A Diagrammatic Approach to the $K(\pi, 1)$ Conjecture

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What is a Coxeter group?

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Definition

A Coxeter group is given by generators $g_1, g_2, \ldots, g_n$ with relations:

- $g_i^2 = 1$ for all i
- $(g_i g_j)^{m_{ij}} = 1$ for all i, j

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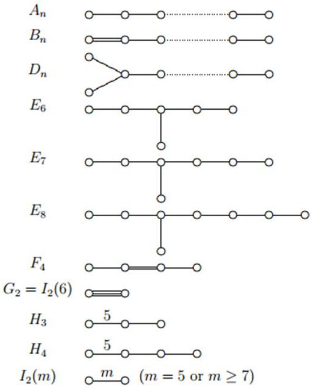
Some examples of Coxeter groups include the symmetric group and reflection groups.

Coxeter Diagrams

Coxeter diagrams can be used to visualize Coxeter groups.

- Each vertex represents a generator
- Edges show the relations between generators



Diagrammatics

We can use Coxeter groups to create certain graphs.

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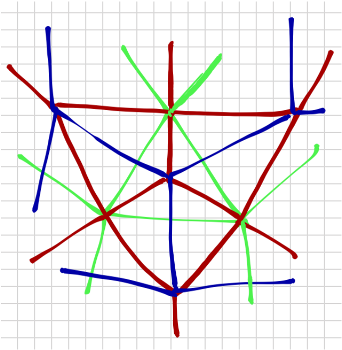
- Each color represents a generator.
- The degree of each vertex is determined by the relations between generators.



$K(\pi, 1)$ Conjecture

There is a conjecture known as the $K(\pi, 1)$ conjecture regarding the second homotopy group of the dual Coxeter complex.

- The dual Coxeter complex is a topological space associated to each Coxeter group
- Elements of second homotopy group correspond to aforementioned graphs

Proving this conjecture is equivalent to proving that all possible graphs for a Coxeter group can be simplfied to the empty graph using a sequence of allowed moves.

MOVES ON DIAGRAMS

How can we simplify a graph?

Moves on Diagrams

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3 allowable moves:

Moves on Diagrams

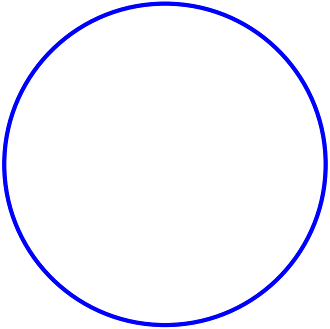
How can we simplify a graph?

3 allowable moves:

- Circle relation
- Bridge relation
- Zamolodchikov relations

Circle Relation

We are allowed to add or remove empty circles of any color.



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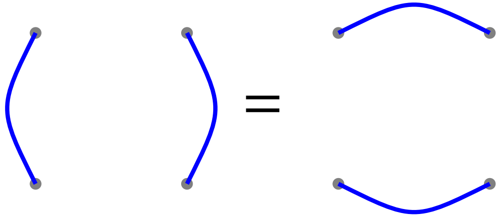
Bridge Relation

If we have two edges of the same color, we can switch around which vertices they connect to, as long as we do not create any new intersections.



Zamolodchikov Relations (ZAM Relations)

Zam relations vary for different coxeter groups. They are found through the reduced expression graph for the longest element.

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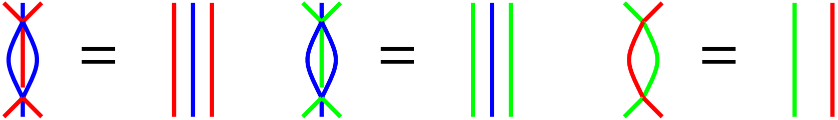

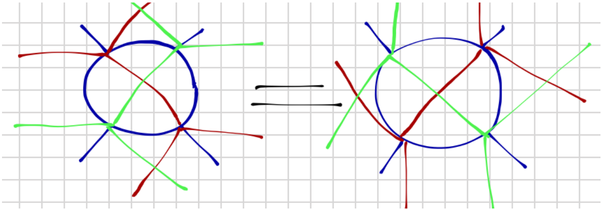

OUR PROJECT

In our project, our primary goal was to prove the $K(\pi, 1)$ conjecture for specific Coxeter groups.

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- $I_2(m)$
- A_3
- B_3
- $G \times H$
- Directed cases
- Working on A_n

Adjacent Vertices

Theorem

We can remove adjacent vertices of the same type.

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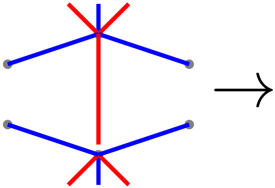

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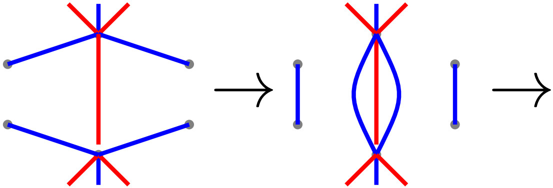

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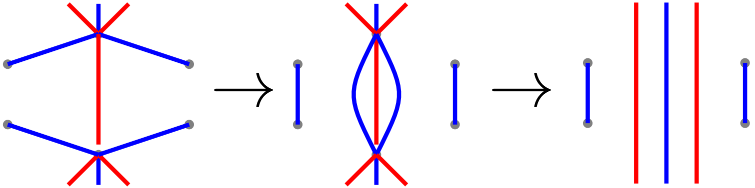

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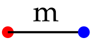
$I_2(m)$

$I_2(m)$: 

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The family of Coxeter groups $I_2(m)$ satisfies the $K(\pi, 1)$ conjecture.

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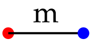
$I_2(m)$: 

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Proof.

- Only 1 type of vertex so necesarily 2 adjacent vertices of same type

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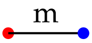
$I_2(m)$: 

Theorem

The family of Coxeter groups $I_2(m)$ satisfies the $K(\pi, 1)$ conjecture.

Proof.

- Only 1 type of vertex so necesarily 2 adjacent vertices of same type
- Use induction on number of vertices

$\square$

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The Coxeter group A_3 satisfies the $K(\pi, 1)$ conjecture.

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Strategy: Look at subgraph of blue color and use Euler characteristic: $V + F = E + 2$ to find a small face.

- Delete the small face.

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The Coxeter group A_3 satisfies the $K(\pi, 1)$ conjecture.

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- Delete the small face.

Using parity, the only nontrivial case is a blue face with 4 edges.

A_3

Any face with 4 edges can be transformed into ZAM for A_3.

- Look at continuation of edges outside of face

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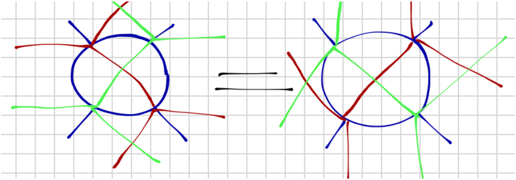

- After using Zam transform, there must be adjacent vertices of the same type.

B_3

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The Coxeter group B_3 satisfies the $K(\pi, 1)$ conjecture.

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- We examine the subgraph of green color.
- Use Euler characteristic to find a small green face.
- Faces with odd number of edges necesarily have adjacent vertices of the same type that can be removed. Using parity and more complicated arguments, faces with 2 or 4 edges also necesarily have vertices that can be removed.

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- We examine the subgraph of green color.
- Use Euler characteristic to find a small green face.
- Faces with odd number of edges necesarily have adjacent vertices of the same type that can be removed. Using parity and more complicated arguments, faces with 2 or 4 edges also necesarily have vertices that can be removed.
- Only nontrivial case is a green face with 6 edges. Vertices of type green-red and green-blue alternate around face.

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Using idea that no adjacent vertices can be of same type, we can manipulate this face into the B_3 ZAM relation.

- Examine continuation of edges outside of face.

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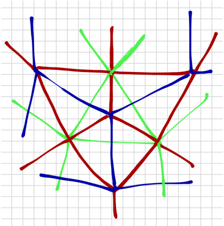
Using idea that no adjacent vertices can be of same type, we can manipulate this face into the B_3 ZAM relation.

- Examine continuation of edges outside of face.
- Only 1 vertex of type red-blue inside face.
- Use bridge relation inside and outside of face to connect edges.



- Use ZAM transformation and get adjacent vertices of the same type.

$G \times H$

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If the $K(\pi, 1)$ conjecture holds for groups G and H, then it holds for the group $G \times H$.

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Strategy: Commutative Colors

$A_1 \times H$

Theorem

If two generators commute, then we can move the edges corresponding to them independently.

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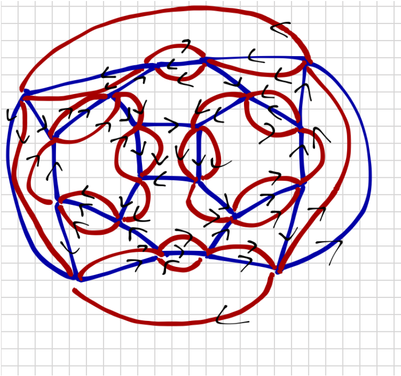
Using this idea, we can solve the general case $G \times H$ by essentially separating the graph formed by the generators of G from the one formed by the generators of H.

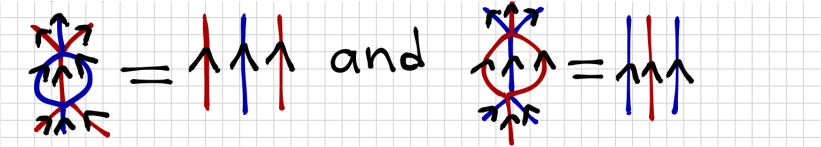
Oriented Graphs

We also solved some cases involving oriented graphs.



ORIENTED A_2

Oriented cases are much more difficult because we cannot necessarily remove adjacent vertices of the same type.



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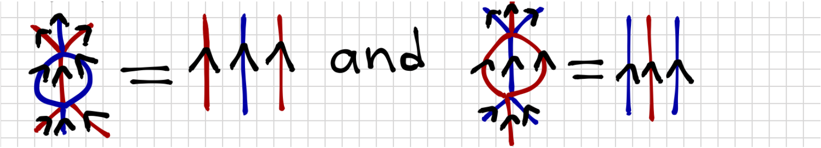

Strategy: Look at the longest cycle

Future Directions

We can take this project in multiple directions in the future.

- We could continue proving the $K(\pi, 1)$ conjecture for other Coxeter groups.
- We could generalize our proofs to classes of Coxeter groups. (For example, we have a nearly-finished proof for A_n.)
- We could also investigate oriented versions of the cases we have already solved.

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