An Index-Type Invariant of Knot Diagrams and Bounds for Unknotting Framed Knots

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Abstract

We introduce a new knot diagram invariant called *self-crossing index*, or SCI. We found that SCI changes by at most ± 1 under framed Reidemeister moves, and specifically provides a lower bound for the number of $\Omega 3$ moves. We also found that SCI is additive under connected sums, and is a Vassiliev invariant of order 1. We also conduct similar calculations with Hass and Nowik's diagram invariant and cowrithe, and present a relationship between forward/backward, ascending/descending, and positive/negative $\Omega 3$ moves.

1 Introduction

A knot is an embedding of the circle in the three-dimensional space without self-intersection. Knot theory, the study of these knots, has applications in multiple fields. DNA often appears in the form of a ring, which can be further knotted into more complex knots. Furthermore, the enzyme topoisomerase facilitates a process in which a part of the DNA can temporarily break along the phosphate backbone, physically change, and then be resealed [13], allowing for the regulation of DNA supercoiling, important in DNA replication for both growing fork movement and in untangling chromosomes after replication [12]. Knot theory has also been used to determine whether a molecule is chiral or not [18]. In addition, knot theory has been found to have connections to mathematical models of statistic mechanics involving the partition function [13], as well as with quantum field theory [23] and string theory [14].

Knot diagrams are particularly of interest because we can construct and calculate knot invariants and thereby determine if knots are equivalent to each other. While this may seem easy to do, knots are not always trivially equivalent, especially since knots can become increasingly complex and since it might be necessary to further complicate the knot before they can reach the desired state (if at all possible), like in the case of "hard unknot diagrams" [7].

In this paper, we introduce a new knot diagram invariant called *self-crossing index*, or SCI. We calculate the behavior of SCI under Reidemeister moves for knots and framed knots, as well as several other properties of the diagram invariant. Three of the main theorems we will prove are as follows:

Theorem 1.1. *For framed Reidemeister moves, the diagram invariant* SCI *only changes under framed Reidemeister Type III moves, where each move changes* SCI *by* ± 1. *Specifically,* SCI *increases by* 1 *under forward* $\Omega 3$ *moves and decreases by* 1 *under backward* $\Omega 3$ *moves.*

Theorem 1.2. *The diagram* D_n *in Figure 14, which has* $8n - 2$ *crossings, satisfies*

$$\frac{1}{2}\left(3n^2 - n + 2\right) \leq d_{fr}(U, D_n)$$

Theorem 1.3. *The family of unknot diagrams L_n can be unknotted in n to $2n$ moves as an unknot diagram, and requires at least $\frac{n(n+1)}{2}$ moves to unknot as framed unknot diagrams. In other words,*

$$n \leq d(U, L_n) \leq 2n$$

and

$$\frac{n(n+1)}{2} \leq d_{fr}(U, L_n).$$

We also conduct similar calculations and determine properties for Hass-Nowik diagram invariant with linking number and cowrithe, and present a relationship between forward-backward, ascending-descending, and positive-negative $\Omega 3$ moves.

2 Definitions

Because three-dimensional objects are generally difficult to imagine and manipulate on paper, a knot in $\mathbb{R}^3$ or S^3 is represented by a knot diagram, a projection of the knot onto a 2D plane or a sphere, S^2. This projection may have places of self-intersection between exactly two points of the knot, known as *crossings*, and a break is made in one of the two strands to indicate which strand is above which in the actual knot. At a given crossing, the part of the diagram on top is the *overstrand*, and the part below is the *understrand*.

It is easy to see that multiple knot diagrams can represent the same knot.

Definition 2.1 (equivalent knot diagrams)**.** Two knot diagrams D and E are considered *equivalent* if they represent the same knot.

We can produce equivalent knot diagrams using *planar isotopies*, deformations of a projection that preserve the number and relative locations of the crossings, and *Reidemeister moves*, shown in Figure 1. In fact, Reidemeister proved the following well-known theorem about Reidemeister moves:

Theorem 2.2 (Reidemeister)**.** *Diagrams D and E are equivalent if and only if D and E are connected by a sequence of Reidemeister moves.*

For simplicity and concision, we will refer to Reidemeister moves of Type I, II, and III as $\Omega 1$, $\Omega 2$, and $\Omega 3$ moves in this paper, respectively.

Definition 2.3 (orientation)**.** Given a knot diagram D, we can assign a direction or *orientation*, denoted by an arrow on the curve, that is consistent throughout the entire diagram.

Due to the orientation, each of the three Reidemeister moves has several different cases, seen in Figures 2-4.

Also, we can define a *forward* oriented Reidemeister move for all three types. Forward $\Omega 1$ moves create a loop and extra crossing. Forward $\Omega 2$ moves overlap two strands, creating two new crossings. For forward $\Omega 3$ moves, we use the definition given by Suwara [19].

Definition 2.4 (forward and backward $\Omega 3$ move)**.** Let the *vanishing triangle* be formed by the three crossings involved in the $\Omega 3$ move. Assign an orientation

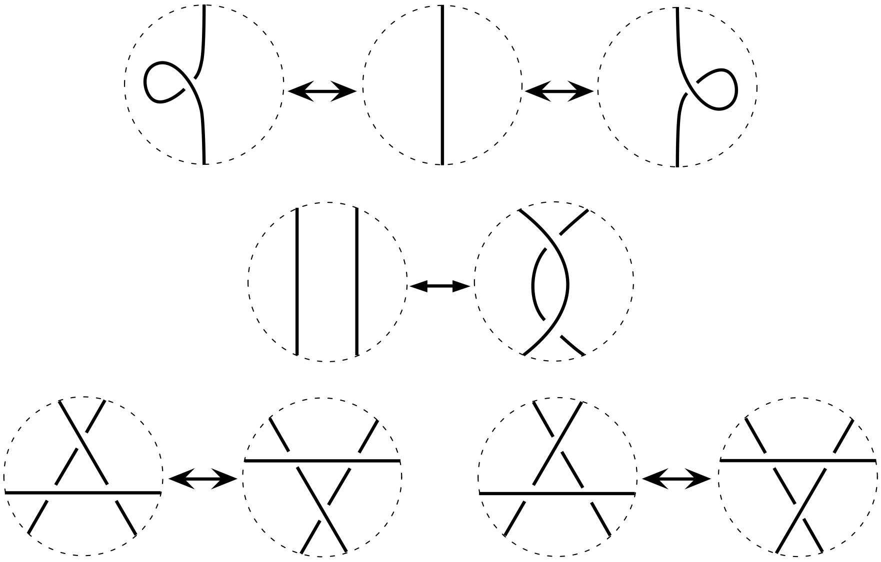


Figure 1: Reidemeister moves of Type I, II, and III (top to bottom).

to the vanishing triangle so that the direction cycles the strands in a cyclic top-middle-bottom order. Let m be the number of sides of the vanishing triangle whose orientation agrees with the orientation of the corresponding segment on the knot diagram, and let $q = (-1)^m$. Any $\Omega 3$ move changes m by ± 1 or ± 3, so q changes under any $\Omega 3$ move. In particular, a *forward* $\Omega 3$ moves changes the value of q from $+1$ to -1. A *backward* oriented Reidemeister move causes the opposite of whatever the forward move does.

Note that the left-to-right Reidemeister moves in Figures 2-4 are the forward moves.

One of the main interests of knot theory is determining if two knot projections represent the same knot. Despite decades of research, mathematicians still do not have an algorithmic method to classify knot diagrams according to equivalence. Thus, being able to find tighter bounds has implications on developing an algorithm to find a series of Reidemeister moves between two diagrams (or the nonexistence of one), and ultimately to determine whether two diagrams are equivalent. A simplified version of this question is how many moves it takes to "unknot" a nontrivial unknot projection, where the trivial unknot projection is simply a circle.

Definition 2.5 (unknotting)**.** Given a projection of the unknot, the process of *unknotting* refers to a sequence of Reidemeister moves that changes the original projection into the trivial projection.

Lackenby [10] showed that any projection of the unknot requires at most $(236n)^{11}$ Reidemeister moves to unknot.

Hass and Nowik [8] introduced a knot diagram invariant involving smoothing (Definition 3.3) and linking number, and subsequently provided an infinite family of unknots with $7n - 1$ crossings that had a quadratic lower bound in n for the minimal number of Reidemeister moves needed to reach the trivial

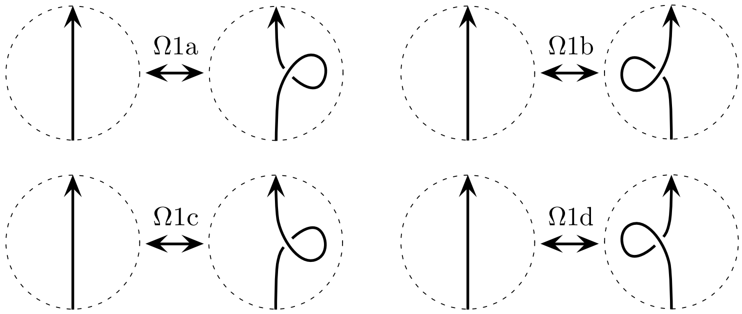


Figure 2: Right (a, c) and left (b, d) oriented Reidemeister moves of type I.

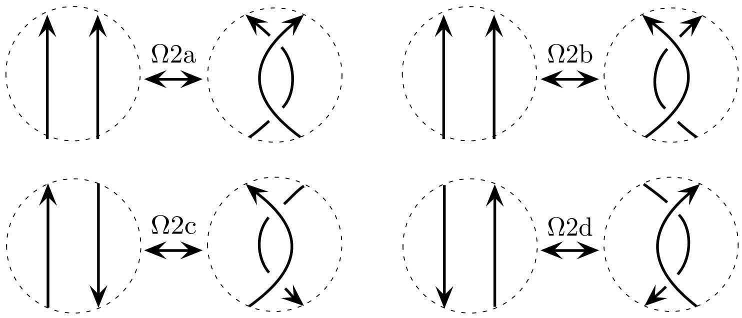


Figure 3: (a-b) Matched (Ω2m) and (c-d) unmatched (Ω2u) oriented Reidemeister moves of type II.

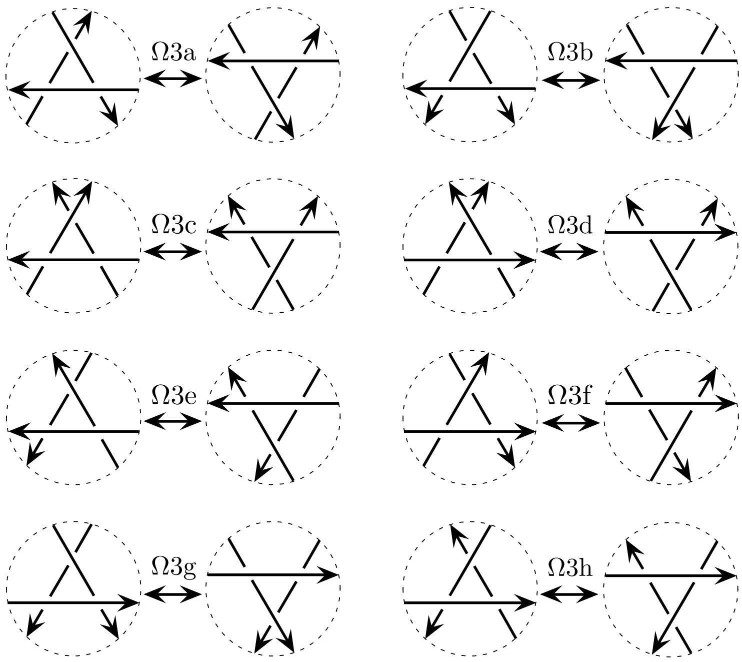


Figure 4: Oriented Reidemeister moves of type III.

unknot diagram [9]. In this paper, we constructed new diagram invariants to further bound the minimal number of Reidemeister moves needed to unknot an oriented diagram.

As seen with Hass and Nowik, knot diagram invariants are useful in determining bounds for the number of Reidemeister moves.

Definition 2.6 (knot diagram invariant)**.** A *knot diagram invariant* is a value (usually an integer) that is assigned to a knot diagram. This value is invariant under any isotopies of the plane.

An example of a diagram invariant is the writhe of a knot diagram.

Definition 2.7 (sign of a crossing)**.** Let $\mathscr{C}$ be the set of all crossings in oriented knot diagram D. Take any crossing $c \in \mathscr{C}$. The *sign* of the crossing c, $\operatorname{sgn}(c)$, when the crossing is turned so the strands are oriented bottom right to top left and bottom left to top right, is 1 if the bottom left to top right strand is the overstrand and -1 if the bottom right to top left strand is the overstrand (see Figure 5).

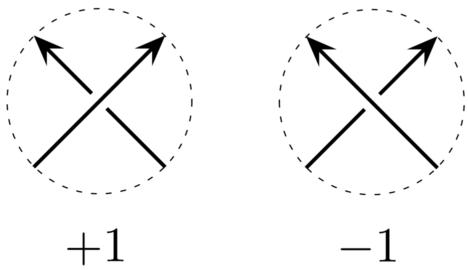


Figure 5: Rule for calculating the sign of a crossing.

Definition 2.8 (writhe)**.** Let $\mathscr{C}$ be the set of all crossings in oriented knot diagram D. Then the *writhe* of D, denoted as $\mathrm{w}(D)$, is

$$\mathrm{w}(D) = \sum_{c \in \mathscr{C}} \operatorname{sgn}(c).$$

3 Curve invariants

In this paper, a (generic) *curve* is a smooth immersion of the circle into the plane with intersecting double points, but has neither self-intersection points with multiplicity greater than 2 nor points of self-tangency.

Reidemeister moves can be applied to curves, with the crossings simply being ignored. Notice that this means that there exist non-equivalent knot diagrams with underlying curves (obtained by removing the distinction between the over- and understrand) that are connected by a series of Reidemeister moves. Curves have curve invariants, the most well-known being the Arnold invariants J^+, J^-, and St.

Definition 3.1 (positive and negative $\Omega 3$ move)**.** Assign an orientation to the sides of the vanishing triangle from the first to the last to appear if we move along the knot beginning at an arbitrary point. Let n be the number of sides of the vanishing triangle whose orientation agrees with the orientation of the corresponding segment on the knot diagram, and let $q' = (-1)^n$. Then a $\Omega 3$ move is considered positive if it changes q' from -1 to $+1$, and negative if the reverse occurs.

Definition 3.2 (Arnold [1])**.** The Arnold invariants J^+, J^-, and St are defined by the following rules:

1. Orientation of the curve does not affect the invariants.
2. J^+ changes by $+2$ under matched $\Omega 2$ moves, and remains unchanged under unmatched $\Omega 2$ moves and $\Omega 3$ moves.
3. J^- changes by $+2$ under unmatched $\Omega 2$ moves, and remains unchanged under matched $\Omega 2$ moves and $\Omega 3$ moves.
4. St changes by $+1$ under positive $\Omega 3$ moves (and -1 for negative $\Omega 3$), and remains unchanged under $\Omega 2$ moves.
5. For curves K_0 and K_i, for $i \in \mathbb{N}_0$ (see Figure 6),
   (a) $\mathrm{J}^+(K_{i+1}) = 2i$, $\mathrm{J}^+(K_0) = 0$;
   (b) $\mathrm{J}^-(K_{i+1}) = -3i$, $\mathrm{J}^-(K_0) = -1$;
   (c) $\mathrm{St}(K_{i+1}) = i$, $\mathrm{St}(K_0) = 0$.

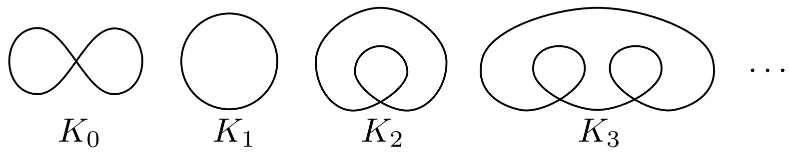


Figure 6: The base cases K_i for which the Arnold invariants are defined.

Hayashi, Hayashi, Sawada, and Yamada [6] introduced several knot diagram invariants based on the Arnold invariants and writhe: $\mathrm{J}^+/2 + \mathrm{St}$, $\mathrm{J}^-/2 + \mathrm{St}$, and $\mathrm{J}^-/2 + \mathrm{St} \pm \mathrm{w}/2$. These diagram invariants were then used to get bounds for unknotting. The description of how they change under Reidemeister moves is in the Appendix, Table 1.

We can define the smoothing operation of a crossing of a curve (or knot diagram) in the following way:

Definition 3.3. Let c be a crossing in the oriented curve C. Then the *smoothing* of c results in a two (potentially intersecting) curves created by removing crossing c and replacing it with two non-intersecting strands that preserve the original orientation, while leaving the rest of C the same.

Note that while D must be oriented, the smoothing operation is independent of the orientation as reversing the orientation of C reverses the orientation for both strands, resulting in the same two non-intersecting strands after smoothing.

In addition, we can define similar "indices" for edges and regions.

Definition 3.4 (edge and region)**.** Let C be a curve. Curve C may be oriented or not. An *edge* of C is any connected segment of the curve that ends at both sides at a crossing and does not contain any crossings. A *region* is any connected component of the complement of C.

Given an oriented curve C, we can assign a *winding number* $\mathrm{ind}(r)$ to each region r in the knot diagram. To define the winding number, we use the following definition:

Definition 3.5 (left and right of a curve)**.** Let l be an oriented curve in the plane P. Let v be another oriented curve in P, such that v intersects l at point A. Let $\vec{n}_l$ and $\vec{n}_v$ be vectors beginning at A tangent to l and v, respectively. Then v *crosses to the left* (*right*) of l if $(\vec{n}_l, \vec{n}_v)$ is a positively (negatively) oriented basis of the plane. Suppose a curve l that divides an (infinite) area into two distinct regions. Then a curve that crosses to the left of l begins in the region on the *right* of l and ends in the region on the *left* of l. Figure 7 illustrates the relative positions of the left and right regions with respect to an oriented edge.

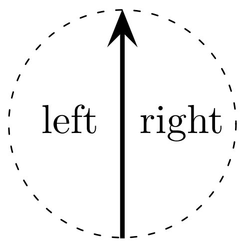


Figure 7: The left and right regions with respect to an oriented curve.

Thus we can define the winding number, also oftentimes called the *index*, with respect to the curve, in the following way:

Definition 3.6 (winding number of a region)**.** Let C be an oriented curve. Let r be a region of D, and let P be any point in r. Draw a oriented half-line l from P directed towards the point at infinity such that it does not pass through any crossings of C. Then $\mathrm{ind}(r)$, the winding number of region r, is the number of times C crosses l to the left minus the number of times C crosses l to the right.

Definition 3.6′. The winding number of region r, $\mathrm{ind}(r)$, is also commonly defined as the net number of counterclockwise turns around any point of the region.

The winding number of a region can also be calculated with the following two rules: the winding number of the unbounded exterior region is 0 and for any edge of D, the winding number of the region on the left is one more than that of the region on the right, as seen in Figure 8.

Using winding number, we can define similar values for edges and crossings.

Definition 3.7. For an edge e in the curve, let R be the set of two regions adjacent the edge. Then let

$$\mathrm{hInd}(e) = \frac{1}{2} \sum_{r \in R} \mathrm{ind}(r).$$

Definition 3.8. For a crossing c in the curve, let R be the set of four regions that surround the crossing. Then let

$$\mathrm{qInd}(c) = \frac{1}{4} \sum_{r \in R} \mathrm{ind}(r).$$

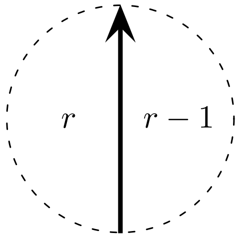


Figure 8: Rule for winding numbers of adjacent regions.

Simply put, hInd(c) is the average of the winding numbers of the two regions adjacent to the edge e, and qInd(c) is the average of the winding numbers of the four regions surrounding the crossing c.

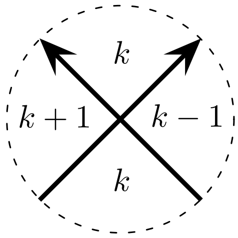


Figure 9: For any integer k, the winding numbers of the four regions surrounding a crossing.

These values were used by Viro [22] to find explicit formulas for J^+ and J^- and by Shumakovitch [17] to find explicit formulas for St.

Theorem 3.9 (Viro [22])**.** *Let C be a curve with n double points. Let $\bar{C}$ be the diagram obtained by smoothing all crossings of C, and let $\mathscr{R}(\bar{C})$ be the set of regions in $\bar{C}$. Then*

$$\mathrm{J}^+(C) = 1 + n - \sum_{r \in \mathscr{R}(\bar{C})} \left(\chi(r)\,\mathrm{qInd}^2(r)\right),$$

$$\mathrm{J}^-(C) = 1 - \sum_{r \in \mathscr{R}(\bar{C})} \left(\chi(r)\,\mathrm{qInd}^2(r)\right),$$

where χ is the Euler characteristic.

Fix p as an arbitrary point on the oriented curve C which differs from all n crossings. Label the edges from 1 to $2n$ by following the orientation of the curve and with the edge containing p being labeled 1.

Definition 3.10 (weight)**.** Weight $\omega(x)$ is defined for crossings, edges, and regions. Take a crossing c, such that the edges e_i and e_j point towards c and have labels i and j, respectively, with e_i crossing to the right of j (see Figure 10). Let $\mathrm{sgn}(k)$ be the sign of the integer k. Then,

$$\omega(c) = \mathrm{sgn}(i - j),$$

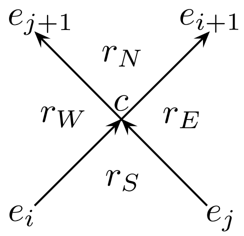


Figure 10: Calculation of weight for crossing c and the edges e_i and e_j pointing towards c, and the contribution to the weight of the surrounding regions.

$$\omega(e_i) = \operatorname{sgn}(i - j),$$
$$\omega(e_j) = -\operatorname{sgn}(i - j),$$

where $\operatorname{sgn}(x)$ of a real number x is 1 if $x > 0$, -1 if $x < 0$, and 0 if $x = 0$.

Let r_W be the region directly to the left of e_i, r_E be the region directly to the right of e_i, r_S be the region directly to the right of e_i and left of e_j, and r_N be the remaining region surrounding c (see Figure 10). The weight of a region is the sum of the contributions by each crossing, $\omega_c(r)$, and

$$\omega_c(r_W) = \omega_c(r_E) = \frac{1}{2}\operatorname{sgn}(i - j),$$

$$\omega_c(r_N) = \omega_c(r_S) = -\frac{1}{2}\operatorname{sgn}(i - j).$$

Theorem 3.11 (Shumakovich [17])**.** *Let C be an oriented curve. Let $\mathscr{R}$, $\mathscr{E}$, $\mathscr{C}$ be the set of regions, edges, and crossings of C, respectively. Then*

$$\operatorname{St}(C) = \sum_{c\in\mathscr{C}} (\omega(c)\operatorname{qInd}(c)) + \delta^2 - \frac{1}{4},$$

$$\operatorname{St}(C) = \frac{1}{2}\sum_{e\in\mathscr{E}} \big(\omega(e)\operatorname{hInd}^2(e)\big) + \delta^2 - \frac{1}{4},$$

$$\operatorname{St}(C) = \frac{1}{3}\sum_{r\in\mathscr{R}} \big(\omega(r)\operatorname{ind}^3(r)\big) + \delta^2 - \frac{1}{4},$$

where δ is equal to the hInd *of the edge that the initial point p lies on.*

One thing that is not as well defined is how J^+, J^-, and St change under one $\Omega 1$ move. For completeness, we will calculate more precise values for the change of each invariant, utilizing the explicit formulas mentioned above.

Theorem 3.12. *Under a forward $\Omega 1$ move, J^- changes by $-2\operatorname{qInd}(c) - 1$ and J^+ changes by $-2\operatorname{qInd}(c)$.*

Proof. One property that is useful for determining the Euler characteristic of a region of $\bar{C}$ is that χ is an invariant under homotopy equivalences, which includes retractions. Therefore, we can take a region with h holes in it and retract it into a graph with one vertex and h edges, each looping around a distinct hole in the region. This implies that $\chi(r) = 1 - h$.

The addition of a loop by a forward $\Omega 1$ move affects J^+ and J^- in two ways: adding a new region, and changing the Euler characteristic for the region the loop is made in. Let c be the crossing created by the $\Omega 1$ move. Then the new region in $\bar{C}$ has Euler characteristic $1 - 0 = 1$ and quarter-index $\mathrm{qInd}(c) + 1$. The Euler characteristic of the region surrounding the loop decreases by 1, say from χ_1 to $\chi_1 - 1$, and the quarter-index remains as $\mathrm{qInd}(c)$. Thus, the change to J^- is

$$\Delta\,\mathrm{J}^- = -(\mathrm{qInd}(c)+1)^2 - ((\chi_1 - 1)\,\mathrm{qInd}^2(c) - \chi_1\,\mathrm{qInd}^2(c)) = -2\,\mathrm{qInd}(c) - 1$$

under one forward $\Omega 1$ move.

Since $\mathrm{J}^+ = \mathrm{J}^- + n$, where n is the number of crossings, it quickly follows that J^+ changes by $-2\,\mathrm{qInd}(c)$ under a forward $\Omega 1$ move. □

Theorem 3.13. *Arnold invariant* St *changes by* $+\,\mathrm{qInd}(c)$ *under a left* $\Omega 1$ *move and* $-\,\mathrm{qInd}(c)$ *under a right* $\Omega 1$ *move.*

Proof. The addition of a loop by a forward $\Omega 1$ move adds one more crossing to evaluate to determine St. While the numbering changes with the addition of two new edges, this does not impact the weights of any other crossing because it does not change the sign of the difference of the two intersecting edges. Let the three edges connected to the new crossing be numbered k, $k+1$, and $k+2$, increasing in the direction of the orientation. If the initial point is on the original edge, we can ensure that the point is on the new edge $k = 1$. Also, if the diagram is the trivial unknot diagram, then the enumeration of edge $k+2$ is actually 1, but this does not affect the following calculations. The change in St is dependent on whether the loop is made on the left or right of the curve. If it is on the left of the curve, the weight of the crossing is $\omega(c) = \mathrm{sgn}(k+1-k) = +1$. On the other hand, if it is on the left of the curve, the weight of the crossing is $\omega(c) = \mathrm{sgn}(k-(k+1)) = -1$. Thus, St changes by $+\,\mathrm{qInd}(c)$ under a left $\Omega 1$ move and $-\,\mathrm{qInd}(c)$ under a right $\Omega 1$ move.

As the sign of the crossing does not affect the change of St under a $\Omega 1$ move, St changes by $+2\,\mathrm{qInd}(c)$ under a left framed $\Omega 1$ move and $-2\,\mathrm{qInd}(c)$ under a right framed $\Omega 1$ move. □

4 The Self-Crossing Index

4.1 The invariant

In this section, we introduce a new knot diagram invariant, called the *self-crossing index* or SCI. The definitions made for curves in Section 2 can also be applied to knot diagrams by simply using the underlying curve of the diagram.

Definition 4.1 (Self-Crossing Index)**.** Let D be an oriented knot diagram and let $\mathscr{C}(D)$ be the set of crossings of D. Then

$$\mathrm{SCI}(D) = \sum_{c \in \mathscr{C}(D)} \mathrm{sgn}(c)\,\mathrm{qInd}(c).$$

An oriented knot diagram D is considered *ascending* if there exists an initial point or *lowest point* p on the diagram such that the understrand is passed first for all crossings when following along D starting at p in the direction of the orientation. For ascending knots diagrams, Shumakovitch [17] shows that

$$\mathrm{St}(C) = \sum_{c\in\mathscr{C}} (\mathrm{sgn}(c)\,\mathrm{qInd}(c)) + \delta^2 - \frac{1}{4}.$$

Therefore, it turns out that

Theorem 4.2. *For an ascending knot diagram D and its underlying curve C,*

$$\mathrm{SCI}(D) = \mathrm{St}(C) - \delta^2 + \frac{1}{4},$$

where δ is equal to the hInd *of the edge that the lowest point p of the ascending diagram lies on.*

This relationship, however, only holds for ascending knot diagrams, in which the sign of a crossing is ensured to be the same as its weight, and not other knot diagrams. However, this relation does motivate alternate definitions of SCI in general. Like St, we can also define SCI is terms of its edges or regions. Note that these formulas do not come directly from Shumakovitch's formulas for St, as $\mathrm{SCI} = \mathrm{St} - \delta^2 + \frac{1}{4}$ only applies for ascending diagrams.

Definition 4.3. Using a set-up similar to that used to define weight (Definition 3.10), for a crossing c,

$$\tilde{\omega}(e_i) = \mathrm{sgn}(c),$$
$$\tilde{\omega}(e_j) = -\,\mathrm{sgn}(c),$$
$$\tilde{\omega}(r_W) = \tilde{\omega}(r_E) = \frac{1}{2}\sum_{c\in\mathscr{C}(r)} \mathrm{sgn}(c),$$
$$\tilde{\omega}(r_N) = \tilde{\omega}(r_S) = -\frac{1}{2}\sum_{c\in\mathscr{C}(r)} \mathrm{sgn}(c),$$

where $\mathscr{C}(r)$ is the set of all crossings around the region

Theorem 4.4 (Alternate formulas of SCI)**.** *Let D be an oriented knot diagram and let $\mathscr{E}(D)$ and $\mathscr{R}(D)$ be the set of edges and regions of D, respectively. Then*

$$\mathrm{SCI}(D) = \frac{1}{2}\sum_{e\in\mathscr{E}(D)} \tilde{\omega}(e)\,\mathrm{hInd}^2(e),$$

$$\mathrm{SCI}(D) = \frac{1}{3}\sum_{r\in\mathscr{R}(D)} \tilde{\omega}(r)\,\mathrm{ind}^3(r).$$

Proof. To show that these three formulas of SCI are equivalent, we will show that the calculations are equivalent around one crossing, c. Using the original definition, that would be $\mathrm{sgn}(c)\,\mathrm{qInd}(c)$. To simplify calculations, let $\alpha = \mathrm{qInd}\, c$. Then $\mathrm{hInd}(e_i) = \alpha + \frac{1}{2}$, $\mathrm{hInd}(e_j) = \alpha - \frac{1}{2}$, $\mathrm{ind}(r_W)\alpha + 1$, $\mathrm{ind}(r_E) = \alpha - 1$, and $\mathrm{ind}(r_N) = \mathrm{ind}(r_S) = \alpha$.

Then the sum for the edges pointing towards c becomes

$$\frac{1}{2}\left(\mathrm{sgn}(c)\left(\alpha + \frac{1}{2}\right)^2 - \mathrm{sgn}(c)\left(\alpha - \frac{1}{2}\right)^2\right) = \alpha\,\mathrm{sgn}(c).$$

Similarly, we can consider the regions around c, and consider the contribution to $\tilde{\omega}(r)$ that c makes (i.e. $\operatorname{sgn}(c)$), since $\operatorname{ind}^3(r)$ is constant for a region r and the contribution to $\tilde{\omega}(r)$ can be separated into each crossing, allowing us to rewrite the definition of SCI using regions as

$$\mathrm{SCI}(D) = \frac{1}{3} \sum_{c \in \mathscr{C}(D)} \sum_{r \in \mathscr{R}(c)} \operatorname{sgn}(c) \operatorname{ind}^3(r),$$

where $\mathscr{C}(D)$ is the set of crossings in D and $\mathscr{R}(c)$ is the set of four regions surrounding a crossing c. Calculating, we get that

$$\frac{1}{3}\left(\frac{\operatorname{sgn}(c)}{2}\left((\alpha+1)^3 + (\alpha-1)^3\right) - \frac{\operatorname{sgn}(c)}{2}\left(\alpha^3 + \alpha^3\right)\right) = \alpha \operatorname{sgn}(c).$$

Thus, all three definitions are equivalent at each crossing, and thus for any knot diagram D, and can be used to define SCI. □

4.2 Notable properties of SCI

SCI behaves in a predictable manner and thus can be used to find bounds for the number of framed Reidemeister moves between two framed knot diagrams. Unfortunately, SCI does not behave in a controlled manner under $\Omega 1$ moves, being able to be changed by any integer value that corresponds to the qInd of the crossing that is created/removed. However, we can make a slight adjustment to the allowed Reidemeister moves to obtain the framed Reidemeister moves:

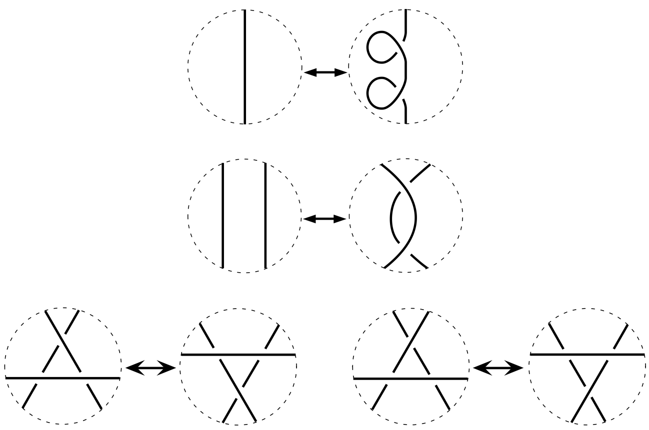


Figure 11: Framed Reidemeister moves of Type I, II, and III (top to bottom).

Forward framed $\Omega 1$ moves are a combination of two $\Omega 1$ moves that create two loops with opposite sign (see Definition 2.7) on the same side of the segment of the knot diagram. Framed $\Omega 2$ and $\Omega 3$ moves are just the usual $\Omega 2$ and $\Omega 3$ moves. Thus, we will continue to refer to $\Omega 2$ and $\Omega 3$ moves without the preceding "framed" term, even when considering framed knots.

Like with Reidemeister moves and knot diagrams, classifications of moves, like forward-backward, generally remain the same, with the conventional $\Omega 1$ move being replaced with its framed counterpart. There is also a similar theorem of equivalence for framed knot diagrams.

Theorem 4.5. *Two framed knot diagrams D and E represent the same framed knot (are equivalent) if and only if there exists a sequence of framed Reidemeister moves connecting them.*

In turns out that there is another way to determine if two framed knot diagrams are equivalent, proven by Trace [20].

Theorem 4.6 (Trace [20])**.** *Two framed knot diagram D and E are equivalent if and only if they have the same writhe and D and E are equivalent knot diagrams.*

Now we can calculate the change of SCI under each framed Reidemeister move and prove Theorem 1.1.

Proof of Theorem 1.1. For framed $\Omega 1$ moves, each move creates or removes two crossings of opposite sign. Also, note that the qInd of each crossing is equal to the winding number of the region the crossing is contained within, and therefore both are equal. Thus, the change to SCI causes by a framed $\Omega 1$ move is 0.

For $\Omega 2$ moves, each move creates or removes two crossings of opposite sign. Similar to framed $\Omega 1$ moves, the quarter-index of the two crossings is equal, meaning that SCI does not change under $\Omega 2$ moves.

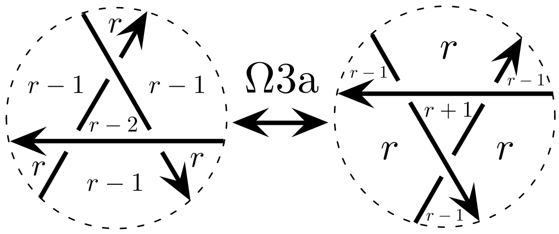


Figure 12: Changes to winding numbers of regions adjacent to the crossings involved in a $\Omega 3$a move.

For $\Omega 3$ moves, there are eight cases of the move that need to be considered [19]. These eight cases are displayed in Figure 4. It turns out for all eight that SCI increases by 1 in the forward direction, meaning that SCI changes by 1 under any forward $\Omega 3$ move and by -1 under any backward $\Omega 3$ move. For example, in the $\Omega 3$a case (see Figure 12), the signs of the crossings remain unchanged as $+1$, -1, and $+1$, and the quarter-indices of the three crossings increases by 1, resulting in a total change of SCI of $+1$. $\square$

In addition to Reidemeister moves, SCI also behaves in a predictable fashion under an operation known as a *connected sum* between knots.

Definition 4.7 (connected sum)**.** A connected sum is an operation conducted on two projections D and E that produces a new knot diagram $D\#E$. To do so, place D and E close to each other, and select a small segment from the curve of each knot, such that it is adjacent to the infinite exterior region, remove them, and connect the two knots together at the endpoints of the removed segments so there is no overlap between the two new strands. An example is shown in Figure 13. Note that for knot diagrams, this operation is not uniquely defined, as it depends on the edges chosen.

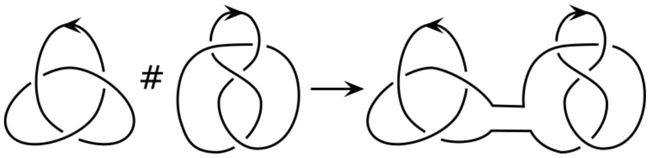


Figure 13: The connected sum of a trefoil and figure-eight knot.

Definition 4.8. A knot diagram invariant I is *additive* if, for any two knot diagrams D and E and any connected sum $D\#E$,

$$I(D\#E) = I(D) + I(E).$$

Theorem 4.9. *The knot diagram invariant* SCI *is additive.*

Proof. The connected sum of D and E leaves the signs of the crossings unchanged, as the operation only changes a very small segment of each diagram. In addition, the winding numbers of the regions do not change, as the operation simply merges together two regions, one from each diagram, and assigns it the same winding number as before. As the winding number of the regions adjacent to each crossing remain constant, so do the quarter-indices for the crossings. □

We will also consider SCI as a Vassiliev diagram invariant.

Definition 4.10. Let I be a knot invariant. Let K be a knot and let D be a diagram of K with n crossings. Let S be the set of those n crossings, and for $m \leq n$, let S_m be a subset of S such that $|S_m| = m$. Then $I_{S_m}(D)$ can be defined recursively such that

$$I_{S_m}(D) = I_{(S_m \setminus \{c\})}(D) - I_{(S_m \setminus \{c\})}(D_c),$$

where c is an arbitrary crossings in S_m and D_c is the same as D except with the crossing c changed. Alternatively, we can define this as

$$I_{S_m}(D) = \sum_{X \subseteq S_m} (-1)^{|X|} I(D_X),$$

where X is the set of crossings that have been changed and $|X|$ is the cardinality of X.

Definition 4.11. Let I be a knot invariant. Let $\mathscr{D}_x$ be the set of all knot diagrams of a knot K with at least the x crossings, and S_k be a subset of k crossings of the diagram. Then I is defined as a Vassiliev invariant of order m if and only if $I_{S_{m+1}}(D) = 0$ for all $D \in \mathscr{D}_{m+1}$ for all knots K, and there exists a diagram $D' \in \mathscr{D}_m$ for some knot K' such that $I_{S_m}(D') \neq 0$. An invariant which has this property is often said as having finite type m.

This easily generalizes to knot diagram invariants. Notice that any curve invariant, when turned into a knot diagram invariant, is a Vassiliev diagram invariant of order 0, since the sign of the crossings does not affect the value of the invariant.

We can calculate the order of SCI as a Vassiliev invariant.

Theorem 4.12. SCI *is a Vassiliev invariant of order 1.*

Proof. Let a be a crossing of a knot diagram D. Let the knot diagram D_a be equivalent to D except at crossing a, which has the opposite sign in D_a with respect to D. Then, for any crossing $c \neq a$ on D and D_a, the sign and quarter-index of c is the same on both diagrams. For crossing a, its quarter-index is the same on both diagrams, as changing the sign of the crossing does not affect the winding number of the regions in the diagram. Therefore,

$$\begin{aligned}\mathrm{SCI}_{\{a\}}(D) &= \mathrm{SCI}(D) - \mathrm{SCI}(D_{\{a\}}) \\ &= \mathrm{SCI}(D) - (\mathrm{SCI}(D) - 2\,\mathrm{sgn}(a)\,\mathrm{qInd}(a)) \\ &= 2\,\mathrm{sgn}(a)\,\mathrm{qInd}(a).\end{aligned}$$

Let $b \neq a$ be another crossing on D. Then

$$\begin{aligned}\mathrm{SCI}_{\{a,b\}}(D) &= \mathrm{SCI}_{\{a\}}(D) - \mathrm{SCI}_{\{a\}}(D_{\{b\}}) \\ &= \big(\mathrm{SCI}(D) - \mathrm{SCI}(D_{\{a\}})\big) - \big(\mathrm{SCI}(D_{\{b\}}) - \mathrm{SCI}(D_{\{a,b\}})\big) \\ &= 2\,\mathrm{sgn}(a)\,\mathrm{qInd}(a) - 2\,\mathrm{sgn}(a)\,\mathrm{qInd}(a) \\ &= 0.\end{aligned}$$

Thus, $\mathrm{SCI}_{\{a_1,a_2\}}(D) = 0$ for all knot diagrams D with at least two crossings. Any $\mathrm{SCI}_{\{a_1,a_2,\cdots,a_k\}}(D) = 0$ for $k \geq 2$ for all knot diagrams D with at least k crossings, which is easily proven by induction since

$$\mathrm{SCI}_{\{a_1,a_2,\ldots,a_n\}}(D) = \mathrm{SCI}_{\{a_1,a_2,\ldots,a_{n-1}\}}(D) - \mathrm{SCI}_{\{a_1,a_2,\ldots,a_{n-1}\}}(D_{\{a_n\}}).$$

Therefore SCI is a Vassiliev invariant of order 1. □

4.3 A family of framed diagrams with quadratic bounds for unknotting

Let U be the trivial projection of the unknot (i.e. a circle). Let $d_{fr}(U, D)$ be the minimal number of framed Reidemeister moves needed to unknot D. In this section we will introduce a family of framed unknot diagrams D_n, adapted from a family presented by Hass and Nowik [9]. Then we will show that as n increases, $d_{fr}(U, D_n)$ grows quadratically with respect to n and the number of crossings in D_n (Theorem 1.2):

Proof of Theorem 1.2. Through computation, we find that

$$\mathrm{SCI}(D_n) = \frac{1}{2}\left(3n^2 - n + 2\right).$$

Since $\mathrm{SCI}(U) = 0$, and SCI changes at most 1 for every framed Reidemeister move, the lower bound for $d_{fr}(U, D_n)$ is $\frac{1}{2}\left(3n^2 - n + 2\right)$. □

Hass and Nowik [9] found a lower bound of $2n^2 + 3n - 2$ using $\mathrm{HN}_{\mathrm{lk}}$ (see Definition 5.1) for regular Reidemeister moves for a similar family of unknots without the $n-1$ forward $\Omega 1$ moves with negative crossings on the right. The quadratic lower bound for D_n from $\mathrm{HN}_{\mathrm{lk}}$ is $2n^2 + 2n - 1$. We can show this using the same homomorphism used by Hass and Nowik [9]. We can calculate that

$$\mathrm{HN}_{\mathrm{lk}}(D_n) = nX_n + nX_{-n} + (2n-1)X_{-1} + (4n-1)Y_0.$$

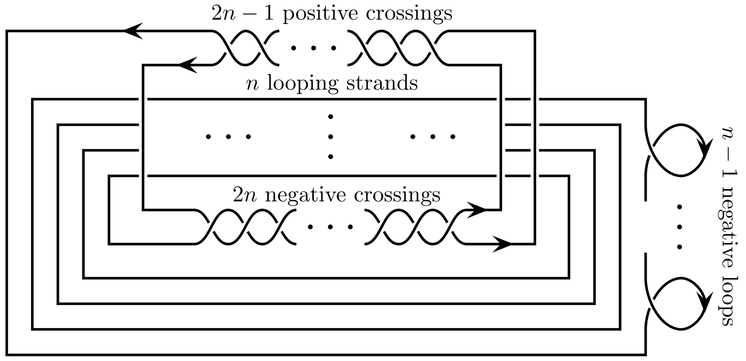


Figure 14: The family of unknots D_n.

Let $g\colon \mathbb{G}_{\mathbb{Z}} \to \mathbb{Z}$ be the homomorphism defined by $g(X_k) = 1 + |k|$ and $g(Y_k) = -1 - |k|$. Then

$$g\left(\mathrm{HN}_{\mathrm{lk}}(D_n)\right) = 2n^2 + 2n - 1.$$

Let R be the set of $X_k + Y_k$, $X_k + Y_{k+1}$, $X_{k+1} - X_k$, and $Y_{k+1} - Y_k$, for all integers k, and their negatives, which represent all possible changes of $\mathrm{HN}_{\mathrm{lk}}$ under framed Reidemeister moves. Since $|g(r)| \leq 1$ for all $r \in R$, the lower bound for the number of framed Reidemeister moves to unknot D_n obtained from $\mathrm{HN}_{\mathrm{lk}}$ is $2n^2 + 2n - 1$.

This lower bound is higher than that found by SCI, which has $\frac{3}{2}$ as the quadratic coefficient. However, SCI is still useful, as it provides bounds on the minimal number of Reidemeister Type III moves. $\mathrm{HN}_{\mathrm{lk}}$ does not provide such a bound, as the homomorphism g still changes under unmatched $\Omega 2$ moves. In fact, $\mathrm{HN}_{\mathrm{lk}}$ should not be expected to provide a lower bound for the minimal number of $\Omega 3$ moves needed for unknotting. Notice that $(X_k + Y_{k+1}) - (X_k + Y_k) = Y_{k+1} - Y_k$, so the change of $\mathrm{HN}_{\mathrm{lk}}$ under some $\Omega 3$ moves is the same as under a forward unmatched $\Omega 2$ move and a backward matched $\Omega 2$ move. Therefore, the value of $\mathrm{HN}_{\mathrm{lk}}$ is not sufficient to distinguish between $\Omega 3$ moves and the previously-mentioned combination of $\Omega 2$ moves.

Moreover, note that the minimal number of framed Reidemeister moves for unknotting can be degrees higher than that for Reidemeister moves. One example is the following family of unknot diagrams L_n and Theorem 1.3.

Proof of Theorem 1.3. Let $d(U, D)$ be the minimal number of Reidemeister moves to unknot D. Unknotting using Reidemeister moves clearly requires at most $2n$ moves when using backwards $\Omega 1$ moves. In addition, calculating $\mathrm{HN}_{\mathrm{lk}}$ (Definition 5.1) gives us

$$\mathrm{HN}_{\mathrm{lk}}(L_n) = nX_0 + nY_0.$$

The set of $X_k + Y_k$, $X_k + Y_{k+1}$, $X_{k+1} - X_k$, and $Y_{k+1} - Y_k$, for all integers k, and their negatives represents all possible changes of $\mathrm{HN}_{\mathrm{lk}}$ under framed Reidemeister moves. Then the minimal construction of $\mathrm{HN}_{\mathrm{lk}}(L_n)$ is done with n additions of $X_0 + Y_0$. Thus, the lower bound is n, and

$$n \leq d(U, L_n) \leq 2n.$$

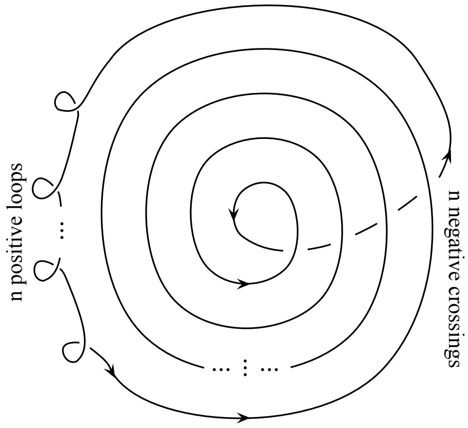


Figure 15: A family of unknots L_n.

However, calculating SCI for L_n gives that

$$\mathrm{SCI}(L_n) = \frac{n(n+1)}{2}.$$

Thus, $d_{fr}(U, L_n) \geq \frac{1}{2}(n^2+n)$, meaning $d_{fr}(U, L_n)$ is at least an entire degree higher than $d(U, L_n)$ □

In fact, we can further strengthen the bound to

$$d_{fr}(U, L_n) \geq \frac{n(n+1)}{2} + n = \frac{n(n+3)}{2}$$

because SCI provides a lower bound for the number of $\Omega3$ moves only, and the $2n$ crossings, which remain unchanged in amount after the $\Omega3$ moves, are removed in at least n Reidemeister moves. (specifically, n moves, each of type I or II).

5 Other diagram invariants

5.1 Hass-Nowik diagram invariant

Just like smoothing a crossing of a curve results in two (potentially intersecting) curves, smoothing one crossing of a knot diagram results in a two-component link diagram. Recall the definition of Hass and Nowik's diagram invariant [8]:

Definition 5.1 (Hass-Nowik diagram invariant [8])**.** Let $\mathscr{C}_+(D)$ be the set of positive crossings and $\mathscr{C}_-(D)$ be the set of negative crossings in oriented knot diagram D. Let $\mathbb{G}_S$ be the free abelian group with basis $\{X_s, Y_s\}_{s\in S}$. Let $\mathfrak{D}$ denote the set of all diagrams of D produced from planar isotopy. Also, let D_c denote the two-component link diagram obtained by smoothing D at crossing c. Then for an invariant $\phi\colon \mathfrak{L} \to S$, for some set S, $\mathrm{HN}_\phi\colon \mathfrak{D} \to \mathbb{G}_\mathbb{Z}$ is

$$\mathrm{HN}_\phi(D) = \sum_{c\in\mathscr{C}_+(D)} X_{\phi(D_c)} + \sum_{c\in\mathscr{C}_-(D)} Y_{\phi(D_c)}.$$

Hass and Nowik's diagram invariant is useful because it can be used to determine a lower bound on the number of Reidemeister moves between two equivalent knot diagrams.

It turns out that this diagram invariant that Hass and Nowik introduced using linking number has the same property as SCI under the connected sum operation.

Theorem 5.2. *The knot diagram invariant* $\mathrm{HN}_{\mathrm{lk}}$ *is additive. In other words, given two knot diagrams* D *and* E*,*

$$\mathrm{HN}_{\mathrm{lk}}(D\#E) = \mathrm{HN}_{\mathrm{lk}}(D) + \mathrm{HN}_{\mathrm{lk}}(E).$$

Proof. Let $\mathscr{C}(D)$ and $\mathscr{C}(E)$ be the sets of crossings in $D\#E$ that were part of the projection of D and E prior to connected sum operation, respectively. Also let d and e be the parts of $D\#E$ that come from D and E, respectively. To calculate the linking number for a two-component link, we simply add the signs of every crossing between the two components and divide by 2. When a smoothing operation is conducted on a crossing $a \in \mathscr{C}(D)$, then e is contained entirely within one of the two components. Thus, none of the crossings in $\mathscr{C}(E)$ contribute to the linking number of the generated two-component link, meaning that link from smoothing $a \in \mathscr{C}(D)$ in $D\#E$ has the same linking number as the link from smoothing a in D. Similarly, the linking number of the link from smoothing crossing $b \in \mathscr{C}(E)$ of $D\#E$ is the same as that of the link from smoothing b in E. Thus, $\mathrm{HN}_{\mathrm{lk}}(D\#E) = \mathrm{HN}_{\mathrm{lk}}(D) + \mathrm{HN}_{\mathrm{lk}}(E)$. □

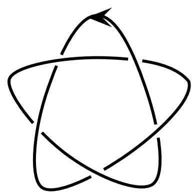


Figure 16: Diagram of the $(2, 5)$-torus knot

Like SCI, we can consider $\mathrm{HN}_{\mathrm{lk}}$ as a Vassiliev diagram invariant. In fact, we will show that $\mathrm{HN}_{\mathrm{lk}}$ is not a Vassiliev diagram invariant. We can prove this with the following evaluation of $\mathrm{HN}_{\mathrm{lk}}$ for $(2, p)$-torus knot and diagrams where some of its crossings are changed, for prime $p \neq 2$. Figure 16 depicts the typical projection of the $(2, 5)$-torus knot. Let $t(2, p)$ refer to the diagram of the $(2, p)$-torus knot with p crossings where following the orientation would lead to an alternating path of being the over- and understand of crossings, like the one in Figure 16.

Theorem 5.3. *Let* S *be the set of the* p *positive crossings of* $t(2, p)$*. Let* S_k *be a subset of* S *with cardinality* k*, where* $0 \leq k \leq p$*, and* $t(2, p)_{S_k}$ *be the knot diagram of* $t(2, p)$ *with the crossings of* S_k *changed. Then*

$$\mathrm{HN}_{\mathrm{lk}}(t(2, p)_{S_k}) = (p - k)X_{\frac{p-2k-1}{2}} + kY_{\frac{p-2k+1}{2}}.$$

Proof. For $t(2, p)$ and $t(2, p)_{S_k} = t(2, -p)$, this can be shown fairly easily, as smoothing any crossing in the diagram results in the $t(2, p-1)$ and $t(2, -p+1)$

torus links, respectively. (While all the crossings of $t(2,p)$ are positive, the crossings of $t(2,-p)$ are all negative.) In each of these links, each of the two components alternate being the over- and understrand if we trace it along its orientation. Thus,

$$\mathrm{HN}_{\mathrm{lk}}(t(2,p)) = pX_{\frac{p-1}{2}}$$

and

$$\mathrm{HN}_{\mathrm{lk}}(t(2,p)_{S_p}) = \mathrm{HN}_{\mathrm{lk}}(t(2,-p)) = pY_{-\frac{p-1}{2}}.$$

To prove the theorem for the remaining $t(2,p)_{S_k}$ for $1 \leq k < p$, we first prove the following lemma.

Lemma 5.4. *Every crossing in the 2-link created by smoothing a crossing of $t(2,p)$ or $t(2,p)_{S_k}$ is a crossing between the two components.*

Proof. This can easily be seen for $t(2,p)$, as a smoothing creates the $t(2,p-1)$ torus link. Since changing the sign of a crossing cannot suddenly make that crossing a self-crossing of one of the components, all crossings of $t(2,p)_{S_k}$ are also crossings between the two components. □

Consider the knot diagram $t(2,p)_{S_k}$, in which $1 \leq k < p$. Then the diagram has $p-k$ positive crossings and k negative crossings. When one of the positive crossings is smoothed, there are $p-k-1$ positive crossings and k negative crossings between the two components of the resulting link. Since the linking number is half of the sum of signs of the crossings between the two components, smoothing any of the $p-k$ positive crossings results in a total contribution of $(p-k)X_{\frac{p-2k-1}{2}}$. Similarly, smoothing any of the k positive crossings results in a total contribution of $kY_{\frac{p-2k+1}{2}}$. Thus,

$$\mathrm{HN}_{\mathrm{lk}}(t(2,p)_{S_k}) = (p-k)X_{\frac{p-2k-1}{2}} + kY_{\frac{p-2k+1}{2}}$$

as desired. □

This allows us to determine if $\mathrm{HN}_{\mathrm{lk}}$ has a finite order as a Vassiliev diagram invariant.

Corollary 5.5. *The Hass-Nowik invariant with linking number,* $\mathrm{HN}_{\mathrm{lk}}$, *is not a Vassiliev diagram invariant.*

Proof. For the sake of contradiction, assume that $\mathrm{HN}_{\mathrm{lk}}$ is a Vassiliev diagram invariant of order n. Let S be the set of crossings of a knot diagram D with n crossings.

Let us consider the diagram of the torus knot $t(2,p)$ for a prime $p > n$. This knot diagram has p crossings, and let $S = S_p$ be the set of all of the crossings. Let S_k denote an arbitrary set of k crossings from S. We know that

$$\mathrm{HN}_{\mathrm{lk}}(t(2,p)) = pX_{\frac{p-1}{2}}.$$

We must get a term of $X_{\frac{p-1}{2}}$ to cancel out with the term given by $\mathrm{HN}_{\mathrm{lk}}(t(2,p))$, so one of the $t(2,p)_{S_k}$ must have a crossing that can be smoothed to create a 2-link with the linking number $\frac{p-1}{2}$. However, Theorem 5.3 clearly shows that none of this knot diagrams have a $\mathrm{HN}_{\mathrm{lk}}$ that can cancel out $\mathrm{HN}_{\mathrm{lk}}(t(2,p))$. Thus $(\mathrm{HN}_{\mathrm{lk}})_{S_i} \neq 0$. But this implies that if $\mathrm{HN}_{\mathrm{lk}}$ is a Vassiliev invariant, then $\mathrm{HN}_{\mathrm{lk}}$

must be a Vassiliev invariant of order $ord > n$, which is a contradiction to our original assumption. Since primes can be infinity large, this contradiction occurs for any n. Thus the only possibility is that $\mathrm{HN}_{\mathrm{lk}}$ is not a Vassiliev invariant. □

5.2 Cowrithe

From Hass and Nowik [8], we get the following definition of cowrithe. We have changed the sign to fit the sign conventions of Hayashi, Hayashi, Sawada, and Yamada [6].

Definition 5.6 (cowrithe)**.** Let D be an oriented knot diagram. Let $f\colon \mathbb{G}_{\mathbb{Z}} \to \mathbb{Z}$ be the homomorphism defined by $f(X_n) = n$ and $f(Y_n) = -n$. Then the cowrithe of D is $f(\mathrm{HN}_{\mathrm{lk}})$.

To determine how cowrithe changes under Reidemeister moves, we require another classification of $\Omega 3$ moves:

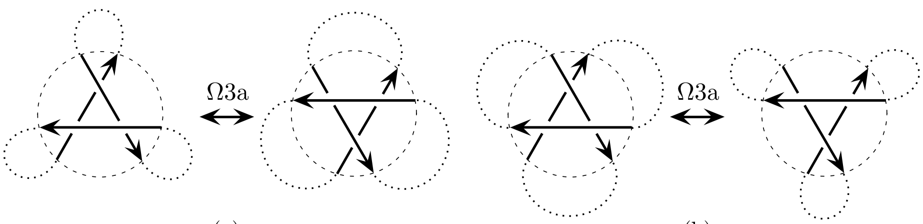
Definition 5.7 (ascending and descending $\Omega 3$ moves [15])**.** Follow the orientation of the knot diagram. An $\Omega 3$ move is *ascending* if the three segments involved are passed in order bottom-middle-top, and *descending* if the three segments involved are passed in order top-middle-bottom.

We can establish a relationship between forward/backward, positive/negative, and ascending/descending $\Omega 3$ moves. Consider a Reidemeister move of type III. Let q equal to 1 for forward $\Omega 3$ moves and -1 for backward $\Omega 3$ moves. Similarly, r equals to 1 for ascending $\Omega 3$ moves and -1 for descending $\Omega 3$ moves.

Theorem 5.8. *A $\Omega 3$ move is positive if $qr = 1$ and negative if $qr = -1$.*

Specifically, forward ascending and backward descending $\Omega 3$ moves are positive, and forward descending and backward ascending $\Omega 3$ moves are negative $\Omega 3$ moves.

Proof. The eight cases of $\Omega 3$ moves can be placed into two groups based on the bottom-middle-top orientation of the vanishing triangle. Cases a, d, e, and g are clockwise, and cases b, c, f, and h are counterclockwise. We will prove Theorem 5.8 for $\Omega 3$a and $\Omega 3$b moves, as the others can be done with similar casework.



(a) (b)

Figure 17: Diagrams for forward (a) ascending and (b) descending $\Omega 3$a moves.

For ascending $\Omega 3$a moves, of the three strands involved in the move, the bottom strand continues around the diagram and eventually connects to the middle strand, the middle to the top, and the top to the bottom. The order-of-appearance orientation of the vanishing triangle, introduced in Definition 3.1, is clockwise. This makes $q' = -1$ for this diagram. Once the $\Omega3$ move is made, the order-of-appearance orientation remains clockwise, and $q' = +1$. Thus, an ascending $\Omega3$a move is positive.

For descending $\Omega3$a moves, of the three strands involved in the move, the bottom strand connects to the top strand, the top to the middle, and the middle to the bottom. The order-of-appearance orientation of the vanishing triangle is counterclockwise. This makes $q' = +1$ for this diagram. Once the $\Omega3$ move is made, q' becomes -1. Thus, a descending $\Omega3$a move is negative.

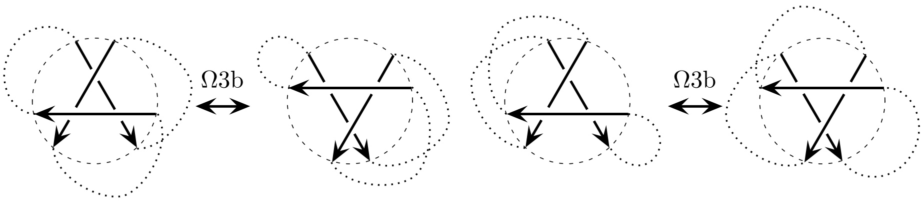


Figure 18: Diagrams for forward (a) ascending and (b) descending $\Omega3$b moves.

For ascending $\Omega3$b moves, of the three strands involved in the move, the bottom strand connects to the middle strand, the middle to the top, and the top to the bottom. The order-of-appearance orientation of the vanishing triangle is counterclockwise. This makes $q' = -1$ for this diagram. Once the $\Omega3$ move is made, q' becomes $+1$. Thus, an ascending $\Omega3$b move is positive.

For descending $\Omega3$b moves, of the three strands involved in the move, the bottom strand connects to the top strand, the top to the middle, and the middle to the bottom. The order-of-appearance orientation of the vanishing triangle is clockwise. This makes $q' = +1$ for this diagram. Once the $\Omega3$ move is made, q' becomes -1. Thus, a descending $\Omega3$a move is negative. □

Using this, we can provide more specific changes to cowrithe under $\Omega3$ moves.

Theorem 5.9. *Cowrithe does not change under (framed)* $\Omega1$ *and unmatched* $\Omega2$ *moves, decreases by* 1 *under forward matched* $\Omega2$ *and forward ascending (i.e. positive)* $\Omega3$ *moves, and increases by* 1 *under forward descending (i.e. negative)* $\Omega3$ *moves.*

Hayashi, Hayashi, Sawada, and Yamada [6] have provided a relationship between cowrithe and the Arnold invariants.

Theorem 5.10 (Hayashi, Hayashi, Sawada, Yamada [6])**.** *Let D be an oriented knot diagram with n crossings, writhe w, and cowrithe x. Then* $x+n/2\mp \mathrm{w}/2 = 4c_2-(\mathrm{J}^-/2+\mathrm{St}\pm\mathrm{w}/2)$*, where c_2 is the coefficient of x^2 in the Alexander-Conway polynomial of D. Furthermore,* $x+n/2-\mathrm{w}/2$ *(resp.* $+\mathrm{w}/2$*) does not change under forward* $\Omega1$ *moves that create a positive (negative) crossings and matched* $\Omega2$ *moves, and increases by* 1 *under forward* $\Omega1$ *moves that create a negative (positive) crossings, unmatched* $\Omega2$ *moves, and negative* $\Omega3$ *moves.*

6 Future investigations

I would be useful to determine if there exists a family of unknot diagram that has a lower bound for unknotting of order at least n^3, where n is the number of crossings. Related to this, are there other integer-valued index-type invariants for (framed) knot diagrams that change by at most one under (framed) Reidemeister moves that can?

Also, Theorem 4.2 provides a relationship between SCI and St for ascending knot diagrams, namely $\mathrm{SCI} - \mathrm{St} = \delta^2 - 1/4$. This naturally poses the question: what is $\mathrm{SCI} - \mathrm{St}$ for other knot diagrams?

Finally, for framed knots, SCI "counts" the number of (forward minus backward) moves of Type III; the number of crossings "counts" the number of forward $\Omega 1$ and $\Omega 2$ moves; and cowrithe "counts" the number of positive $\Omega 3$ moves minus the number of forward matched $\Omega 2$ moves; half of winding number counts forward $\Omega 1$ moves; (again, positive minus negative and forward minus backward). It would be interesting to determine how many linearly independent invariants that "count" some kinds of moves are there?

7 Acknowledgments

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8 Appendix

Below is a table of knot diagram invariants and how they change under different types of forward Reidemeister moves: type 1, framed type 1, matched and unmatched type 2, and ascending and descending type 3.

	Ω1	framed Ω1	Ω2m	Ω2u	Ω3asc.	Ω3desc.
writhe	± 1	0	0	0	0	0
number of crossings	+1	+2	+2	+2	0	0
winding number	± 1	0	0	0	0	0
SCI	$\mathrm{sgn}(c)\,\mathrm{qInd}(c)$	0	0	0	+1	+1
$\mathrm{HN}_{\mathrm{lk}}$	X_0 if $\mathrm{sgn}(c) = +1$, Y_0 if $\mathrm{sgn}(c) = -1$	$X_0 + Y_0$	$X_k + Y_{k+1}$	$X_k + Y_k$	$X_k - X_{k+1}$, $Y_{k+1} - Y_k$	$X_{k+1} - X_k$, $Y_k - Y_{k+1}$
cowrithe	0	0	-1	0	-1	+1
J^+	$-2\,\mathrm{qInd}(c)$	$-4\,\mathrm{qInd}(c)$	+2	0	0	0
J^-	$-2\,\mathrm{qInd}(c) - 1$	$-4\,\mathrm{qInd}(c) - 2$	0	+2	0	0
St	$\mathrm{qInd}(c)$ if on left $-\,\mathrm{qInd}(c)$ if on right	$2\,\mathrm{qInd}(c)$ if on left $-2\,\mathrm{qInd}(c)$ if on right	0	0	+1	-1
$\mathrm{J}^+/2 + \mathrm{St}$	0	0	+1	0	+1	-1
$\mathrm{J}^-/2 + \mathrm{St}$	$-1/2$	-1	0	-1	+1	-1
$\mathrm{J}^-/2 + \mathrm{St} + \mathrm{w}/2$	0 if $\mathrm{sgn}(c) = +1$ -1 if $\mathrm{sgn}(c) = -1$	-1	0	-1	+1	-1
$\mathrm{J}^-/2 + \mathrm{St} - \mathrm{w}/2$	-1 if $\mathrm{sgn}(c) = +1$ 0 if $\mathrm{sgn}(c) = -1$	-1	0	-1	+1	-1
$x + n/2 - \mathrm{w}/2$	0 if $\mathrm{sgn}(c) = +1$ +1 if $\mathrm{sgn}(c) = -1$	+1	0	+1	-1	+1
$x + n/2 + \mathrm{w}/2$	+1 if $\mathrm{sgn}(c) = +1$ 0 if $\mathrm{sgn}(c) = -1$	+1	0	+1	-1	+1

Table 1: Multiple knot diagram invariants and their changes under Reidemeister moves for knots and framed knots. The crossing c is the crossing created by the Ω1 move. In a Ω1 move, St changes by $+\,\mathrm{qInd}(c)$ if the loop is on the left of the curve, and by $-\,\mathrm{qInd}(c)$ if the loop is on the right of the curve.

Results for $\mathrm{HN}_{\mathrm{lk}}$ come from Hass and Nowik [8], although we distinguished between changes from ascending and descending Ω3 moves. Change for cowrithe follows from $\mathrm{HN}_{\mathrm{lk}}$. Changes for J^+ and J^- follow from Viro's formulas [22] and Theorem 3.12. Similarly, changes for St follow from Shumakovitch's formula [17] and Theorem 3.13. Changes for $\mathrm{J}^+/2 + \mathrm{St}$, $\mathrm{J}^-/2 + \mathrm{St}$, and $\mathrm{J}^-/2 + \mathrm{St} \pm \mathrm{w}/2$ were taken from Hayashi, Hayashi, Sawada, and Yamada [6].

References

[1] Vladimir I. Arnold, *Plane curves, their invariants, perestroikas and classifications*, Singularities and bifurcations, Adv. Soviet Math., vol. 21, Amer. Math. Soc., Providence, RI, 1994, pp. 33–91.

[2] Colin C. Adams, *The Knot Book*, Amer. Math. Soc., Providence, RI, 1994.

[3] S. Chmutov, S. Duzhin, and J. Mostovoy, *Introduction to Vassilev Knot Invariants*, Cambridge University Press, 2012.

[4] Chuichiro Hayashi and Miwa Hayashi, *Irregular smoothing and the number of Reidemeister moves*, available at `http://arxiv.org/abs/1103.5173`.

[5] Chuichiro Hayashi, Miwa Hayashi, and Tahl Nowik, *Unknotting number and number of Reidemeister moves needed for unlinking*, Topology Appl. **159** (2012), no. 5, 1467–1474.

[6] Chuichiro Hayashi, Miwa Hayashi, Minori Sawada, and Sayaka Yamada, *Minimal unknotting sequences of Reidemeister moves containing unmatched RII moves*, J. Knot Theory Ramifications **21** (2012), no. 10, 1250099, 13 pages.

[7] Allison Henrich and Louis H. Kauffman, *Unknotting unknots*, available at `http://arxiv.org/abs/1006.4176`.

[8] Joel Hass and Tahl Nowik, *Invariants of knot diagrams*, Math. Ann. **342** (2008), no. 1, 125–137.

[9] ———, *Unknot diagrams requiring a quadratic number of Reidemeister moves to untangle*, Discrete Comput. Geom. **44** (2010), no. 1, 91–95.

[10] Marc Lackenby, *A polynomial upper bound on Reidemeister moves*, Ann. Math. **182** (2015), no. 2, 491–564.

[11] ———, *Elementary knot theory*, available at `https://arxiv.org/abs/1604.03778`.

[12] Harvey Lodish, Arnold Berk, S. Lawrence Zipursky, Paul Matsudaira, David Baltimore, and James Darnell, *Molecular Cell Biology*, 4th ed., W. H. Freeman, New York, 2000.

[13] Kunio Murasugi, *Knot Theory and Its Applications*, translated by Bohdan Kurpita, Birkhäuser Boston, Cambridge, MA, 1996.

[14] Hirosi Ooguri and Cumrun Vafa, *Knot invariants and topological strings*, Nuclear Phys. B **577** (2000), no. 3, 419–438.

[15] Olof-Petter Östlund, *Invariants of knot diagrams and relations among Reidemeister moves*, J. Knot Theory Ramifications **10** (2001), no. 8, 1215–1227.

[16] Michael Polyak, *Minimal generating sets of Reidemeister moves*, Quantum Topol. **1** (2010), no. 4, 399–411.

[17] Alexander Shumakovich, *Explicit formulas for strangeness of plane curves*, St. Petersburg Math. J. **7** (1996), no. 3, 445-472.

[18] Jonathan Simon, *Topological chirality of certain molecules*, Topology **25** (1986), no. 4, 229–235.

[19] Piotr Suwara, *Minimal generating sets of directed oriented Reidemeister moves*, available at `http://arxiv.org/abs/1601.00559`.

[20] Bruce Trace, *On the Reidemeister moves of a classical knot*, Proc. Amer. Math. Soc. **89** (1983), no. 4, 722–724.

[21] Victor A. Vassiliev, *Invariants of ornaments*, Singularities and bifurcations, Adv. Soviet Math., vol. 21, Amer. Math. Soc., Providence, RI, 1994, pp. 225–262.

[22] Oleg Viro, *First degree invariants of generic curves on surfaces*, UUDM report, Department of Mathematics, Uppsala University, 1994.

[23] Edward Witten, *Quantum Field Theory and the Jones Polynomial*, Commun. Math. Phys. **121** (1989), no. 3, 351–399.