Algebraicity regarding Graphs and Tilings

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Abstract

Given a planar graph G, we prove that there exists a tiling of a rectangle by squares such that each square corresponds to a face of the graph and the side lengths of the squares solve an extremal problem on the graph. Furthermore, we provide a practical algorithm for calculating the side lengths. Finally, we strengthen our theorem by restricting the centers and side lengths of the squares to algebraic numbers and explore the application of our technique in proving algebraicity in packing problems.

1 Introduction

The art of tiling is thousands of years old, but it still remains relevant. Ancient builders used tilings to create intricate floor patterns and complex mosaics that complement the aesthetics of buildings. Tilings are not only created, but also found in the natural world. Regular patterns similar to tilings have been found in molecular structures [12].

A tiling problem asks us to cover a given region with a set of tiles, completely and with no overlap. Furthermore, each tile in the set is similar to some base tile, most commonly a square. Although tiling problems seem to be solely geometric, these problems have implications for various fields such as Complexity Theory [2, 10], Mathematical Logic [14], and Molecular Chemistry [12]. Given these implications, a method to encode the tilings into an easily analyzable "blueprint" becomes necessary.

The most common "blueprint" for a tiling is its contact graph. Each vertex of a contact graph corresponds to a square in the tiling, and two vertices are connected by an edge if their corresponding squares share a vertex or an edge.

Creating a blueprint from a tiling is simple, but how do we reverse the process and create a tiling from a "blueprint"? In Circle Packing (where the region is not covered completely), the famous Koebe-Andreev-Thurston Theorem states that for every connected simple planar graph G, there exists a circle packing whose contact graph is G [13, 1]. O. Schramm explored the analog of the theorem in square tilings and discovered a similar theorem using conformal maps [16]. Furthermore, he proved that the square tiling solves an extremal problem on the contact graph. The use of conformal maps to solve extremal problems has been further explored by M. Bonks [3].

The contact graph is not the only "blueprint" for a tiling. Brooks, Smith, Stone and Tutte [5] constructed graphs where the vertices correspond to horizontal edges of the square tiling, and two vertices are connected if there is a square with one of them as the top edge and the other as the bottom edge. Then, borrowing Kirchoff's loop rule and junction rule from electrical networks, they constructed tilings where all of the squares have different side lengths. In this paper, we explore another type of "blueprint", a planar embedding whose faces correspond to squares in the tiling and the side lengths correspond to the oscillation of each face. As shown by projects in the past years, the oscillation function is closely related to tiling harmonic functions [18] and electrical networks [7].

2 Extremal Problem

We introduce our extremal problem on a finite planar graph with technicality.

Definition (Graphs and boundaries)**.** Let $G = (V, E)$ be a planar, finite, connected graph. Embed the graph in the complex plane $\mathbb{C}$.

The set of **faces** F is the set of closed, bounded, connected components of $\mathbb{C} \setminus G$.

The set of **boundary vertices** dV of G is the set of vertices that belong to the unbounded connected component U^* of $\mathbb{C} \setminus G$. Furthermore, when we write $v \in f$ for some face f, we are considering the vertices both on the boundary and in the interior of f.

Let A, B be disjoint, connected subsets of dV. The **weight function** $w : V \to [0, \infty)$ is a nonnegative function that satisfies $w(v) = 1$ for all $v \in A$, and $w(v) = 0$ for all $v \in B$. We call set A the top vertices, and set B the bottom vertices.

We now introduce the oscillation function on the planar graph and the quantity we are hoping to minimize. The oscillation is a discrete version of the upper gradient $\oint dw$ from analysis. Specifically, it is the lower bound to the upper gradient of a particle traveling from $w_{\min}(v)$ to $w_{\max}(v)$ through any path.

Definition (Oscillation)**.** The **oscillation** of a face $f \in F$ is defined as

$$\mathrm{osc}_w(f) = \max(w(v) : v \in f) - \min(w(v) : v \in f).$$

Finally, the **area** of the weight w is defined as

$$\mathrm{Area}_w = \sum_{f \in F} \mathrm{osc}_w(f)^2,$$

and the **extremal area** of $(G; A, B)$ is defined as

$$\mathrm{Area}(G; A, B) = \inf(\mathrm{Area}_w),$$

where the infimum is taken over all weights w.

We show that for any $(G; A, B)$ defined as above, this extremal area can be achieved by some weight function.

Lemma 2.1. *If $w(v) > 1$ for some $v \in V$, then there exists weight w' such that $A_{w'} < A_w$.*

Proof. Let $S \subset V$ be the set of vertices with the maximum weight.

Let ϵ be a positive real number such that for $v \in S$, $w(v) - \epsilon > 1$ and $w(v) - \epsilon$ remains the largest weight in the graph.

Consider the weight function w' such that $w'(v) = w(v) - \epsilon$ for $v \in S$ and $w'(v) = w(v)$ for $v \notin S$. Since $w(v) > 1$, we have not changed the weight of vertices on the "top and bottom edge". Furthermore, the oscillation on faces containing vertices $v \in S$ is necessarily smaller. Therefore, w' is a valid weight function with a smaller area. $\square$

Theorem 2.2. *Define $(G; A, B)$ as above. There exists weight w such that* $\text{Area}_w = \text{Area}(G; A, B)$.

Proof. By Lemma 2.1, all weight functions with $w(v) > 1$ do not produce minimum area. Therefore, minimum area must be produced when $w(v) \in [0, 1]$ for all vertices v.

The set of such weights w is a compact space. By applying the Extreme Value Theorem on A_w, the area function A must attain its minimum. The weight function where A attains the minimum is the desired weight function. $\square$

Therefore, we define the **extremal weight** as the weight function that achieves the minimum.

We are now ready to state our main theorem.

Theorem 2.3. *Define $(G; A, B)$ and F as above, and let w be the extremal weight. There exists a square tiling T_f, indexed by F, of rectangle $R = [0, l] \times [0, 1]$ such that T_f and T_g are in contact for all pairs of faces $f, g \in F$ in contact, and T_f has side length* $\text{osc}_{w,f}$ *for all $f \in F$.*

The theorem will be proven at the end of Section 3.

Figure 1 is an example of an extremal weight function and its corresponding tilling. The red vertices are the top vertices A, and the blue vertices are the bottom vertices B. The numbers on the left are the weights on each vertex, and the oscillations on the faces are, from left to right, $1, \frac{1}{3}, \frac{1}{3}, \frac{1}{3}, 1$. Using these oscillations, we find the tiling on the right, consisting of two 1×1 squares and three $\frac{1}{3} \times \frac{1}{3}$ squares.

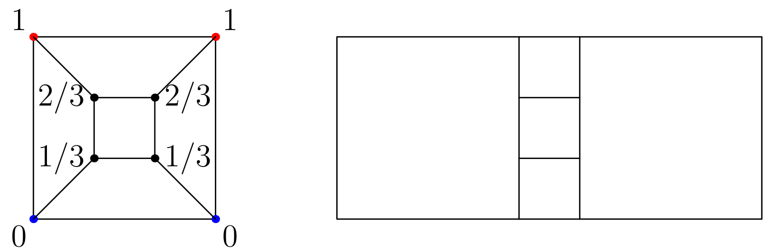

Figure 1: Extremal problem for a simple graph and its corresponding tiling

3 Duality

The extremal problem we consider is, loosely speaking, a dual of the extremal problem proposed by Schramm [16].

Let $(G; A, B)$ be defined as above, and consider its planar embedding as a tiling of $\mathbb{C} \setminus U^*$ with polygons. Then, consider the contact graph G' of this tiling. Furthermore, let A' be the faces containing vertex $v \in A$, and B' be the faces containing a vertex $v \in B$. In figure 2, the graph on the right is the contact graph of that on the left.

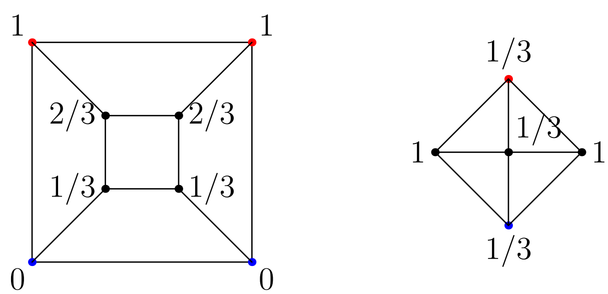

Figure 2: The dual of the graph in Figure 1

Then, we find that our problem on $(G; A, B)$ is similar to the extremal problem Schramm considered on $(G'; A', B')$. However, there are still major differences in our problems. In particular, Schramm's metric can assign any real value to osc_f, while our values of osc_f are evaluated from the vertices. Therefore, our problem is more restrictive than the one Schramm considers.

Our extremal problem is a non-trivial addition to Schramm's work because the formulation with oscillations demonstrates that square tilings minimize important functions in analysis, such as tiling-harmonic functions [18]. Furthermore, square tilings themselves may be considered as planar graphs. Elegantly, when we solve the extremal problem on such a planar graph, the corresponding tiling is the original square tiling.

Schramm [16] built additional structure on the planar graph to discover a relationship between square tilings and his extremal problem.

Definition. Let $G = (V, E)$ be a planar graph, with top vertices A and bottom vertices B. Then, for a weight function $w : V \to [0, \infty)$, let the norm of the weight $||w||$ be the maximum sum of weights $w(v)$ for any path from the bottom edge to the top edge, and the normalized area of the weight is defined as

$$\text{Area}_w = \sum_{v \in V} \frac{w(v)^2}{||w||^2}.$$

Theorem 3.1 (Schramm's tiling theorem [16]). *Let w be the weight function such that $Area_w$ is minimal. Then, there exists a square tiling T_v, indexed by V, of rectangle $R = [0, l] \times [0, ||w||]$ such that T_u and T_v are in contact for all connected vertices $u, v \in V$, and T_v has side length $w(v)$ for all $v \in V$.*

Schramm's theorem provides a new perspective that allows us to prove Theorem 2.3. We will first establish the many apparent similarities between Theorem 2.3 and Schramm's tiling theorem. Then, we will generate a tiling using Schramm's tiling theorem and assign weights based on the tiling. From this perspective, we are able to exploit the geometry of the tilings and the apparent similarities to solve the original extremal problem. Finally, we will show that the y-coordinates of the vertices in the tiling correspond to the extremal weights of the planar graph.

Proof of Theorem 2.3. Let $G = (V, E)$ be a planar graph. Consider the contact graph $G' = (V', E')$ of G. Let the top vertices A' be the set of vertices whose corresponding faces in G contain vertices in A, and the top vertices B' be the set of vertices whose corresponding faces in G contain vertices in B.

By Schramm's tiling theorem, there is a weight function $w'(v')$ that solves Schramm's extremal problem on graph G'.

Notice that the area function is invariant when we multiply all values of $w'(v')$ by a scalar. Therefore, by dividing w' by $||w'||$, we can assert without loss of generality that $||w'|| = 1$. By Schramm's tiling theorem, this weight function corresponds to some square tiling $T_{v'}$ of a rectangle $R : [0, l] \times [0, 1]$.

We assign weights $w(v)$ to our original graph using the y-coordinate of a point on the square tiling, with the following five cases:

1. If a vertex v is inside a face, use any point inside the corresponding square.
2. If the vertex v is between two faces, use any point along the edge where the two corresponding squares are connected. However, if v is a top vertex, use the highest point among the points where the squares are connected. Similarly, if v is a bottom vertex, use the lowest points among the points where the squares are connected.
3. If the vertex v is between three faces, use the point where the three corresponding squares meet.
4. If the vertex v is between four faces, then the four corresponding squares are all mutually in contact. Thus, either the corresponding squares meet at one point or one or more of the squares are degenerate. In the first case, use the point where the four squares meet; in the second case, ignore the degenerate squares and apply the first three cases.
5. If the vertex v is between five or more faces, then one or more of the corresponding squares must be degenerate. Ignore the degenerate squares and apply the first four cases.

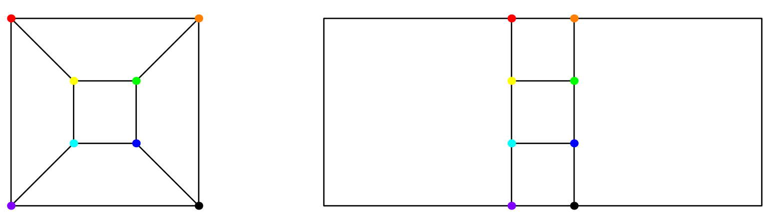

Figure 3: The vertices are color-coded to indicate which point is used.

In figure 3, each vertex of the original graph is assigned to a point on the tiling based on the rules above. We claim that the weight function w created solves the extremal problem on the original graph.

First, we show that the top vertices are assigned 1. All the faces containing a top vertex correspond to top vertices in the contact graph. Thus, their corresponding squares have their top edges at $y = 1$. Since a top vertex is along the boundary, it must fall in cases $2-5$. In all 4 cases, the top vertex is assigned a weight of 1.

Similarly, the bottom vertices are assigned a weight of 0.

Then, we show that the oscillations are equal to the weights w' on the contact graph. Notice that the vertices on the top edge of a square are connected to additional squares above or are the highest vertices; the vertices on the bottom edge are connected to additional squares below or are the lowest vertices. Therefore, at least one vertex on the face is assigned to the top edge of the corresponding square and at least one is assigned to the bottom edge. Thus, $\mathrm{osc}(f) = \max(w(v) : v \in f) - \min(w(v) : v \in f) = y_{\max} - y_{\min}$, which is the side length of the square. By Schramm's Theorem, $\mathrm{osc}_w(f) = w'(v')$.

Furthermore, the area Schramm considers is equal to

$$\sum_{v' \in V'} \frac{w'(v')^2}{||w'||^2} = \sum_{f \in F} \mathrm{osc}(f)^2,$$

which is the area we consider.

Finally, our problem involves assigning oscillations based on weights around the boundary. This restricts the value of oscillations we can assign. Since Schramm's theorem gives the extremal values when we assign oscillations directly, the weights are also extremal in the restricted case.

Thus, we have constructed a square tiling such that each square corresponds to a vertex on the contact graph — a face on the original graph. Furthermore, we have constructed a weight function that solves our extremal problem and corresponds to the square tiling. Our proof by construction is complete. □

4 Calculating Extremal Weights

Because of the similarity between our extremal problem and Schramm's as described in Section 3, we are also able to apply previous results by Schramm to calculate our extremal weight function.

Algorithm. Let $G = (V, E), (G; A, B)$ be defined above. Suppose we want to calculate the extremal weights with an error of $\pm\epsilon$.

1. Find the contact graph $G' = (V', E')$, and the set of faces A', B' as described in Section 3.

2. Apply Schramm's algorithm in [16] on $(G'; A', B')$ to find the extremal weights $w'(v')$ for all $v' \in V'$ with an error of $\pm \frac{\epsilon}{|V|}$.

3. Assign weights $w(v)$ to the original vertices using the five cases described in the proof of Theorem 2.3. This is the extremal weight function.

5 Algebraicity of Tilings

Although our algorithm is able to find an increasingly accurate approximation of the weights, it does not compute the exact value. Moreover, the extremal problems associated with this paper and both the paper of Schramm [15] and Bonk [3] are examples of mathematical optimization. In areas of mathematical optimization, such as Linear Programming, the algorithms used to solve the optimization problem, such as the Ellipsoid Method, only produce an approximation. Is an algorithm that finds the exact value impossible?

Probably not. In this section we prove that the optimal weights are algebraic. An algebraic number, solution of some polynomial

$$\frac{p_n}{q_n}x^n + \frac{p_{n-1}}{q_{n-1}}x^{n-1} + \ldots + \frac{p_0}{q_0},$$

can be stored by a computer as the array $[p_n, q_n, \ldots, p_0, q_0]$. If the solution to the extremal problem is algebraic, it can be expressed by a computer, intuitively suggesting that an exact algorithm likely exists.

To prove Algebraicity of the side lengths, we apply a theorem by Alfred Tarski regarding real closed fields.

Theorem 5.1 (Tarski 1951 [14, 17]). *Let S be a first-order statement in the theory of real-closed fields. If S is true in one real-closed field, then it is true in every real-closed field.*

Since both the real numbers and the algebraic numbers are real-closed fields, if the square-tiling problem we solved can be expressed in first-order language and proved in $\mathbb{R}$, it is also true for the algebraic numbers, $\mathbb{R} \cap \overline{\mathbb{Q}}$.

Our plan is to translate the statements of Schramm's theorem into appropriately constructed first-order statements and apply Tarski's theorem to obtain the truth of the theorem over algebraic numbers. We need to be careful with the operators provided by a first-order statement in the theory of real-closed fields.

Roughly speaking, we may use the usual logic operators and quantifiers so that all variables in the statement are quantified, symbols from set theory, as well as symbols from the theory of real-closed fields $(1, 0, \times, +, =, >)$. In particular, we can only quantify over variables, not relations or sets.

With these tools, we are able to prove the algebraicity of our theorem. Notice that by the proof of 3.2, the solution of the extremal problem follows from the creation of the square tiling. Thus, we strengthen all of theorem 3.2 by proving that a tiling with algebraic coordinates exists.

Theorem 5.2. *Given a simple planar graph $G = (V, E)$, there exists a square tiling with contacts graph G, whose centers and side lengths are algebraic numbers.*

Proof. We write the square tiling problem in first-order language as an elementary statement. The lack of many operators make the language difficult to use, so we will rewrite common operators in term of ones that are allowed.

We rewrite the statement $a < b$ as $b > a$, the statement $a \leq b$ as $\neg(a > b)$, and the statement $a \geq b$ as $\neg(b > a)$.

For constant positive integer powers, we rewrite x^n as $x \times x \times \ldots \times x$, where x is repeated n times.

For $i \in V$, let x_i, y_i represent the x- and y-coordinate of the center of square i; let a_i represent the side length of square i. For $i, j \in V$, we define the following statement:

- Let $E_{i,j}$ be the statement $(4(x_j - x_i)^2 = (a_i + a_j)^2 \cap 4(y_j - y_i)^2 \leq (a_i + a_j)^2) \cup (4(y_j - y_i)^2 = (a_i + a_j)^2 \cap 4(x_j - x_i)^2 \leq (a_i + a_j)^2)$;
- Let $N_{i,j}$ be the statement $4(x_j - x_i)^2 > (a_i + a_j)^2 \cup 4(y_j - y_i)^2 > (a_i + a_j)^2$;
- Let I_i be the statement $2x_i \geq a_i \cap 2y_i \geq a_i \cap 2x_i \leq 2w - a_i \cap 2y_i \leq 2 - a_i$;
- Let C_i be the statement $(4(x - x_i)^2 \leq (a_i)^2) \cap (4(y - y_i)^2 \leq (a_i)^2)$.

The statements $E_{i,j}$ ensure that the squares required to be adjacent by the original theorem are indeed adjacent. The first clause checks if the squares are stacked beside each other, and the second one checks if the squares are stacked on top of each other.

The statements $N_{i,j}$ ensure that the squares that cannot be adjacent according to the theorem are not adjacent. The first clause checks if the squares are too far away horizontally, and the second one checks if they are too far away vertically.

The statements I_i ensure that every square is contained in the rectangle.

The statements $C_{i,x,y}$ ensure that every point in R is covered by some squares.

Then, for each pair of (V, E), our theorem can be stated as

$$\exists w, x_1, \ldots, x_{|V|}, y_1, \ldots, y_{|V|}, a_1, \ldots, a_{|V|} \in \mathbb{R}, (\bigwedge_{(i,j)\in E} E_{i,j})$$

$$\cap(\bigwedge_{(i,j)\notin E} N_{i,j}) \cap (\forall x, y(x > w \cup x < 0) \cup (y > 1 \cup y < 0) \rightarrow I_1 \cap \ldots \cap I_{|V|})$$

$$\cap(\forall x, y(x \leq w \cap x \geq 0) \cap (y \leq 1 \cap y \geq 0) \rightarrow C_1 \cup C_2 \cup \ldots \cup C_n).$$

In the statement, $\forall(i, j) \in E$ is an abbreviation for enumerating all edges of E; $\forall(i, j) \notin E$ is an abbreviation for enumerating over all vertices v_i, v_j of V that do not share an edge.

Since we have proven that the theorem is a true first-order statement in $\mathbb{R}$, it is also true in $\mathbb{R} \cap \overline{\mathbb{Q}}$. Having found an algebraic tiling, we assign weights using the procedure in the proof of Theorem 2.3. $\square$

Tarski's theorem is difficult to apply to square tilings because the square is not smooth. As a result, statements that define whether two squares are in contact, $E_{i,j}$ and $N_{i,j}$, are especially lengthy. With smooth objects such as circles, the application is much easier.

Brooks et. al were able to argue in [5] that the side lengths of square tilings are not only algebraic but also rational. However, we approach the problem from a different field than Brooks et. al. They instead built a connection between tilings and electrical networks. Moreover, our approach can be applied to other theorems in the field of packings and tilings.

6 Further generalizations regarding algebraicity

Tarski's theorem can be further applied to theorems in packing problems. First, we generalize circle packing theorem to circles in 2-dimensional L^p space, a function space with a different definition of norm.

Definition (Circles in L^p space)**.** Let $p > 0$. A *circle in L^p space* with center (a, b) and radius r is the set of points (x, y) such that $(x - a)^p + (y - b)^p = r^p$.

We will apply the following theorem by M. Brandt and A. Harrington to prove results regarding circles in L^p space.

Theorem 6.1 ([4, 11])**.** *Let $G = (V, E)$ be a finite planar graph and let K be a region in $\mathbb{R}$ which is homeomorphic to the closed unit disk and whose boundary is C^1 smooth. Then there is a packing $P = (K_v, v \in V)$ such that G is the contact graph of P up to isomorphism, and K_v is obtained from K through translation and scaling.*

If we can prove that disks in L^p (the region bounded by a circle in L^p) are smooth and homeomorphic to the unit disk, we can prove that the circle packing theorem can be generalized to circles in L^p space.

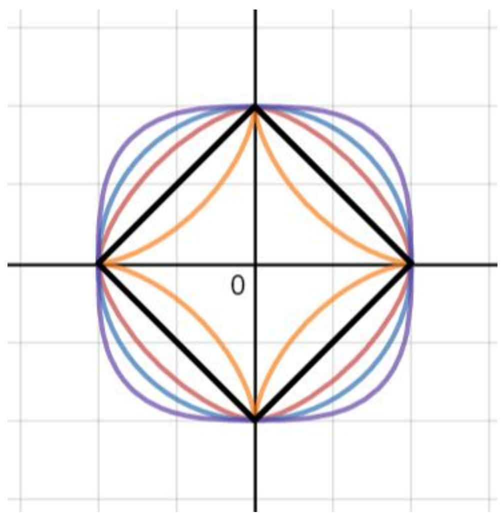

Figure 4: Unit circle in $L^{2/3}, L^1, L^{3/2}, L^2, L^3$.

We notice in figure 4 that Euclidean circles and squares are circles in L^2 and L^1, respectively. Thus, proving a theorem about circles in L^p space is a natural generalization of the circle packing theorem and the Schramm's tiling theorem. Furthermore, we notice in figure 4 that when $p \leq 1$ the circle is not smooth. Therefore, we can only hope for smoothness when $p > 1$.

Theorem 6.2. *Let $G = (V, E)$ be a finite planar graph and let $p > 1$. Then there is a packing $P = (K_v, v \in V)$ of circles in L^p space such that G is the contact graph of P.*

Proof. Since smoothness does not change when an object is scaled or translated, we will examine the circle centered at $(0, 0)$ with radius 1. Such a circle has formula

$$x^p + y^p = 1.$$

Implicitly differentiating in the first quadrant, we find

$$px^{p-1} + py^{p-1}\frac{dy}{dx} = 0, \frac{dy}{dx} = \left(\frac{x}{y}\right)^{p-1}.$$

The derivative $\left(\frac{x}{y}\right)^{p-1}$ is continuous when $x, y > 0$. Furthermore, since $p-1 > 0$, there is a horizontal tangent at $(x, y) = (0, 1)$ since $\frac{dy}{dx} = 0$ and a vertical tangent at $(1, 0)$ since $\frac{dy}{dx}$ approaches ∞.

Therefore, a circle in L^p is smooth when $p > 1$.

We prove that the disk is homeomorphic to the unit disk by constructing an explicit homeomorphism.

Let f from the unit disk to the L^2 disk be the mapping such that for every point (x, mx) on the unit disk, f scales (x, y) by $\frac{(m^p+1)^{1/p}}{(m^2+1)^{1/2}}$. For points $(0, y)$ on the unit disk, f is the identity mapping.

The inverse of f the mapping that scales (x, mx) by $\frac{(m^2+1)^{1/2}}{(m^p+1)^{1/p}}$. Furthermore, the scaling factor is continuous when m changes, and approaches 1 as m approaches infinity. Therefore, both f and f^{-1} are continuous maps, the disk in L^p is homeomorphic to the closed unit disk. □

Louder et. al proved that the circle packing theorem remains true when we restrict the centers and the radii to algebraic numbers [14]. A similar generalization exists for our theorem.

Theorem 6.3. *Let $G = (V, E)$ be a finite planar graph and let $p > 1$ be a rational number with an even numerator. Then there is a packing $P = (K_v, v \in V)$ of circles in L^p space such that G is the contact graph of P and the center and radii of every circle K_v is algebraic.*

Proof. We rewrite positive integer powers x^n as we did in Theorem 5.2, and rewrite positive rational powers $x^{\frac{p}{q}}$ by asserting $\exists y\ y^q = x$ and replacing all appearance of $x^{\frac{p}{q}}$ with y^p.

We define the following statements:

- Let R_i be the statement $r_i > 0$. This ensures that the radii are positive.
- Let $E_{i,j}$ be the statement $(x_i - x_j)^p + (y_i - y_j)^p = (r_i + r_j)^p$.
- Let $N_{i,j}$ be the statement $(x_i - x_j)^p + (y_i - y_j)^p > (r_i + r_j)^p$.

Statements R_i ensure that the radii are positive. $E_{i,j}$ states that two disks K_i and K_j are tangent to each other: When the condition is satisfied, the point $(\frac{r_i x_j + r_j x_i}{r_i + r_j}, \frac{r_i y_j + r_j y_i}{r_i + r_j})$, a "weighted average" of the two centers, lie on both circles and is the point of tangency. Finally, since the shapes are convex, that is the only point of contact. Similarly, $N_{i,j}$ states that two disks K_i and K_j are not in contact.

Then, our theorem is the statement

$$S = R_1 \cap R_2 \cap \ldots \cap R_n \cap (\bigwedge_{(i,j) \in E} E_{i,j}) \cap (\bigwedge_{(i,j) \notin E} N_{i,j}).$$

Since S is true in the language of real numbers, by Tarski's theorem it is true in the language of algebraic numbers. $\square$

In the case where p is a transcendental number, it is easy to see that the theorem is not true. The case when p is algebraic but irrational remains neither proven or disproven. This case may be explored further in future research.

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