The Discrete Curve Shortening Flow

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Outline

Outline

Differential Geometry

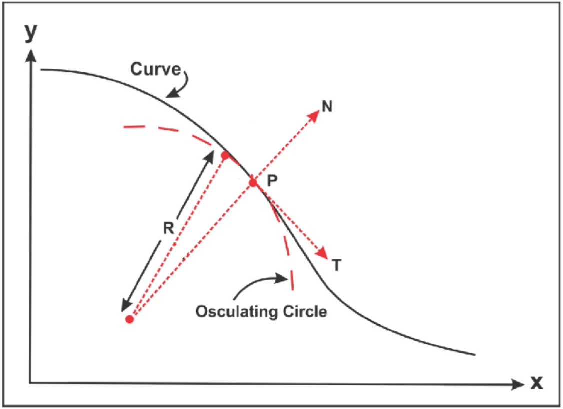

Figure: Geometric meaning ($R = \frac{1}{k}$)

Differential Geometry
cont

Definition

Given a smooth curve $(x(s), y(s))$, we define the **unit tangential vector** at every point as:

$$\vec{t} = \frac{1}{\sqrt{x'^2(s) + y'^2(s)}} \langle x'(s), y'(s) \rangle$$

We define the **unit normal vector** at every point as:

$$\vec{n} = \frac{\vec{t'}}{|\vec{t'}|}$$

The **curvature** k such that:

$$k\vec{n} = \vec{t'}$$

Smooth Curve Shortening Flow

Differential equation

Define the motion of a curve such that every point $\mathbf{x}$ moves according to the following differential equation:

$$\frac{d\mathbf{x}}{dt} = -k(\mathbf{x})\vec{n}(\mathbf{x})$$

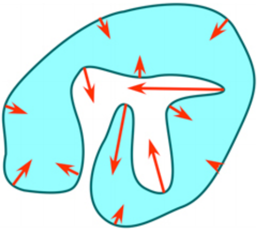

Figure: A curve with $\frac{d\mathbf{x}}{dt}$ vectors drawn in

Smooth Curve Shortening Flow
cont

Ecker-Huisken Result

All smooth curves that are a graph of some function will converge to a straight line, if initially the graphs aren't too "weird".

Gage-Hamilton-Grayson Result

All smooth, closed curves will flow to a point under curve shortening flow, and become more and more circular.

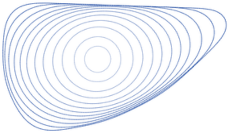

Figure: A curve undergoing curve shortening flow

Discrete Curve Shortening Flow

Curvature

Curvature $k(\mathbf{x})$ at point $\mathbf{x}$ is $\pi - \alpha$, where α is the interior angle at $\mathbf{x}$.

Normal vectors

The normal vector $\vec{n}(\mathbf{x})$ at point $\mathbf{x}$ is in the direction of the angle bisector at $\mathbf{x}$.

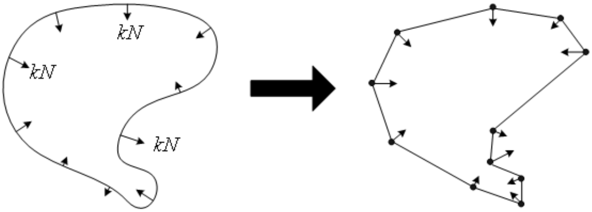

Figure: A smooth curve and a discrete analogue

Discrete Curve Shortening Flow

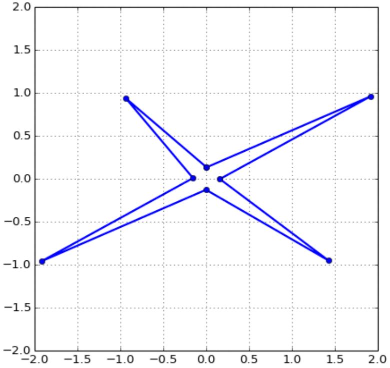

Figure: A shape undergoing discrete curve shortening flow

Outline

Isosceles Triangles

- Top angle $< \frac{\pi}{3}$: flows to a line (Ramanujam)
- Top angle $> \frac{\pi}{3}$: flows to a point (Rowley and Cohen)

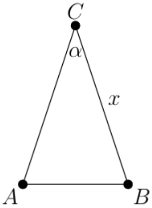

Figure: Isosceles triangle

Isosceles Triangles
cont.

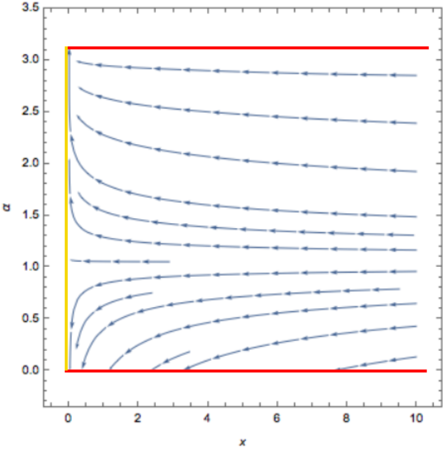

Figure: Phase plane diagram for isosceles triangles

General Triangles

- All triangles except the isosceles specified before go to lines (Rowley and Cohen)

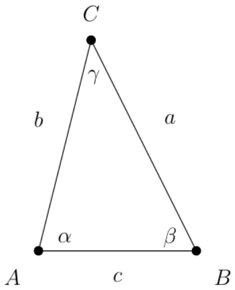

Figure: General triangle and phase plane diagram for general triangles

General Triangles
cont.

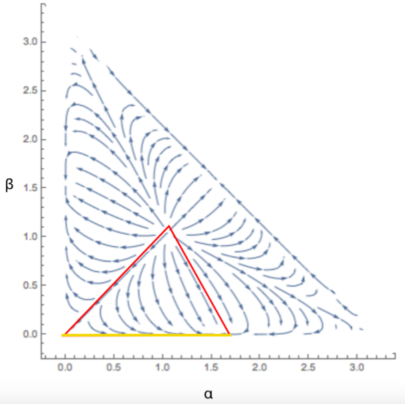

Figure: Phase plane diagram for general triangles

Outline

Discrete Generalization

- Does Ecker and Huisken's result hold for discrete open curves
- Specifically graphs
- Even if we were to consider just a linear approximation of the flow, it would be incredibly complex, infinite system

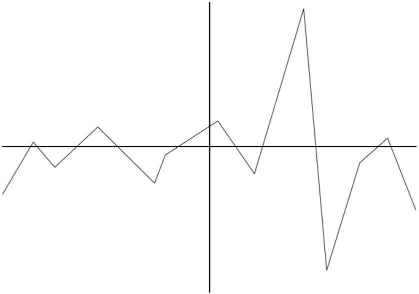

Figure: A section of an infinite piecewise linear curve

Useful Restriction

- General infinite curves are very hard, so we can restrict conditions to allow for easier analysis.
- Had the idea of periodic curves with points that remain fixed between the repeating periods
- These curves we found would be of the type $\ldots C\ C^T\ C\ C^T \ldots$
- C^T: Construct fixed points every n s.t. $\theta_{an-m} = \theta_{an+m}$ for $m < n$

Figure: Section of an infinite piecewise linear curve of this type

Outline

Description

By showing a result for finite curves, we can then show one for infinite curves

Finite piecewise linear curve

A collection of points $\mathbf{x_0}, \mathbf{x_1}, ..., \mathbf{x_n}$ defining a discrete curve, with $\mathbf{x_0}$ and $\mathbf{x_n}$ being fixed under DCSF

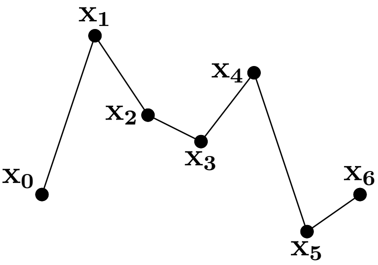

Figure: Example of a finite piecewise linear curve

Equations

- The velocity of each point $\mathbf{x_0}, \mathbf{x_1}, \ldots, \mathbf{x_n}$ under, using the equation $\frac{d\mathbf{x}}{dt} = -k(\mathbf{x})\vec{n}(\mathbf{x})$ is

$$\frac{d\mathbf{x_i}}{dt} = \cos^{-1}\left(\frac{(\mathbf{x_{i-1}}-\mathbf{x_i})\cdot(\mathbf{x_{i+1}}-\mathbf{x_i})}{|\mathbf{x_{i-1}}-\mathbf{x_i}||\mathbf{x_{i+1}}-\mathbf{x_i}|}\right)\left(\frac{(\mathbf{x_{i-1}}-\mathbf{x_i})|\mathbf{x_{i+1}}-\mathbf{x_i}|+(\mathbf{x_{i+1}}-\mathbf{x_i})||\mathbf{x_{i-1}}-\mathbf{x_i}|}{|((\mathbf{x_{i-1}}-\mathbf{x_i})|\mathbf{x_{i+1}}-\mathbf{x_i}|+(\mathbf{x_{i+1}}-\mathbf{x_i})||\mathbf{x_{i-1}}-\mathbf{x_i}|)|}\right)$$

Equations

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- This isn't very helpful...

Geometry

- Instead of analyzing the equations, we analyze the geometry

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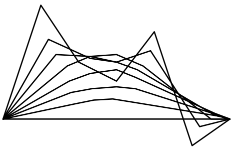
Geometry

- Instead of analyzing the equations, we analyze the geometry
- More specifically, the movement of the maximum and minimum points
- Clear to see the maximum will always decrease and minimum increase (unless one is one of the endpoints)
- With this, we can determine the end behavior!

End Behavior

With this behavior of a constantly decreasing maximum and increasing minimum, we showed that all these curves result in a line!



Meaning then that infinite curves of the type $\ldots C\ C^T\ C\ C^T \ldots$ also go to lines

An Animation

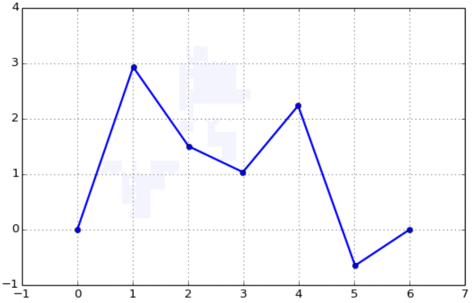


Figure: Evolution of a finite piecewise linear curve

Discrete Generalization

- Is there an analogue to the Gauge-Hamilton-Grayson Result?
- Will all polygons collapse to a point under the DCSF?
- Will polygons become more and more convex under the DCSF?
- Will all polygons become convex before collapsing under the DCSF?

Outline

Convexity

Theorem

Under the DCSF, every convex polygon will remain convex until it collapses.

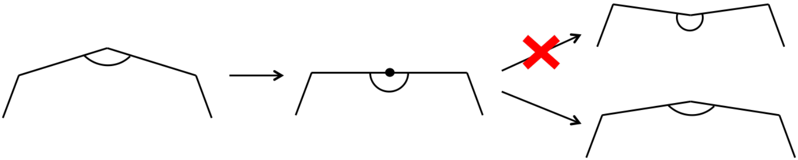


Figure: A sketch of the proof

Convexity

cont

- However, a polygon will not necessarily become more convex:

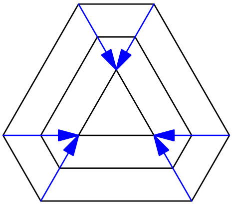


Figure: An equiangular hexagon under the DCSF

Outline

Symmetric Concave Quadrilateral

- Simplest concave polygon

Theorem

Every symmetric concave quadrilateral will become convex before collapsing under the DCSF.

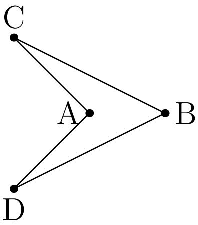


Figure: A symmetric concave quadrilateral

Exploit Symmetry

- C and D will evolve symmetrically, only consider one of them
- Define $\angle CAB = \alpha$, $\angle CBA = \beta$, and $AB = x$

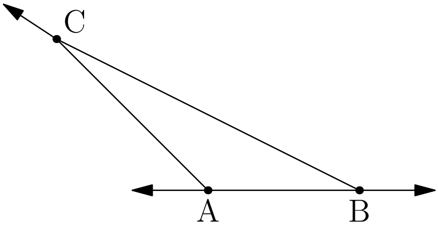


Figure: Three points whose evolution we will consider, normal vectors drawn in

The Evolution

The Differential Equations

Three differential equations dictate the evolution of the points:

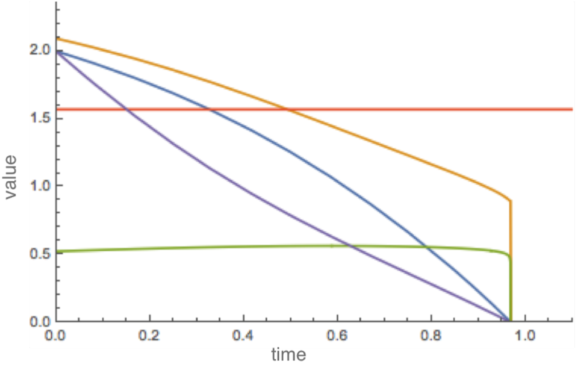
$$\frac{dx}{dt} = 2\alpha + 2\beta - 2\pi$$

$$\frac{d\alpha}{dt} = -\frac{((\alpha+\beta)\cos(\frac{\alpha+\beta}{2}) - (-2\alpha+\pi)\sin(\alpha))\csc(\beta)\sin(\alpha+\beta)}{x}$$

$$\frac{d\beta}{dt} = -\frac{((\alpha+\beta)\cos(\frac{\alpha+\beta}{2}) - (-2\beta+\pi)\sin(\beta))\csc(\alpha)\sin(\alpha+\beta)}{x}$$

- Different initial conditions will lead to different results

Examples



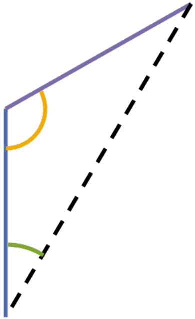


Figure: $\alpha = \frac{2\pi}{3}$ and $\beta = \frac{\pi}{6}$ and $x = 2$

Examples

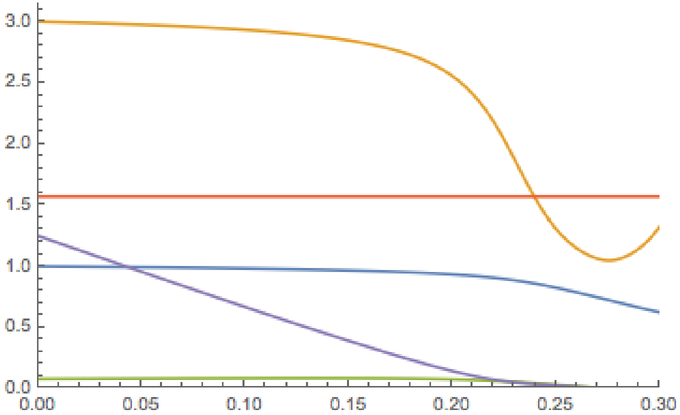


Figure: $\alpha = \frac{191\pi}{200}$ and $\beta = \frac{\pi}{40}$

Examples

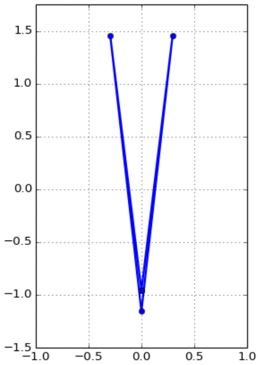


Figure: A shape undergoing discrete curve shortening flow

Boundary Cases

$\alpha(0)$ near $\frac{\pi}{2}$ and $\beta(0)$ near 0

$\alpha(0)$ very near π and $\beta(0)$ very near 0

$\alpha(0)$ and $\beta(0)$ very near $\frac{\pi}{2}$

- Want to show that α becomes less than $\frac{\pi}{2}$ before $\beta = 0$ (Case 1) or $x = 0$ (Case 2)

Case 1: Phase Plane Portrait

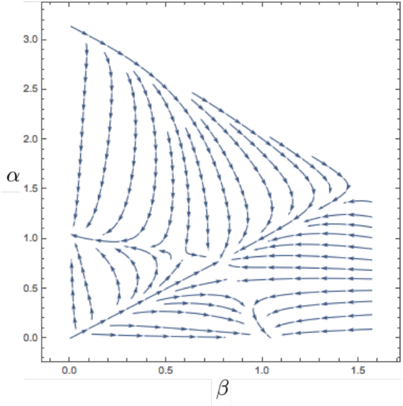
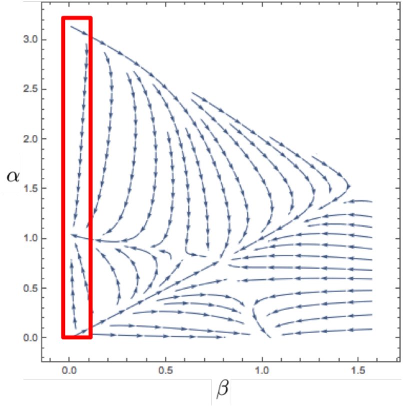


Figure: α vs β

Case 1: Phase Plane Portrait



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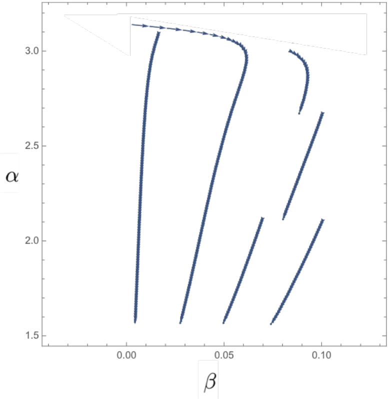

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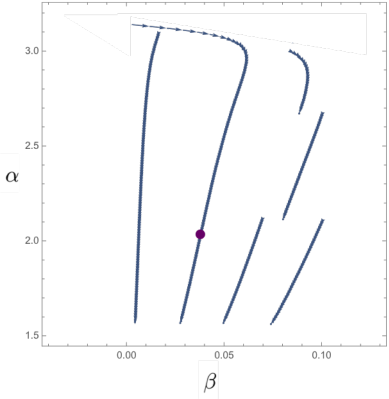

Figure: α vs β

Case 1: Phase Plane Portrait

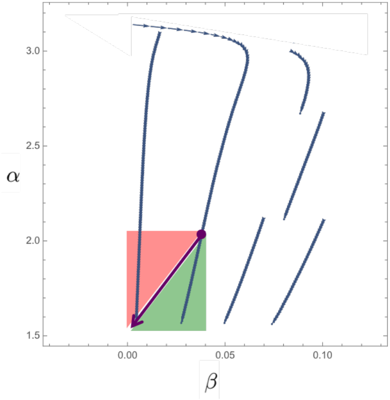

Figure: α vs β

Case 2: A similar approach

- Similar reasoning
- Use PPP of α vs x
- Algebraic manipulation yields proof

Generalization

The Differential Equations

Three differential equations dictate the evolution of the points:

$$\frac{dx}{dt} = 2\alpha + 2\beta - 2\pi$$

$$\frac{d\alpha}{dt} = -\frac{((\alpha+\beta)\cos(\frac{\alpha+\beta}{2}) - (-2\alpha+\pi)\sin(\alpha))\sin(\alpha+\beta)}{x\sin\beta}$$

$$\frac{d\beta}{dt} = -\frac{((\alpha+\beta)\cos(\frac{\alpha+\beta}{2}) - (-2\beta+\pi)\sin(\beta))\sin(\alpha+\beta)}{x\sin\alpha}$$

- What features of the equations make the result true?

Next Steps

- Does the geometry dictate the singular behavior of the derivatives when the figure is about to collapse?
- Will analogous dependencies hold for all quadrilaterals, implying that every quadrilateral will become convex?

Acknowledgements

- Ao Sun, Project Mentor
- PRIMES Program