The Probabilistic Method

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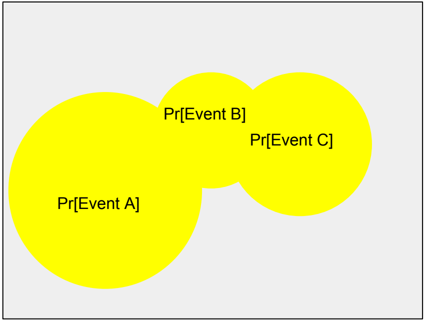
What is the Probabilistic Method?

- A method of proving that a certain structure must exist.
- The structure exists if a randomly chosen element in the probability space has the desired structure with **positive** probability.
- We often look at the probability that a random element does **not** have the desired structure.

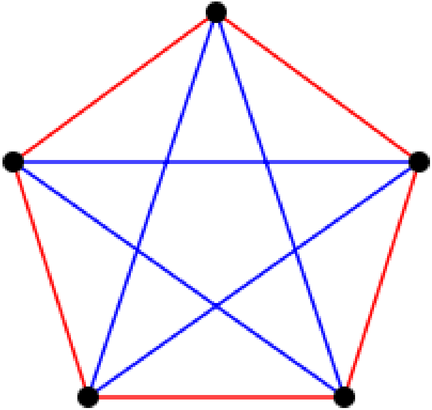
A Useful Rule

$$\sum_{e \in E} Pr(A_e) \geq Pr(\bigvee_{e \in E} A_e)$$



Ramsey Numbers

The *Ramsey Number* $R(k, l)$ is the smallest number n such that given any coloring of the complete graph K_n by red and blue, there is either a red K_k or blue K_l. For example, $R(3, 3) \neq 5$ because K_5 can be colored as



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Ramsey Numbers

One of the first applications of the Probabilistic method was to prove the following theorem:

Theorem (Erdös)

For integers $k < n$, if $\binom{n}{k}2^{1-\binom{k}{2}} < 1$ then $R(k,k) > n$ so $R(k,k) > \lfloor 2^{k/2} \rfloor$.

- Each edge is colored red with probability $\frac{1}{2}$ and blue otherwise
- A_R is the event that the induced complete subgraph on R is monochromatic where R is a subset of V of size k

Then the probability that K_n contains a monochromatic subgraph of size k is at most

$$\sum_{\substack{R \subset V \\ |R|=k}} Pr\left[A_R\right] = \binom{n}{k} 2^{1-\binom{k}{2}} < 1$$

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Expected Value

- Linearity of Expectaion: $E[A+B] = E[A] + E[B]$.
- Often counted using indicator random variables.
- If the expected number of events occuring is less than k, there must be a case in which at most $k-1$ events occur.

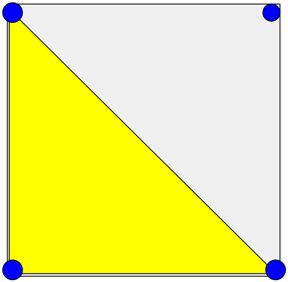
The Smallest Triangle Problem

Given a set n points in a unit square (the set S),

Let T(S) = the area of the smallest triangle defined by three of the n points in S.

What is the maximum value of T(S) over all sets S of n points?

For example, the max over 4 points is $\frac{1}{2}$.



The Smallest Triangle Problem

Theorem (Komlós, Pintz, Szemerédi)

There is a set S of n points in the unit square U such that $T(S) \geq 1/(100n^2)$.

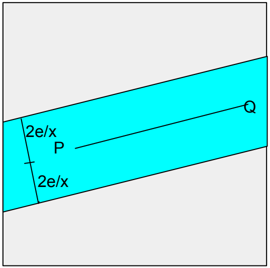
Overview

- Determine the probibility that a random 3 points make a triange of area less than $1/(100n^2)$.
- Find the expected number of "too small" triangles in a random set of $2n$ points.
- Remove points from "too small" triangles to create optimal set.

The Smallest Triangle Problem

Given 3 random points P, Q, R in U, what is the probability that area of $\Delta PQR \leq 1/(100n^2)$?

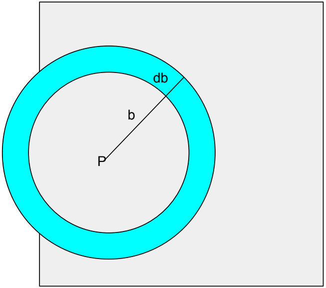
- Area $\Delta PQR = \mu$
- Condition on the distance x between P and Q.
- $\Pr[\mu \leq \epsilon] \leq 4\sqrt{2} \cdot \frac{\epsilon}{x}$



The Smallest Triangle Problem

What if the distance x is random?

$$Pr[b \le x \le b + db] \le 2\pi b db$$



$$Pr[\mu \le \epsilon] \le \int_0^{\sqrt{2}} (4\sqrt{2} \cdot \frac{\epsilon}{b})(2\pi b) db = 16\pi\epsilon \le \frac{0.6}{n^2}$$

The Smallest Triangle Problem

Making Alterations

Randomly Place $2n$ points.

Expected number of triangles with area $\leq 1/(100n^2) =$

$$\binom{2n}{3} \cdot \frac{0.6}{n^2} < \frac{8n^3}{6} \cdot \frac{0.6}{n^2} < n$$

There must be a case with less than n "bad" triangles of area less than $\frac{1}{100n^2}$.

Remove one point from each at most n "bad" triangles. We are left with a set of n points and no triangles with area less than $\frac{1}{100n^2}$.

Hardy - Ramanujan Theorem

In 1934 Turán proved the following result:

- $\nu(n)$ denotes the number of primes dividing n
- $\omega(n) \to \infty$ arbitrarily slowly
- $\pi(n)$ is the number of x in $\{1, \ldots, n\}$ such that

$$|\nu(x) - \ln \ln n| > \omega(n)\sqrt{\ln \ln n}$$

Theorem (Turán 1934)

$\pi(n) = O(n)$, *meaning that*

$$\lim_{n \to \infty} \frac{\pi(n)}{n} = 0.$$

Variance and Covariance

Variance is the measure of how spread out a specific random variable is.

- Definition: for a random variable X,

$$\mathrm{Var}\left[X\right] = E\left[\left(X - E[X]\right)^2\right]$$

- By linearity of expectation, this is equivalent to

$$\mathrm{Var}\left[X\right] = E\left[X^2\right] - E\left[X\right]^2$$

- *Covariance*, of which variance is a special case, is a measure of correlation between two random variables
- For random variables X and Y,

$$\mathrm{Cov}\left[X, Y\right] = E\left[XY\right] - E\left[X\right]E\left[Y\right]$$

Variance and Covariance

- X_i is series of indicator variables
- Random variable $X = \sum_i X_i$
- Variance is

$$\mathrm{Var}[X] = \sum_i \mathrm{Var}[X_i] + \sum_{i \neq j} \mathrm{Cov}[X_i, X_j]$$

If $Pr[X_i] = p_i$,

$$\mathrm{Var}[X_i] = p_i(1 - p_i) \leq p_i = E[X_i]$$

So

$$\mathrm{Var}[X] \leq E[X] + \sum_{i \neq j} \mathrm{Cov}[X_i, X_j]$$

Chebyschev's Inequality

Theorem (Chebyshev)

- *X is a random variable*
- $\mu = E[X]$
- $\sigma = Var[X]$
- $\lambda > 0$

$$Pr[|X - \mu| \geq \lambda\sigma] \leq \frac{1}{\lambda^2}.$$

This is derived from Markov's inequality, which states that

$$Pr[X \geq \lambda] \leq \frac{E[X]}{\lambda}.$$

Hardy - Ramanujan Theorem

- x is randomly chosen from $\{1, \ldots, n\}$
- X_p is 1 if $p|x$ and 0 otherwise
- $X = X_2 + X_3 + X_5 + \cdots + X_p$ where $p \leq M = n^{1/10}$
- No x may have more than 10 prime factors exceeding M

Hence,

$$\nu(x) - 10 \leq X(x) \leq \nu(x).$$

This roughly translates to asymptotic bounds on the variation of $\nu(x)$ by bounding the variation of $X(x)$:

$$X(x) \leq \nu(x) \leq X(x) + 10.$$

Hardy - Ramanujan Theorem

- Begin by finding $E[X]$
- $E[X_p]$ is

$$\frac{\#\text{ of multiples of } p \text{ less than } n}{n} = \frac{\lfloor n/p \rfloor}{n}$$

- $t - 1 \leq \lfloor t \rfloor \leq t$, so

$$E[X_p] = \frac{1}{p} + O(1/n)$$

- Then the total expectation is

$$E[X] = \sum_{p \leq M} \left(\frac{1}{p} + O(1/n) \right) = \ln \ln n + O(1)$$

Hardy - Ramanujan Theorem

- Now find the variance
- Start with

$$\text{Var}[X] = \sum_{p \leq M} \text{Var}[X_p] + \sum_{p \neq q} \text{Cov}[X_p, X_q]$$

- Since $\text{Var}[X_p] = \frac{1}{p}\left(1 - \frac{1}{p}\right) + O(1/n)$,

$$\sum_{p \leq M} \text{Var}[X_p] = \sum_{p \leq M} \left(\frac{1}{p} - \frac{1}{p^2} + O(1/n)\right) = \ln\ln n + O(1)$$

- This leaves only covariances

Hardy - Ramanujan Theorem

The covariance of X_p and X_q (p and q prime) is

$$\begin{aligned}
\mathrm{Cov}\left[X_p, X_q\right] &= \frac{\lfloor n/pq \rfloor}{n} - \frac{\lfloor n/p \rfloor}{n}\frac{\lfloor n/q \rfloor}{n} \\
&\leq \frac{1}{pq} - \left(\frac{1}{p} - \frac{1}{n}\right)\left(\frac{1}{q} - \frac{1}{n}\right) \\
&\leq \frac{1}{n}\left(\frac{1}{p} + \frac{1}{q}\right)
\end{aligned}$$

Hardy - Ramanujan Theorem

- Then

$$\begin{aligned}\sum_{p\neq q} \operatorname{Cov}[X_p, X_q] &\leq \frac{1}{n}\sum_{p\neq q}\left(\frac{1}{p}+\frac{1}{q}\right)\\ &\leq \frac{2M}{n}\sum_{p}\frac{1}{p}\\ &= O(n^{-9/10}\ln\ln n)\\ &= o(1)\end{aligned}$$

- Likewise $\sum_{p\neq q} \operatorname{Cov}[X_p, X_q] \geq -o(1)$
- $\operatorname{Var}[X] = \ln\ln n + O(1)$.

Hardy - Ramanujan Theorem

- Apply Chebyshev's inequality with $\mu = \ln \ln n + O(1)$, $\sigma = \sqrt{\ln \ln n + O(1)}$:

$$Pr\left[|X - \ln \ln n| \geq \lambda\sqrt{\ln \ln n}\right] \leq \frac{1}{\lambda^2} \qquad (1)$$

- Letting $\lambda = \omega(n)$, the probability goes to 0 as $n \to \infty$ and the theorem is proven. □

Thank You

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