Generalizations of the Joints Problem

Fourth Annual MIT PRIMES Conference

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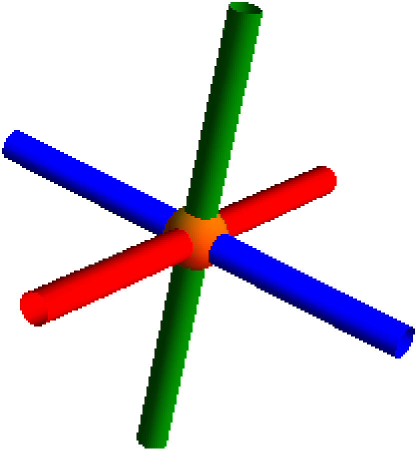
May 17, 2014

Joints

Definition

Let S be a collection of lines in $\mathbb{R}^3$. We say that the point $p \in \mathbb{R}^3$ is a **joint** *if there exist lines $\ell_1, \ell_2, \ell_3 \in S$ such that $\ell_1 \cap \ell_2 \cap \ell_3 = p$ and ℓ_1, ℓ_2, ℓ_3 do not lie in a common plane.*

Joints



The Joints Problem

Given n lines, how many joints can we make?

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Their bound: $cn^{\frac{7}{4}}$

Recent Work

2009: Guth and Katz proved that $J \leq cn^{\frac{3}{2}}$.

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Lemma

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What about c?

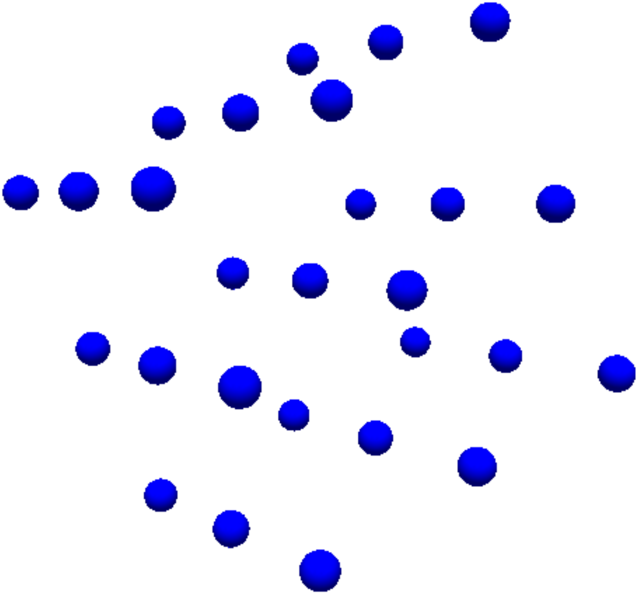
Joints Problem, generalized

What happens if we change the parameters of the problem? Can we bound these cases as well?

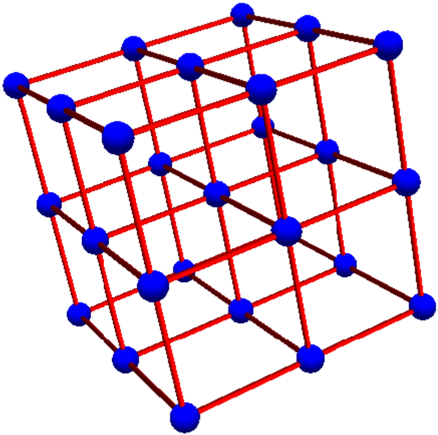
Lower Bounds

Consider a $k \times k \times k$ grid.

Lower Bounds



Lower Bounds



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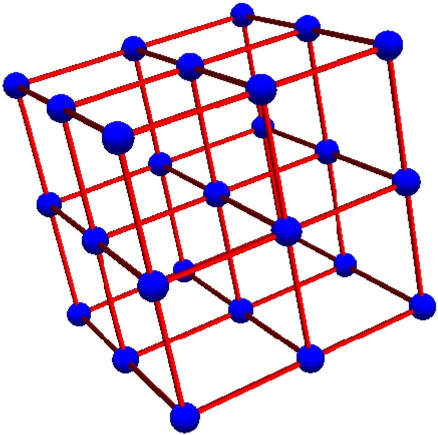
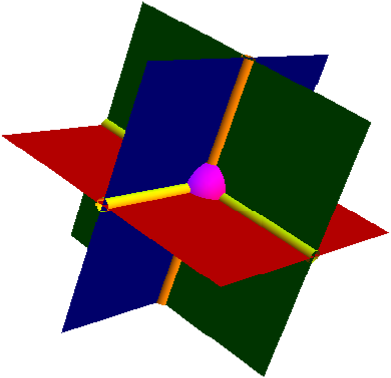


$3k^2$ lines make k^3 joints, so $J = (\frac{1}{3})^{\frac{3}{2}} n^{\frac{3}{2}}$

Lower Bounds

Consider k planes in general position.

Lower Bounds



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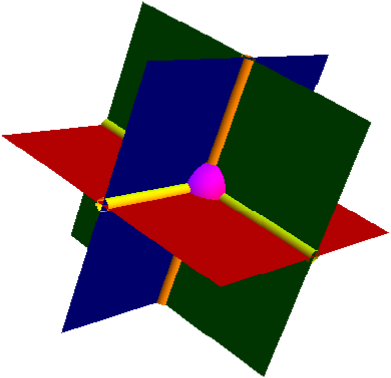

$\binom{k}{2}$ lines make $\binom{k}{3}$ joints, so $J = \frac{\sqrt{2}}{3} n^{\frac{3}{2}}$

The Constant

Weak upper bound: $10n^{\frac{3}{2}}$

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An observation:

$$\frac{4}{3}\left(\frac{n}{2}\right)^{\frac{3}{2}} = \frac{\sqrt{2}}{3}n^{\frac{3}{2}}$$

Line Removal Conjecture

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Conjecture

Suppose we have a set S of n lines $\{\ell_i\}$ in $\mathbb{R}^3$. Given any such set S, let $f(S)$ be the number of joints formed by lines in S. Also, let $g(\ell_0, S)$ be the number of joints formed by ℓ_0 and two members of S. Then there exists a sequence $\{a_i,\ i \leq k\}$ with the following properties:

1. $k \leq \frac{n}{2}$
2. $\forall\ 0 \leq x \leq k-1\ g(\ell_{a_{x+1}}, S \backslash \{\ell_{a_i},\ i \leq x\}) \leq (6f(S \backslash \{\ell_{a_i},\ i \leq x\}))^{\frac{1}{3}}$

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Generalized to $\mathbb{R}^m$, the numbers work out if we use $\frac{1}{m-1}$ instead of $\frac{1}{2}$

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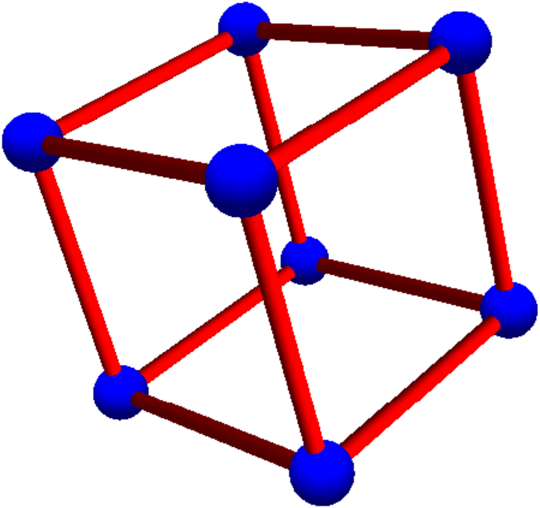

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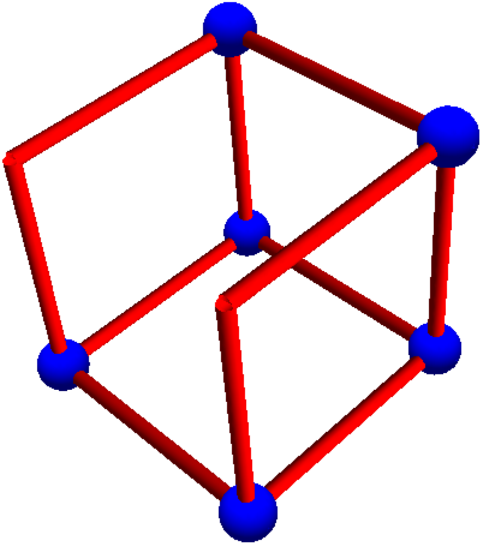

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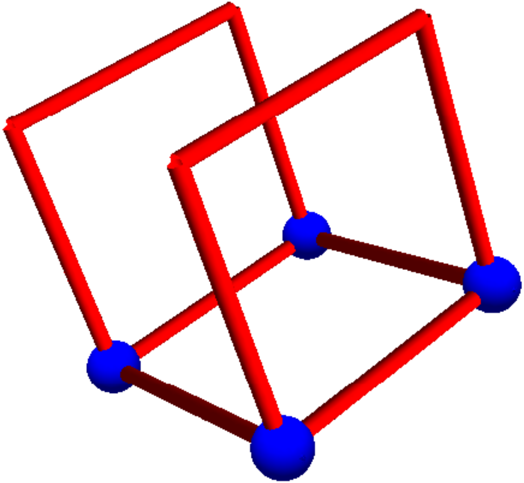

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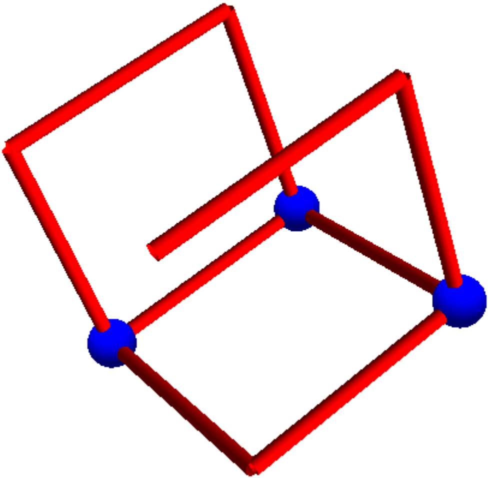

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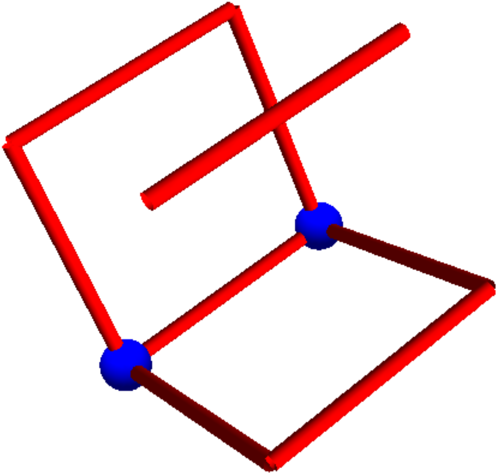

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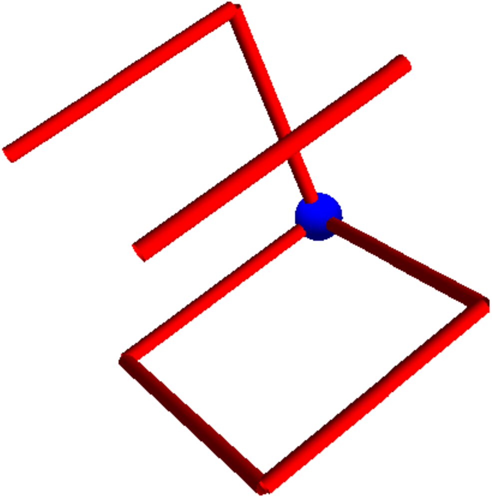

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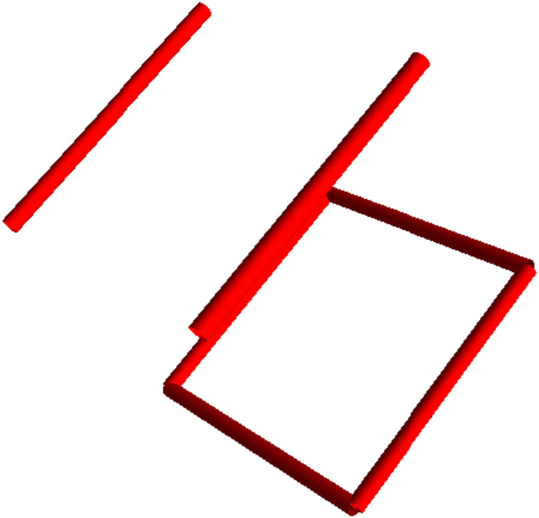

The General Problem

The parameters are:

- The dimension of the space $\mathbb{R}^n$
- The dimension of the objects P_a that are intersecting
- The dimension of their intersection P_b
- The number k of P_a that must intersect to make a joint

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Need $n = b + k(a - b)$

Lower Bound

$$\frac{(k-1)!^{\frac{1}{k-1}}}{k} x^{\frac{k}{k-1}}$$

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Letting $p + \ell = n$, let's bound the number of joints as a function of n.

Upper Bound

Suppose the lines are held in some set of *containing planes* such that each line is contained in exactly one containing plane.

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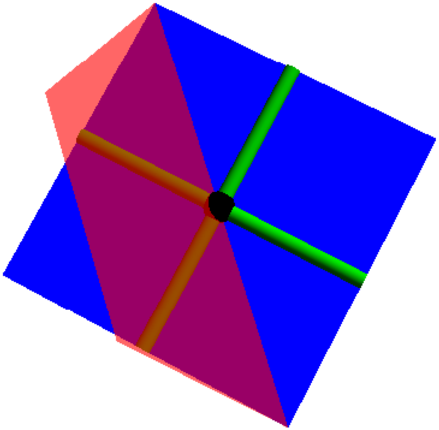

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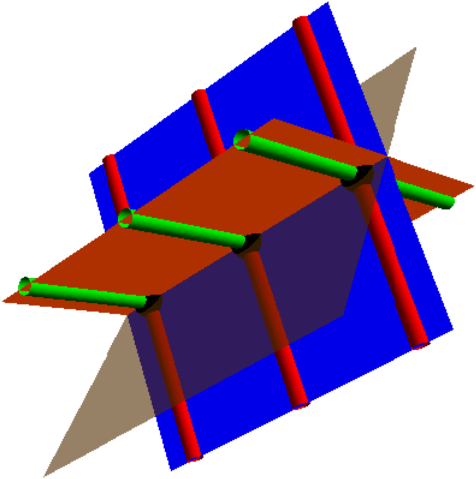

Upper Bound

Theorem

The number of joints is $\leq kn^{\frac{3}{2}}$.

Significance

Lemma

Given a set of S lines in $\mathbb{R}^3$, there exists a set $K \subset S$ with $|K| \leq \frac{1}{3}|S|$ such that given any three lines in S intersecting in a joint, exactly one is in K.

Significance

Lemma

Given a set of S lines in $\mathbb{R}^3$, there exists a set $K \subset S$ with $|K| \leq \frac{1}{3}|S|$ such that given any three lines in S intersecting in a joint, exactly one is in K.

We can use this lemma to give a new proof of the joints theorem $J \leq kn^{\frac{3}{2}}$.

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- Conjecture in $\mathbb{R}^3$
- Generalization with homogeneous dimension of intersecting objects
- Generalization of the idea in $\mathbb{R}^4$, with objects of different dimensions determining joints

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Thank you for listening!