Decomposing Tensor Products of Verma Modules

Shashwat Kishore
Mentor: Gus Lonergan
Fourth Annual MIT PRIMES Conference

May 17, 2014

Outline

Background & Goals

- Introduction to $sl(2)$
- Construction of Verma modules
- Construction of signature characters
- Decomposition of tensor product

Results & Discussion

- Complete solutions for two special cases
- Asymptotically correct approximation for the general case
- Current & future work

Introduction to $sl(2)$

Definition: $sl(2)$

The Lie algebra $sl(2)$ consists of the set of 2×2 matrices over $\mathbb{C}$ with trace 0. The standard basis for $sl(2)$ is:

$$e = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$f = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$$

$$h = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

Representation theory of $sl(2)$

Representation of $sl(2)$

- A representation of $sl(2)$ is a vector space V equipped with three operators, E, F, H, that satisfy:

$$HE - EH = 2E$$

$$HF - FH = -2F$$

$$EF - FE = H.$$

Representation theory of $sl(2)$

Representation of $sl(2)$

- A representation of $sl(2)$ is a vector space V equipped with three operators, E, F, H, that satisfy:

$$HE - EH = 2E$$

$$HF - FH = -2F$$

$$EF - FE = H.$$

- This representation has an associated linear homomorphism $\rho : sl(2) \to \text{End } V$. The homomorphism maps e to E, h to H, and f to F.

Combining representations

Combining representations

Direct sum of representations

Given two representations V and W of $sl(2)$, the direct sum $V \oplus W$ is also a representation. Its homomorphism is given by:

$$\rho_{V \oplus W}(l) = \begin{bmatrix} \rho_V(l) & 0 \\ 0 & \rho_W(l) \end{bmatrix}.$$

Combining representations

Direct sum of representations

Given two representations V and W of $sl(2)$, the direct sum $V \oplus W$ is also a representation. Its homomorphism is given by:

$$\rho_{V \oplus W}(l) = \begin{bmatrix} \rho_V(l) & 0 \\ 0 & \rho_W(l) \end{bmatrix}.$$

Tensor product of representations

Given two representations V and W of a $sl(2)$, the tensor product $V \otimes W$ is also a representation. Its homomorphism is given by:

$$\rho_{V \otimes W}(l) = \rho_V(l) \otimes \mathrm{Id} + \mathrm{Id} \otimes \rho_W(l).$$

Representation theory of $sl(2)$

Infinite dimensional representations of $sl(2)$

- A common class of infinite dimensional representations of $sl(2)$ is the class of Verma modules.
- For any complex λ, there exists a unique Verma module, denoted Δ_λ.
- Δ_λ is the union of 1-d weight spaces $V_\lambda, V_{\lambda-2}, V_{\lambda-4}, \ldots$ corresponding to H. Here each V_i is a weight space with weight i.
- The operator E moves any $v \in V_i$ to a vector in V_{i+2} (and moves $v \in V_\lambda$ to 0).
- The operator F moves any $v \in V_i$ to a vector in V_{i-2}.
- We will only deal with real, nonintegral λ.

Signature characters of Verma modules

Signature characters of Verma modules

- Every real highest weight Verma module has a signature character that encodes (a) information about its weight spaces and (b) the signature of its Hermitian form.
- This signature character is an element of $\mathbb{Z}[s]/(s^2 - 1)$.

Signature characters of Verma modules

Signature characters of Verma modules

- Every real highest weight Verma module has a signature character that encodes (a) information about its weight spaces and (b) the signature of its Hermitian form.
- This signature character is an element of $\mathbb{Z}[s]/(s^2 - 1)$.
- For λ negative the signature character of Δ_λ is:

$$\sum_{i \geq 0} e^{\lambda - 2i} \cdot s^i.$$

Signature characters of Verma modules

Signature characters of Verma modules

- Every real highest weight Verma module has a signature character that encodes (a) information about its weight spaces and (b) the signature of its Hermitian form.
- This signature character is an element of $\mathbb{Z}[s]/(s^2 - 1)$.
- For λ negative the signature character of Δ_λ is:

$$\sum_{i \geq 0} e^{\lambda - 2i} \cdot s^i .$$

- For λ positive the signature character of Δ_λ is:

$$\sum_{i \geq 0}^{\lfloor \lambda \rfloor} e^{\lambda - 2i} + \sum_{i \geq \lceil \lambda \rceil} e^{\lambda - 2i} \cdot s^{\lceil \lambda \rceil - i} .$$

More on signature characters

Denote the signature character (if it exists) of a representation V of $sl(2)$ by $\mathrm{ch}_s(V)$. The signature character obeys some natural rules for direct sums and tensor products.

Relations for the signature character

- $\mathrm{ch}_s(V \oplus W) = \mathrm{ch}_s(V) + \mathrm{ch}_s(W)$.
- $\mathrm{ch}_s(V \otimes W) = \mathrm{ch}_s(V) \cdot \mathrm{ch}_s(W)$.

In particular, tensor products of Verma modules admit a signature character.

Tensor products of Verma modules

Decomposition of the tensor product

Consider the tensor product of the Verma modules $\Delta_{\lambda_1}, \Delta_{\lambda_2}, ..., \Delta_{\lambda_n}$. It decomposes uniquely as a direct sum:

$$\bigotimes_i \Delta_{\lambda_i} \cong \bigoplus_{k \geq 0} \Delta_{(\sum \lambda_i)-2k} \otimes E_k$$

where each multiplicity space E_k has a signature character in $\mathbb{Z}[s]/(s^2 - 1)$ and experiences the null action in the representation.

Tensor products of Verma modules

Decomposition of the tensor product

Consider the tensor product of the Verma modules $\Delta_{\lambda_1}, \Delta_{\lambda_2}, ..., \Delta_{\lambda_n}$. It decomposes uniquely as a direct sum:

$$\bigotimes_i \Delta_{\lambda_i} \cong \bigoplus_{k \geq 0} \Delta_{(\sum \lambda_i)-2k} \otimes E_k$$

where each multiplicity space E_k has a signature character in $\mathbb{Z}[s]/(s^2-1)$ and experiences the null action in the representation.

Motivating question

For a given tensor product decomposition, which multiplicity spaces have definite signature characters? Is there a formula for the signature characters of the multiplicity spaces?

Results: Two special cases

Consider the tensor product of $\Delta_{\lambda_1}, \Delta_{\lambda_2}, \ldots, \Delta_{\lambda_n}$, where each λ_i is negative.

Theorem 1 (Decomposition for negative factors case).

In the decomposition

$$\bigotimes_i \Delta_{\lambda_i} \cong \bigoplus_{k \geq 0} \Delta_{(\sum \lambda_i)-2k} \otimes E_k$$

each multiplicity space E_k has signature character $s^k \cdot \binom{k+n-2}{n-2}$.

Idea of Proof. Standard counting argument.

Results: Two special cases

Consider the tensor product of two arbitrary Verma modules $\Delta_{\lambda_1}, \Delta_{\lambda_2}$.

Theorem 2 (Decomposition for two factors case).

In the decomposition

$$\Delta_{\lambda_1} \otimes \Delta_{\lambda_2} \cong \bigoplus_{k \geq 0} \Delta_{(\sum \lambda_i)-2k} \otimes E_k$$

the signature character of each E_k is given by a known piecewise defined function.

Idea of proof. For λ positive, define $L_\lambda = \mathrm{ch}_s(\Delta_\lambda) - \mathrm{ch}_s(\Delta_{\lambda - 2\lceil \lambda \rceil})$. Compute $L_\lambda \cdot \mathrm{ch}_s(\Delta_\mu)$ for λ positive and μ negative.

Results: The general case

Consider the tensor product of Verma modules $\Delta_{\lambda_1}, \Delta_{\lambda_2}, \ldots, \Delta_{\lambda_n}$, where λ_i is positive for $i \leq p$ and negative for $i > p$.

Theorem 3 (Polynomial behavior in general case).

In the decomposition

$$\bigotimes_i \Delta_{\lambda_i} \cong \bigoplus_{k \geq 0} \Delta_{(\sum \lambda_i) - 2k} \otimes E_k$$

there exist polynomials P and Q such that for all sufficiently large k, the signature character of E_k is $s^{n+k}(P(k) + sQ(k))$. If the number of even floor positive weights is even, then P has degree $n-2$ and Q has degree $n-3$. Otherwise, P has degree $n-3$ and Q has degree $n-2$.

Results: The general case

Theorem 4 (Asymptotic approximation in general case).

The leading terms of the polynomials $P(x)$ and $Q(x)$ from Theorem 3 are

$$\frac{1}{(n-2)!} \cdot x^{n-2}$$

and

$$\frac{\sum_{i \leq p} \lceil \frac{\lambda_i}{2} \rceil}{(n-3)!} \cdot x^{n-3}$$

in some order.

Corollary. In an arbitrary tensor product of Verma modules, there are finitely many definite multiplicity spaces iff $n \geq 3$ and $p \geq 1$.

Summary and future work

Summary of results

- Computed decomposition in two specific cases
- Described asymptotic behavior in general case

Summary and future work

Summary of results

- Computed decomposition in two specific cases
- Described asymptotic behavior in general case

Current and future work

- Currently working on explicitly computing the number of definite multiplicity spaces in the general case
- In the future, it would be nice to describe the short term behavior

Acknowledgements

- Thanks to my mentor, Gus Lonergan, for providing guidance and resources throughout.
- Thanks to Professor Pavel Etingof for suggesting the problem.
- Thanks to the PRIMES program for giving me the opportunity to conduct this research.
- Thanks to my parents for always being there.