Proving the Trefoil is Knotted

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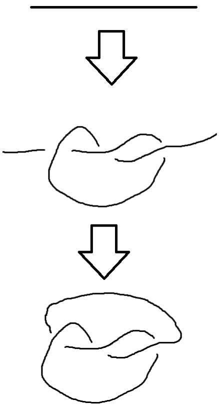
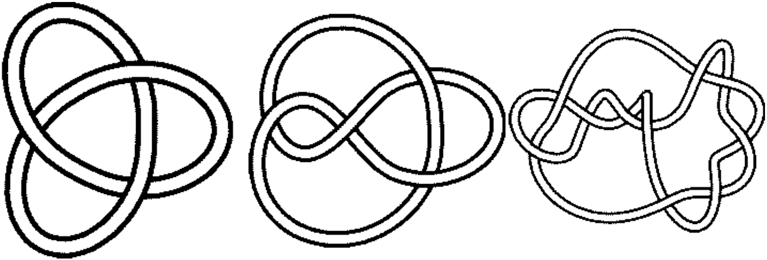
How To Build a Knot

1. Take 1 piece of string
2. Tangle it up
3. Glue ends together

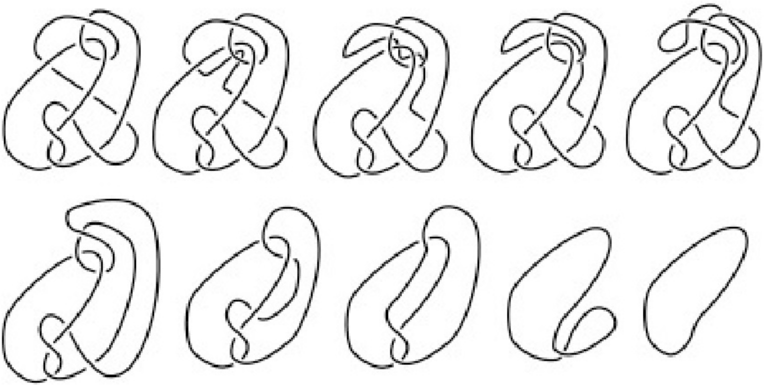


Example Knots



(a) Trefoil (b) 4_1 (c) 10_6

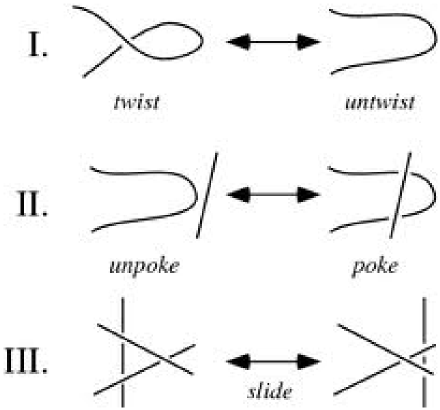
Some knots are the same



Knot diagrams can be deformed, like in real life.

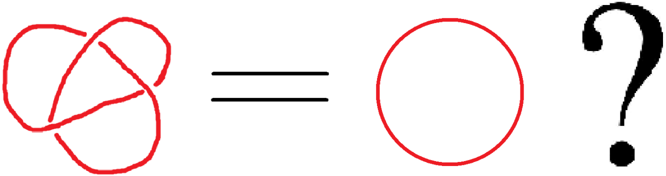
Reidemeister Moves

- We think of knots as knot diagrams:



The big question

Question: How can we show that two knots **aren't** equal?

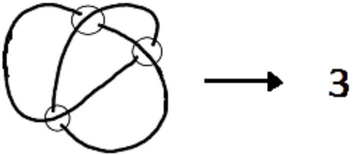


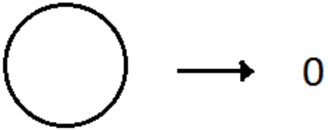
Answer: Find an Invariant.

Assign a number to each knot diagram so that two knot diagrams that are equivalent have same assigned number.

Example Invariant: Crossing Number

- Find an equivalent knot diagram with the fewest crossings
- Crossing number = fewest number of crossings

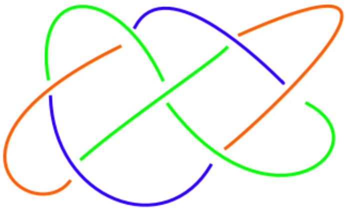




- But this is hard to compute in reality

A Better Invariant: Tricoloring Knots

- Assign one of three colors, a, b, c to each strand in a knot diagram (red, blue, green)
- At any crossing, strands must either have **all different or all same color**



- **We are interested in the number of tricolorings of a knot.**

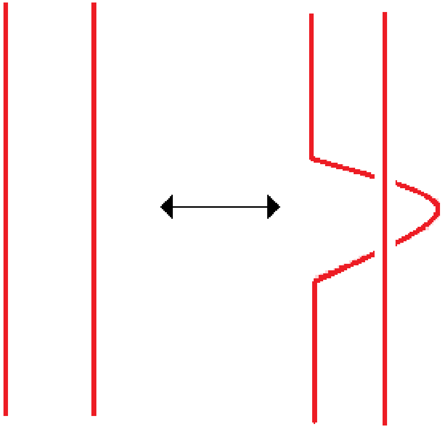
Why Tricolorability Matters

Theorem

If a knot diagram has k tricolorings, then all equivalent knot diagrams have k tricolorings.

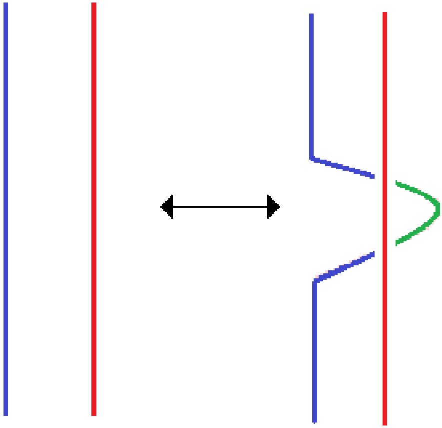
How to Prove:
Show number of tricolorings is maintained by Reidemeister moves.

Example: Bijecting Tricolorings For Second Reidemeister Move



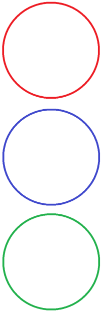
Case 1: Same Color

Example: Bijecting Tricolorings For Second Reidemeister Move

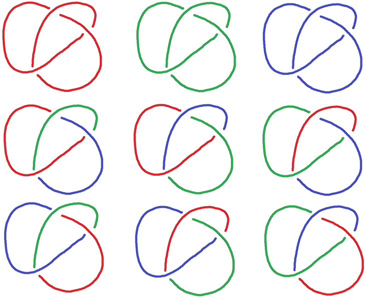


Case 2: Different Colors

Trefoil is Knotted!



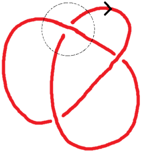
(a) 3 Unknot Tricolorings



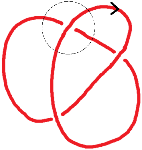
(b) 9 Trefoil Tricolorings

$\Longrightarrow$ Trefoil$\neq$Unknot

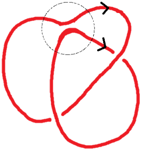
A Different Approach: Focus on One Crossing



Trefoil Knot



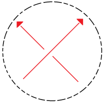
(a) Crossing Change 1

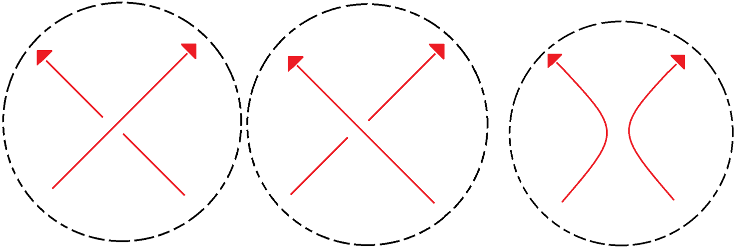


(b) Crossing Change 2

How can we take advantage of this?

A Weird Polynomial: Jones Polynomial $J(K)$

1. Pick a crossing: 
2. Look at its rearrangements:



(a) K_1 (b) K_2 (c) K_3

3. Use recursion:

$$t^{-2}J(K_1) - t^2 J(K_2) = (t - t^{-1})J(K_3)$$
$$J(\mathrm{O}) = 1$$

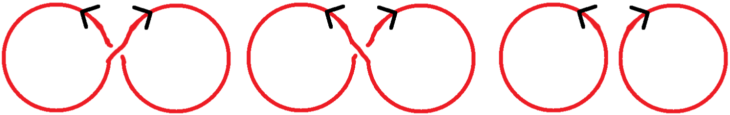
Why Jones Polynomial Matters: It's an Invariant!

Theorem

If knot diagrams A and B are equivalent, then $J(A) = J(B)$.

How to Prove: Show Jones Polynomial is unchanged by Reidemeister Moves.

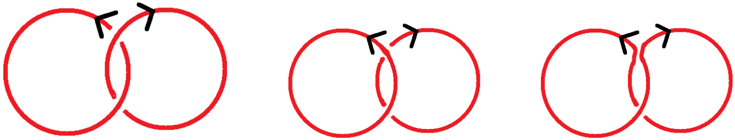
EXAMPLE: 2 UNKNOTS AT ONCE



(a) A_1 (b) A_2 (c) A_3 (2 Unknots)

$$\begin{aligned} J(\text{2 Unknots}) &= J(A_3) \\ &= \frac{t^{-2}J(A_1) - t^2 J(A_2)}{t - t^{-1}} \\ &= \frac{t^{-2}J(\text{Unknot}) - t^2 J(\text{Unknot})}{t - t^{-1}} \\ &= \frac{t^{-2}(1) - t^2(1)}{t - t^{-1}} \qquad = -t - t^{-1}. \end{aligned}$$

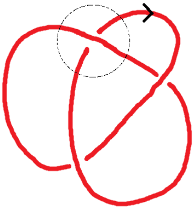
A Weirder Example: Linked Unknots



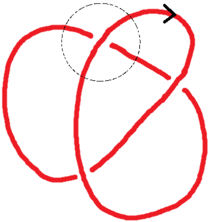
(a) (Linked Unknots) B_1 (b) B_2 (c) B_3

$$\begin{aligned} J(\text{Linked Unknots}) &= J(B_1) \\ &= t^4 J(B_2) + (t^3 - t) J(B_3) \\ &= t^4 J(\text{Separate Unknots}) + (t^3 - t) J(\text{Unknot}) \\ &= t^4(-t - t^{-1}) + (t^3 - t)(1) \\ &= -t^5 - t. \end{aligned}$$

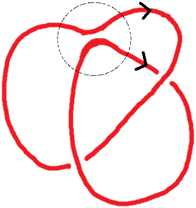
Jones Polynomial of Trefoil



(a) T_1 (Trefoil)



(b) T_2



(c) T_3

$$\begin{aligned} J(\text{Trefoil}) &= J(T_1) \\ &= t^4 J(T_2) + (t^3 - t) J(T_3) \\ &= t^4 J(\text{Unknot}) + (t^3 - t) J(\text{Linked Unknots}) \\ &= t^4(1) + (t^3 - t)(-t^5 - t) \\ &= -t^8 + t^6 + t^2. \end{aligned}$$

Completing Second Proof

$J\Big($  $\Big) = 1$

$J\Big($  $\Big) = -t^8 + t^6 + t^2$

$\implies$ Trefoil$\neq$Unknot

Acknowledgements

We want to thank

1. Our mentor Umut Varolgunes for working with us every week.
2. MIT PRIMES for setting up such a fun reading group.