Braid Groups

4.8 Center of the Braid Group

4.8.1 Theorem

The center of the braidgroup B_n is the cyclic group generated by τ^n where $\tau = \sigma_1\sigma_2\sigma_3...\sigma_{n-1}$.

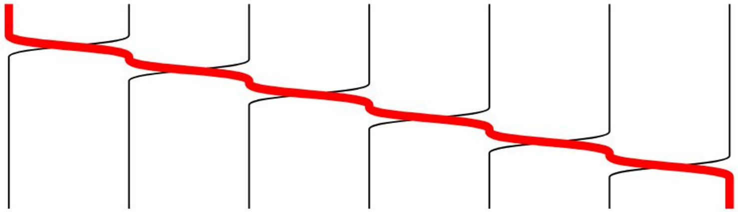


Figure 14: τ

τ^n is equivalent to the full twist on n strands.

If τ^n commutes with every generator, it is in the center of B_n.

First, when $i > 1$

$$\begin{aligned}
\sigma_i \tau &= \sigma_i \sigma_1 \sigma_2 \sigma_3 \ldots \sigma_{i-1} \sigma_i \ldots \sigma_{n-1} \\
&= \sigma_1 \sigma_2 \sigma_3 \ldots \sigma_i \sigma_{i-1} \sigma_i \ldots \sigma_{n-1} \\
&= \sigma_1 \sigma_2 \sigma_3 \ldots \sigma_{i-1} \sigma_i \sigma_{i-1} \ldots \sigma_{n-1} \\
&= \sigma_1 \sigma_2 \sigma_3 \ldots \sigma_{i-1} \sigma_i \ldots \sigma_{n-1} \sigma_{i-1} \\
&= \tau \sigma_{i-1}
\end{aligned} \tag{5}$$

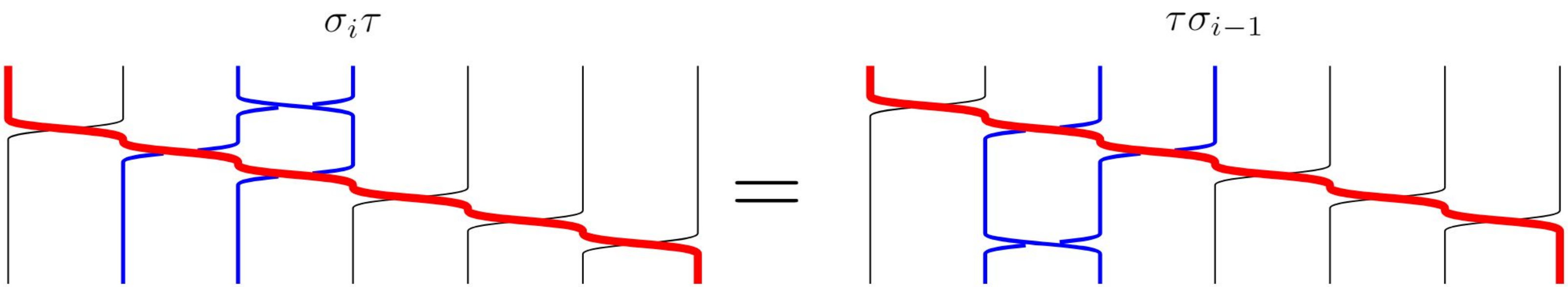
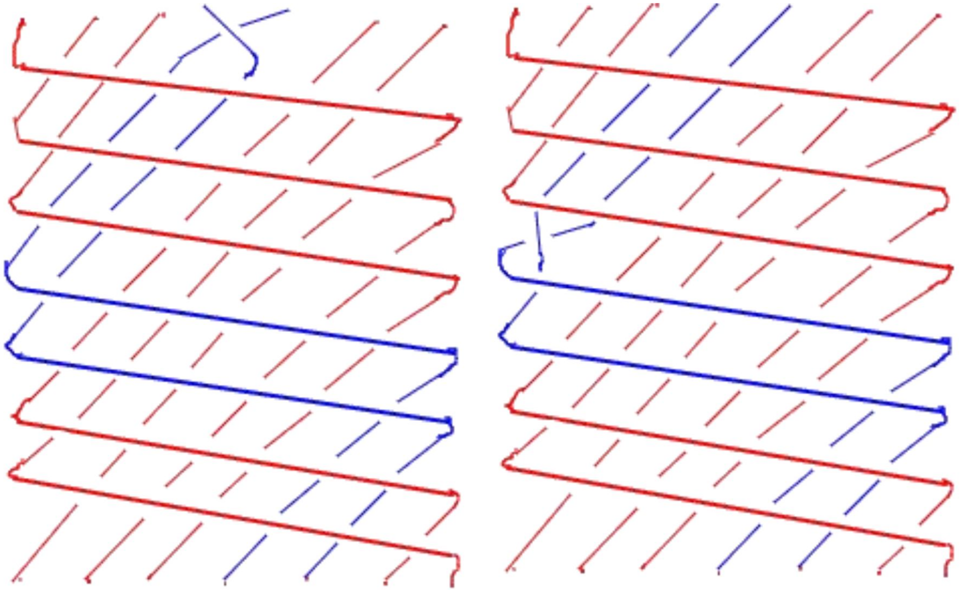


Figure 15: This has a visual effect of σ_i sliding past τ and moving to the left to become σ_{i-1}.

When we continue this "sliding past" process, we get the following:

$$\begin{aligned}\sigma_i\tau^n &= \tau \quad \sigma_{i-1} \quad \tau^{n-1} = \tau^2 \quad \sigma_{i-2} \quad \tau^{n-2} \\ &\dots \\ &= \tau^{i-1} \quad \sigma_1 \quad \tau^{n-i+1}\end{aligned} \tag{6}$$



(a) $\sigma_i\tau^n$

(b) $\tau^{i-1} \quad \sigma_1 \quad \tau^{n-i+1}$

Figure 16: equivalent braids that illustrate eq (6)