_{p,q}$.

It is possible this lower bound may be improved substantially, as it seems likely that even for fixed p, $\delta(T_{p,q})$ goes to infinity as q increases. Currently, there is no known way to show this sort of lower bound, and in [5] Gromov and Guth conjectured that $\delta(T_{p,q}) \sim \max(p, q)$. A few years ago in [7], Studer gave a better upper bound in the case $p = 2$, showing that

$$\delta(T_{2,q}) < \frac{7q}{\log(q)}$$

for $q \geq 50$ by packing the crossings of the knot into spirals of small diameter with relatively low distortion.

Our result is a generalization of Studer's construction to all p. More precisely, we prove the following theorem.

Theorem 1. *For any knot K, and any $p \in \mathbb{N}$, there exists Q depending on p and K so that for $q \geq Q$ with* $\gcd(p, q) = 1$

$$\delta(K_{p,q}) < \frac{7q}{\log(q)}.$$

With this bound, a few things are apparent. This result has room for improvement, improving the bound on Q and improving the constant 7, as well as perhaps improving further sublinear bounds. These may be worth looking into, as it seems as if the distortion of the spirals may be slightly decreased in all these cases to decrease the bound itself, though this may increase the bound on q beyond what it is now.

2. Acknowledgements

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3. Construction

We begin by defining key terms, namely distortion, $T_{p,q}$, and $K_{p,q}$.

Definition 1. Given a curve $\gamma \subset \mathbb{R}^n$, we define the *distortion* of γ to be

$$\delta(\gamma) = \sup_{u,v \in \gamma} \frac{d_\gamma(u,v)}{|u-v|}$$

where $d_\gamma(u,v)$ is the arc length of γ from u to v. For a knot K, the distortion is defined as the infimal distortion over all curves in the isotopy class of K.

Definition 2. The (p,q)-*torus knot*, denoted $T_{p,q}$, is the knot living on the surface of the torus whose standard embedding is given by the parametrization

$$\begin{aligned} x &= (\cos(q\theta) + 2)\cos(p\theta) \\ y &= (\cos(q\theta) + 2)\sin(p\theta) \\ z &= -\sin(q\theta) \end{aligned}$$

for $0 \leq \theta < 2\pi$.

Definition 3. Given a tame knot K, γ be an embedding of K, and T_K an extension of γ to an embedding of the solid torus $S^1 \times D^2$. Then the $(p,q)-$*cabling of* K, denoted $K_{p,q}$, is the copy of $T_{p,q}$ on the torus T_K. Essentially, $K_{p,q}$ is a copy of $T_{p,q}$ which follows K as its central curve.

In [7] Studer constructs a low-distortion embedding of $T_{2,q}$ by using a logarithmic spiral S that is the image of

$$\phi : [0, \pi q] \to \mathbb{R}^3, \phi(s) = e^{\frac{s\log(q)}{2\pi q}}(\cos(s), 0, \sin(s)).$$

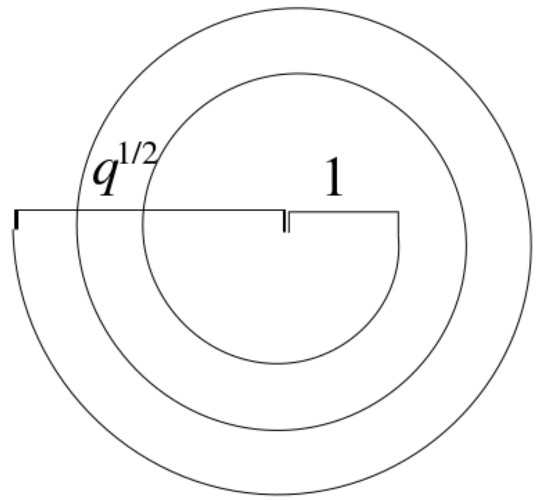

Figure 1: The spiral for $q = 5$.

This has radii ranging from 1 to $\sqrt{q}$, as seen in Figure 1 and we claim it has distortion less than $\frac{7q}{\log(q)}$, as corollary of the following lemma.

Lemma 1. *The distortion of $\phi(s) = e^{ks}(cos(s), 0, sin(s))$, a logarithmic spiral, is at most $\frac{\sqrt{1+k^2}}{k}$.*

Proof. We wish to show that for any two points u, v on the spiral,

$$\frac{d(u,v)}{|u-v|} \leq \frac{\sqrt{1+k^2}}{k},$$

where d is the distance along the curve from u to v. To see this, we represent the points u, v as $\phi(a), \phi(b)$. Then, we compute arc length via integral to get:

$$\begin{aligned} d(u,v) &= \int_a^b \sqrt{|\phi(s)|^2 + |\phi'(s)|^2} ds \\ &= \int_a^b \sqrt{e^{2ks} + k^2 e^{2ks}} ds \\ &= \frac{\sqrt{1+k^2}}{k}(e^{kb} - e^{ka}). \end{aligned}$$

We can also bound the distance $|u-v|$ from below by $|u|-|v| = e^{kb}-e^{ka}$. Thus,

$$\begin{aligned}\frac{d(u,v)}{|u-v|} &= \frac{\frac{\sqrt{1+k^2}}{k}(e^{kb}-e^{ka})}{|u-v|}\\ &\leq \frac{\frac{\sqrt{1+k^2}}{k}(e^{kb}-e^{ka})}{e^{kb}-e^{ka}} = \frac{\sqrt{1+k^2}}{k}.\end{aligned}$$

□

In the case of this particular spiral, $k = \frac{\log(q)}{2\pi q}$. This value for k will be used throughout. So, by the lemma, the distortion of the spiral is at most

$$\begin{aligned}\frac{\sqrt{1+k^2}}{k} &= \frac{2\pi q}{\log(q)}\sqrt{1+\left(\frac{\log(q)}{2\pi q}\right)^2}\\ &< \left(2\pi\sqrt{1+\frac{1}{4\pi^2}}\right)\frac{q}{\log(q)}\\ &< \frac{7q}{\log(q)}.\end{aligned}$$

To complete the embedding, Studer connects the ends of this spiral with a piecewise linear segment that passes through the center of the spiral on its way around. Since the spiral has diameter $\sqrt{q}$, this segment can stay far away from the spiral while maintaining a length sublinear in q, allowing for its contribution to the distortion to occur at only a couple of key points.

For $p > 2$, it is natural to generalize this by packing all the crossings of the (p,q)-torus knot into $p-1$ concentric spirals. The parameterization used is, for $1 \leq n \leq p-1$,

$$\phi_n : \left[0, \frac{2\pi q}{p}\right] \to \mathbb{R}^3, \phi_n(s) = nq^{\frac{n-1}{p}} e^{\frac{s\log(q)}{2\pi q}}(\cos(s), 0, \sin(s)).$$

The same computation shows that these individual spirals have distortion less than $\frac{7q}{\log(q)}$.

Lemma 2. *The total length of the spirals is at most* $(p-1)q^{\frac{p-1}{p}}\frac{\sqrt{1+k^2}}{k}$.

Proof. To see this, we first compute the length of ϕ_n as

$$\begin{aligned}
&\int_0^{\frac{2\pi q}{p}} \sqrt{(nq^{\frac{n-1}{p}}e^{ks})^2 + k^2(nq^{\frac{n-1}{p}}e^{ks})^2}ds \\
&= \int_0^{\frac{2\pi q}{p}} nq^{\frac{n-1}{p}}e^{ks}\sqrt{1+k^2}ds \\
&= nq^{\frac{n-1}{p}}e^{\frac{\log(q)}{2\pi q}s}\Big|_0^{\frac{2\pi q}{p}}\frac{\sqrt{1+k^2}}{k} \\
&= nq^{\frac{n-1}{p}}(q^{\frac{1}{p}}-1)\frac{\sqrt{1+k^2}}{k}.
\end{aligned}$$

So, the length of all the spirals together is

$$\begin{aligned}
&\sum_{n=1}^{p-1} nq^{\frac{n-1}{p}}(q^{\frac{1}{p}}-1)\frac{\sqrt{1+k^2}}{k} \\
&= \left(\sum_{n=1}^{p-1} nq^{\frac{n}{p}} - \sum_{n=0}^{p-2}(n+1)q^{\frac{n}{p}}\right)\frac{\sqrt{1+k^2}}{k} \\
&< \left(\sum_{n=1}^{p-1} nq^{\frac{n}{p}} - \sum_{n=1}^{p-2} nq^{\frac{n}{p}}\right)\frac{\sqrt{1+k^2}}{k} \\
&= (p-1)q^{\frac{p-1}{p}}\frac{\sqrt{1+k^2}}{k}.
\end{aligned}$$

□

Thus, so long as the connecting pieces have length proportional to $q^{\frac{p-1}{p}}$, we may write the total length of the curve as at most

$$(p-1)q^{\frac{p-1}{p}}\frac{\sqrt{1+k^2}}{k} + Cq^{\frac{p-1}{p}}.$$

So long as points u, v on the curve are at least $(p-1)q^{\frac{p-1}{p}}$ apart, the distortion they cause is at most a constant term above the bound given. Thus, we would like to keep the connecting pieces abiding to this length, while also remaining distant enough from the spirals to not contribute to the distortion for a point on a connector and a point on a spiral. This is possible due to the diameter of the spirals being sublinear in q.

To connect the spirals, we construct the following segment: Starting from the end of the k^{th} spiral for $1 \leq k < p-1$, form a segment of length $(2k+2)D$ perpendicular to the xz-plane in the $-y$ direction. Next, take a straight line segment to the point $((2k+2)D, -(2k+2)D, 0)$. Then, take a line segment completely in the $+y$ direction of length $(4k+6)D$. Next, travel in the $-x$ direction until at the same x-coordinate as the start of the next spiral. Add on a semicircular arc of radius $(2k+2)D$ going into the $-z$ direction, which will end at $z=0$, $y=2D$. Finally, complete the connection to the next spiral with a segment of length $2D$ in the $+y$ connection, resulting in connectors resembling the example in Figure 2. Note that this causes the connectors to maintain a distance of more than D from the spirals outside of the segments going to the ends of spirals at the start and end, and from each other in the same circumstances in addition to the semicircular arcs.

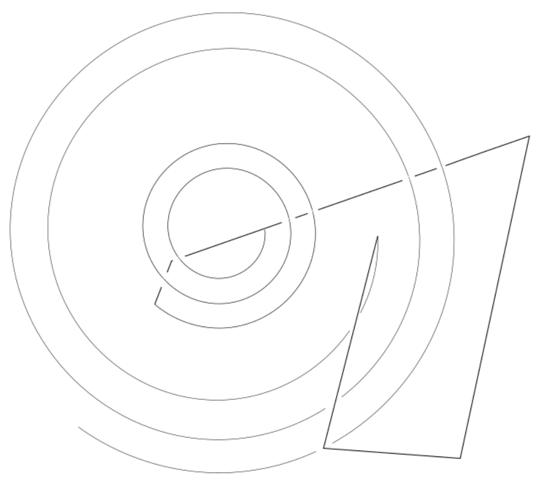

Figure 2: First connector for $T_{3,7}$.

However, the connector from the final spiral to the first still remains. For this, we have $k=p-1$. As before, begin with a perpendicular segment of length $(2k+2)D$ in the $-y$ direction. Then, take a straight line segment to the point $(-2D,-(2k+2)D,0)$, followed by a line segment in the $+y$ direction of length $(2k+4)D$, ending at the point $(-2D,2D,0)$. Then go straight to the point $(0,2D,0)$, and go in the $+y$ direction to $(0,-2D,0)$. After, go straight to the point $(2D,-2D,0)$, then straight to the point $(2D,4D,0)$, then straight to $(1,4D,0)$, and finally take a straight segment of length $4D$ to the start of the first spiral, resulting in the connector present in Figure 3. Note that this connector stays completely away from the other connectors besides again the perpendicular segments, but does come to a distance 1 from itself at distant points, in particular when passing through the middle of the spirals compared to the final connector.

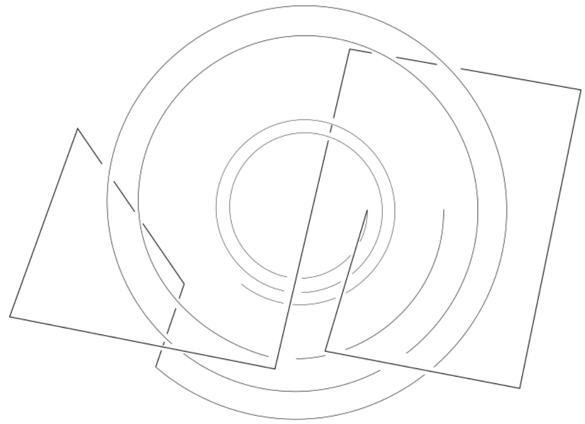

Figure 3: Second connector for $T_{3,7}$.

In the case of a cabling of a knot K, we keep the perpendicular segments and the semicircular arcs. Then, consider a embedding T of the torus knotted in the shape of K such that the inner radius is pD. To get the connectors, we move the spiral construction inside of T, and then connect the ends of the perpendicular segments to the corresponding semicircles while the connector goes along T.

We note that the interior of T is large enough that we may space the connectors $2D$ apart, and $2D$ away from the spirals. Finally, by keeping the connectors relatively smooth, we ensure that each has length still sublinear in q.

Observation. These connectors will have a total length of $Cq^{\frac{p-1}{p}}$ for some C constant with respect to q. We will use C in computations when this constant arises.

Lemma 3. *The described curve is an embedding of $K_{p,q}$.*

Proof. To see that this is indeed a embedding of $K_{p,q}$, we may begin with the standard embedding. Then, we may twist the strands so that all of the windings occur in a short section of the embedded torus, with the strands going straight around the torus besides this one region, as in Figure 4.

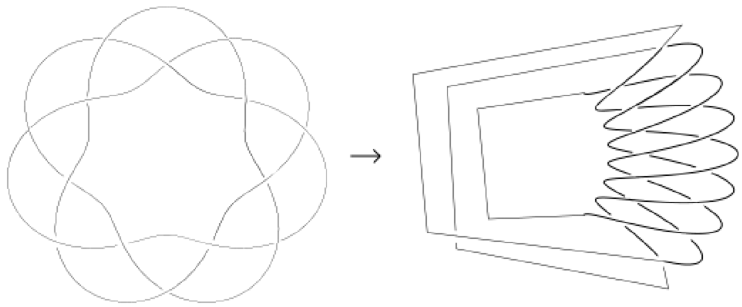

Figure 4: The first step for $T_{3,7}$.

Then, we may label the strands going into this section from 1 to p. Within the section with all of the windings, each of these strands may be viewed as forming spirals of equal radius. We may expand these radii in the order of the labellings so that the radius of the spiral from strand n is larger than the radii of all the earlier strands.

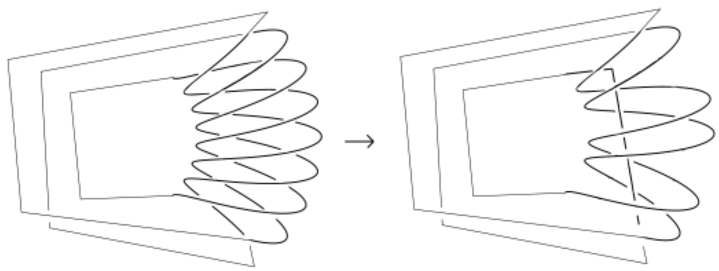

Figure 5: The next step for $T_{3,7}$.

We then may flatten out the spiral from strand 1, as all of the other spirals are outside of it at this point, giving us the central connector from the last spiral to the first, like in Figure 5. Finally, for each remaining spiral, we may have the radius slightly increase as the spiral goes on, so that we can compress it fully into the single plane used in the given embedding. This results in the curve described here, showing that it is equivalent to $K_{p,q}$.

□

4. Proof of Sublinear Distortion Bound

Given this embedding of $K_{p,q}$, we would like to show that its distortion is sublinear in q. Since the distortion of $K_{p,q}$ is infimal over all its embeddings, showing the bound in this one case will prove the distortion bound for $K_{p,q}$.

Given the parametrization, we may compute for pairs of points u, v the ratio of the distance along the curve to Euclidean distance, and show that the bound holds regardless of the two points.

For this, we break up the computations into separate cases. There are 3 relevant cases: points both on spirals, having one point on a spiral and one on a connector, and points both on connectors.

Case 1. *Both points lie on spirals.*

First, consider points u, v with $u \in \phi_n$, $v \in \phi_m$, $n > m$. Then for some $a, b \in [0, \frac{2\pi q}{p}]$,

$$u = \phi_n(a) = nq^{\frac{n-1}{p}} e^{\frac{a \log(q)}{2\pi q}} (\cos(a), 0, \sin(a)),$$

$$v = \phi_m(b) = mq^{\frac{m-1}{p}} e^{\frac{b \log(q)}{2\pi q}} (\cos(b), 0, \sin(b)).$$

Thus, we can compute that

$$\begin{aligned} \frac{d(u,v)}{|u-v|} &\leq \frac{n\frac{\sqrt{1+k^2}}{k} q^{\frac{n-1}{p}} e^k a - m\frac{\sqrt{1+k^2}}{k} q^{\frac{m-1}{p}} e^k b + Cq^{\frac{p-1}{p}}}{nq^{\frac{n-1}{p}} e^{ka} - mq^{\frac{m-1}{p}} e^{kb}} \\ &\leq \frac{\sqrt{1+k^2}}{k} + Cq^{\frac{p-1-(n-m)}{p}}. \end{aligned}$$

Note that for sufficiently large q, $Cq^{\frac{p-1-(n-m)}{p}} \ll \frac{\sqrt{1+k^2}}{k}$, so remembering $k = \frac{\log(q)}{2\pi q}$, the bound $\frac{7q}{\log(q)}$ holds in the limit. We may also extend this computation to include the distortion between a point on the spiral and the origin, which lies on the connection from the last spiral to the first, through setting $m = 0$.

Case 2. *One point lies on a spiral, and one on a connector.*

Because of where the connectors are at least a distance D apart, we may simplify this to the second point lying on one of the perpendicular segments at the end of a connector. For one point u on spiral ϕ_n, and a specific perpendicular segment that intersects the xz-plane at a point w, we may parameterize any point v on the segment by its distance b from the plane of the spirals. Denote the distance $|u - v| = a$. Then, since the line segment from u to w is perpendicular to the segment from w to v, we get that $|u - w| = \sqrt{a^2 + b^2}$. Since the distance from u to v is a, and we have already computed that the distortion caused by points on the spirals (or the origin, in the case of v lying on the segment passing through the origin) is at most $\frac{\sqrt{1+k^2}}{k} + Cq^{\frac{p-1}{p}}$, we know the distance along the curve from u to v is at most $a(\frac{\sqrt{1+k^2}}{k} + Cq^{\frac{p-1}{p}})$.

Thus, we can compute:

$$\begin{aligned}\frac{d(u,w)}{|u-w|} &\leq \frac{d(u,v)+d(v,w)}{\sqrt{a^2+b^2}}\\ &\leq \frac{a(\frac{\sqrt{1+k^2}}{k}+Cq^{\frac{p-1}{p}})}{\sqrt{a^2+b^2}}+\frac{b}{\sqrt{b^2+a^2}}\\ &\leq \frac{\sqrt{1+k^2}}{k}+Cq^{\frac{p-1}{p}}+1.\end{aligned}$$

Again, the terms besides $\frac{\sqrt{1+k^2}}{k}$ become negligible in the limit, and the bound of $\frac{7q}{\log(q)}$ again holds.

Case 3. *Both points lie on connectors.*

For points u, v both on connectors, we need only consider the case where the points lie on the perpendicular segments near the spirals, as in all other cases, the spirals are more than $(p-1)q^{\frac{p-1}{p}}$ apart.

Between u, v, the only distance along the curve not on a connector would be due to spirals. To do this, let a and b be the projections of u and v respectively onto the xz-plane. Then, the distance along the spirals is at most the length of a spiral from a to b, as such a spiral would contain all the ones along the curve between the two points. Let θ_a, θ_b be such that $e^{k\theta_a}=|a|$, and $e^{k\theta_b}=|b|$. Then, the length of this spiral would be

$$\begin{aligned}&\int_{\theta_a}^{\theta_b}\sqrt{(e^{k\theta})^2+(ke^{k\theta})^2}d\theta\\ &=\frac{\sqrt{1+k^2}}{k}(e^{k\theta_b}-e^{k\theta_a})\\ &=\frac{\sqrt{1+k^2}}{k}(|b|-|a|).\end{aligned}$$

Also, note that if the points are on different segments (the only interesting case, as otherwise the distortion is at most $\frac{\pi}{2}$ due to the semicircle), the Euclidean distance is at least 1. Thus, for large enough q, the distortion is bounded above by

$$\begin{aligned}&\frac{(|b|-|a|)\frac{\sqrt{1+k^2}}{k}+Cq^{\frac{p-1}{p}}}{|b|-|a|}\\ &<\frac{\sqrt{1+k^2}}{k}+Cq^{\frac{p-1}{p}}\\ &<\frac{7q}{\log(q)}.\end{aligned}$$

However, there is one case in which the distortion between points on the same connector may be greater. On the final connector from the last spiral to the first, there are points as close as distance 1 apart; however, since there are no spirals between, the same inequality as above holds with the first term disappearing.

As the bound holds in all cases, this embedding of $K_{p,q}$ has distortion bounded above by $\frac{7q}{\log(q)}$ if q is sufficiently large and K and p are fixed.

5. Future Work

One thing that comes to mind in relation to the distortion of knots is the definition of distortion itself. In particular, the use of supremum instead of some kind of averaging seems somewhat odd. One may try and define an average distortion of a curve S by something of the form

$$\frac{1}{\ell(S)^2}\left(\int\int_{(u,v)\in S^2}\left(\frac{d_S(u,v)}{|u-v|}\right)^p dudv\right)^{\frac{1}{p}}$$

for a scaling factor p, in the style of the L^p norms, noting that for a fixed curve, Gromov's supremal distortion will be the limit of these p-distortions as p goes to infinity.

However, taking the infimal average distortion over the isotopy class is relatively uninteresting. Because it averages over the entire embedding, by simply increasing the length of the lower-distortion arcs, we can decrease the average distortion to essentially that of a circle, a constant value.

One potential workaround for this issue would be to place somewhat restrictive geometric constraints on the considered embeddings to avoid large circles creating a constant distortion. Finding an approach with a balance between a strong definition and a computable value is likely to be some combination of different types of restrictions, and since the average distortion for any curve is necessarily bounded above by its supremal distortion, may induce new lower bounds.

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