Towards the classification of unital 7-dimensional commutative algebras

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Abstract

An *algebra* is a vector space with a compatible product operation. An algebra is called *commutative* if the product of any two elements is independent of the order in which they are multiplied. A basic problem is to determine how many unital commutative algebras exist in a given dimension and to find all of these algebras. This classification problem has its origin in number theory and algebraic geometry. For dimension less than or equal to 6, Poonen has completely classified all unital commutative algebras up to isomorphism. For dimension greater than or equal to 7, the situation is much more complicated due to the fact that there are infinitely many algebras up to isomorphism. The purpose of this work is to develop new techniques to classify unital 7-dimensional commutative algebras up to isomorphism. An algebra is called *local* if there exists a unique maximal ideal $\mathfrak{m}$. Local algebras are basic building blocks for general algebras as any finite dimensional unital commutative algebra is isomorphic to a direct sum of finite dimensional unital commutative local algebras. Hence, in order to classify all finite dimensional unital commutative algebras, it suffices to classify all finite dimensional unital commutative local algebras. In this article, we classify all unital 7-dimensional commutative local algebras up to isomorphism with the exception of the special case $k_1 = 3$ and $k_2 = 3$, where, for each positive integer i, $\mathfrak{m}^i$ is the subalgebra generated by products of i elements in the maximal ideal $\mathfrak{m}$ and k_i is the dimension of the quotient algebra $\mathfrak{m}^i/\mathfrak{m}^{i+1}$. When $k_2 = 1$, we classify all finite dimensional unital commutative local algebras up to isomorphism. As a byproduct of our classification theorems, we discover several new classes of unital finite dimensional commutative algebras.

1 Introduction

Algebras are fundamental structures in mathematics. An *algebra* over a field K is a vector space over K with a product operation compatible with the vector space structure. An algebra is called *commutative* if the product of any two elements a and b is independent of the order in which they are multiplied, i.e., $ab = ba$. A basic problem is to determine how many unital commutative algebras exist in a given dimension and to find all of these algebras, here the *dimension* of an algebra is defined to be the dimension of its underlying vector space. The motivation for this classification problem comes from number theory and algebraic geometry, e.g., the solutions to systems of polynomial equations, the parametrization and enumeration of number fields, and their connections to moduli space of commutative algebras as in the work of Bhargava and Poonen [1,2,3,4,9]. For a system of multivariable polynomial equations, there is a corresponding unital commutative algebra generated by 1 and the variables of the polynomials subject to relations given by the system of polynomial equations. When the solution set of the system of polynomial equations is finite, the corresponding unital commutative algebra is finite dimensional [11]. A field K is said to be *algebraically closed* if every polynomial with coefficients in K has a solution (e.g., $\mathbb{C}$, the field of all complex numbers). For systems of polynomial equations with r variables over an algebraically closed field whose solution set consists of n points (counted with multiplicity), there is a geometric object, called the Hilbert scheme of n points in r-space, that parametrizes the possibilities. Understanding the classification of unital commutative algebras with dimension n is essentially equivalent to understanding the Hilbert scheme of n points in r-space when r is large enough [9]. Classifying unital commutative algebras of dimension n over a field is also a first step towards understanding unital commutative algebras of dimension n over a ring, e.g., $\mathbb{Z}$, the ring of integers. Classifying such algebras over $\mathbb{Z}$ and estimating how many there are is a classical problem in number theory.

In [7], Mazzola classifies nilpotent commutative associative algebras without the identity up to dimension 5 when the characteristic of the field is not 2 or 3 (an algebra is called *nilpotent* if each element a in the algebra satisfies $a^k = 0$ for some positive integer k). For dimension less than or equal to 6, Poonen completely classifies unital commutative algebras over any algebraically closed field [8]. For dimension 7 or above, the situation becomes more complicated because there are infinitely many non-isomorphic unital commutative algebras over any algebraically closed field [8, 10]. The main purpose of this research is to develop new techniques to classify 7-dimensional unital commutative local algebras up to isomorphism. The basic building blocks for finite dimensional unital commutative algebras are local algebras since any finite dimensional unital commutative algebra is isomorphic to a direct sum of finite dimensional unital commutative local algebras (an algebra is called *local* if there exists a unique maximal ideal $\mathfrak{m}$). Hence, in order to classify all finite dimensional unital commutative algebras, it suffices to classify all finite dimensional unital commutative local algebras. In this article, we classify all unital 7-dimensional commutative local algebras over any algebraically closed field with characteristic 0 up to isomorphism except the special case when $k_1 = 3$ and $k_2 = 3$, where, for each positive integer i, $\mathfrak{m}^i$ is the subalgebra generated by products of i elements in the maximal ideal $\mathfrak{m}$ and k_i is the dimension of the quotient algebra $\mathfrak{m}^i/\mathfrak{m}^{i+1}$. When $k_2 = 1$, we classify all finite dimensional unital commutative local algebras up to isomorphism. As a byproduct of our work, we discover several new classes of finite dimensional unital commutative local algebras.

For the rest of this article, we assume that K is an algebraically closed field with characteristic 0. A good example of such a field is $\mathbb{C}$, the field of all complex numbers.

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2 Classification of Poonen-Suprunenko algebras

In this section, we completely classify a family of 7-dimensional unital commutative algebras up to isomorphism. This family of algebras is discovered by Poonen and Suprunenko in [8, 10]. It contains infinitely many non-isomorphic 7-dimensional unital commutative algebras over a field K. As a consequence of our classification theorem, we prove that if K is uncountable (e.g., $K = \mathbb{C}$), then there exist uncountably many non-isomorphic 7-dimensional unital commutative algebras over K.

Definition 2.1. *Let $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ be scalars in K such that $\alpha_i \neq \alpha_j$ for some pair i and*

j. We define the Poonen-Suprunenko algebra $A(\alpha_1, \alpha_2, \alpha_3, \alpha_4)$ *to be the 7-dimensional unital commutative algebra over K: $\{c_0 + c_1x_1 + c_2x_2 + c_3x_3 + c_4x_4 + c_5B_1 + c_6B_2 : c_i \in K\}$, where $x_1, x_2, x_3, x_4, B_1, B_2$ are the generators satisfying the relations: $x_iB_j = 0$ for all i and j, $B_iB_j = 0$ for all i and j, $x_ix_j = 0$ for all $i \neq j$ and $x_i^2 = B_1 + \alpha_iB_2$ for all i;*

The following theorem gives a complete classification of Poonen-Suprunenko algebras. This theorem will be proved in the next section.

Theorem 2.2. *Let $\alpha_1, \alpha_2, \alpha_3, \alpha_4, \beta_1, \beta_2, \beta_3, \beta_4$ be scalars in K. Assume that $\alpha_i \neq \alpha_j$ for some pair i and j and $\beta_{i'} \neq \beta_{j'}$ for some i' and j'. The Poonen-Suprunenko algebras $A(\alpha_1, \alpha_2, \alpha_3, \alpha_4)$ and $A(\beta_1, \beta_2, \beta_3, \beta_4)$ are isomorphic if and only if there exists an invertible matrix $\begin{pmatrix} q_{11} & q_{12} \\ q_{21} & q_{22} \end{pmatrix}$ and a permutation σ of $\{1, 2, 3, 4\}$ such that $\beta_i = \frac{q_{21}+q_{22}\alpha_{\sigma(i)}}{q_{11}+q_{12}\alpha_{\sigma(i)}}$.*

The following result gives an easily verifiable necessary condition for two Poonen-Suprunenko algebras to be isomorphic.

Theorem 2.3. *Assume that $\alpha_i \neq \alpha_j$ for some pair i and j. If $A(\alpha_1, \alpha_2, \alpha_3, \alpha_4)$ is isomorphic to $A(\beta_1, \beta_2, \beta_3, \beta_4)$, then there exists a permutation σ of $\{1, 2, 3, 4\}$ such that*

$$\det \begin{pmatrix} \beta_1 & \beta_1\alpha_{\sigma(1)} & 1 & \alpha_{\sigma(1)} \\ \beta_2 & \beta_2\alpha_{\sigma(2)} & 1 & \alpha_{\sigma(2)} \\ \beta_3 & \beta_3\alpha_{\sigma(3)} & 1 & \alpha_{\sigma(3)} \\ \beta_4 & \beta_4\alpha_{\sigma(4)} & 1 & \alpha_{\sigma(4)} \end{pmatrix} = 0.$$

Proof. By Theorem 2.2, the assumption of Theorem 2.3 implies that there exist an invertible matrix $\begin{pmatrix} q_{11} & q_{12} \\ q_{21} & q_{22} \end{pmatrix}$ and a permutation σ of $\{1, 2, 3, 4\}$ such that $\beta_i = \frac{q_{21}+q_{22}\alpha_{\sigma(i)}}{q_{11}+q_{12}\alpha_{\sigma(i)}}$. It follows that the following linear system has a nonzero solution (with q_{11}, q_{12}, q_{21}, and q_{22} as the unknowns and $1 \leq i \leq 4$): $\beta_i\, q_{11} + \beta_i\alpha_{\sigma(i)}\, q_{12} - q_{21} - \alpha_{\sigma(i)}\, q_{22} = 0$. This implies Theorem 2.3. $\square$

Corollary 2.4. *If K is uncountable (e.g., $\mathbb{C}$), then there exist uncountably many non-isomorphic 7-dimensional unital commutative algebras over K.*

Proof. For any nonzero α, we choose $\alpha_1 = 1, \alpha_2 = \alpha, \alpha_3 = \alpha^2, \alpha_4 = \alpha^3$. Assume that $\alpha^n \neq 1$ for any $1 \leq n \leq 3$.

For any nonzero β, we choose $\beta_1 = 1, \beta_2 = \beta, \beta_3 = \beta^2, \beta_4 = \beta^3$. We have

$$\det \begin{pmatrix} \beta_1 & \beta_1\alpha_{\sigma(1)} & 1 & \alpha_{\sigma(1)} \\ \beta_2 & \beta_2\alpha_{\sigma(2)} & 1 & \alpha_{\sigma(2)} \\ \beta_3 & \beta_3\alpha_{\sigma(3)} & 1 & \alpha_{\sigma(3)} \\ \beta_4 & \beta_4\alpha_{\sigma(4)} & 1 & \alpha_{\sigma(4)} \end{pmatrix} = \det \begin{pmatrix} 1 & \alpha^{\sigma(1)-1} & 1 & \alpha^{\sigma(1)-1} \\ \beta & \beta\alpha^{\sigma(2)-1} & 1 & \alpha^{\sigma(2)-1} \\ \beta^2 & \beta^2\alpha^{\sigma(3)-1} & 1 & \alpha^{\sigma(3)-1} \\ \beta^3 & \beta^3\alpha^{\sigma(4)-1} & 1 & \alpha^{\sigma(4)-1} \end{pmatrix}.$$

For each α, the above determinant is a polynomial in β with degree 5. The coefficients of β^5 in this polynomial is $\alpha^{-2}(\alpha^{\sigma(1)} - \alpha^{\sigma(2)})(\alpha^{\sigma(3)} - \alpha^{\sigma(4)})$ which is nonzero by the assumption on α. It follows that, for each α, there exists at most five values of β such that the above determinant is zero. Let K_α be the set of β such that $A(1, \beta, \beta^2, \beta^3)$ is isomorphic to $A(1, \alpha, \alpha^2, \alpha^3)$. The above discussion implies that each K_α is finite. We choose one element k_α from each K_α. Let S be the set of all k_α such that α is nonzero and $\alpha^n \neq 1$ for all $1 \leq n \leq 3$. If α and β are two distinct elements in S, then $A(1, \beta, \beta^2, \beta^3)$ is not isomorphic to $A(1, \alpha, \alpha^2, \alpha^3)$. The assumption that K is uncountable implies that S is uncountable. □

We can use the same argument in the above proof and the fact that an algebraically closed field is infinite to give a new proof of the following result of Poonen and Suprunenko in [8, 10].

Corollary 2.5. *There exist infinitely many non-isomorphic 7-dimensional unital commutative algebras over K.*

3 Classification of generalized Poonen-Suprunenko algebras

In this section, we construct a new family of 7-dimensional unital commutative algebras called generalized Poonen-Suprunenko algebras and classify these algebras up to isomorphism.

Definition 3.1. *Let $A = (a_{ij})$ be a 4×4 symmetric matrix with entries in a field K. Assume that A is not a scalar multiple of the identity matrix. We define the* generalized Poonen-Suprunenko algebra *$\mathcal{A}(A)$ associated to the matrix A to be the 7-dimensional unital commutative algebra over K:* $\{c_0 + c_1x_1 + c_2x_2 + c_3x_3 + c_4x_4 + c_5B_1 + c_6B_2 : c_i \in K\}$, *where $x_1, x_2, x_3, x_4, B_1, B_2$ are the generators satisfying the relations: $x_iB_j = 0$ for all i and j, $B_iB_j = 0$ for all i and j, and $x_ix_j = \delta_{ij}B_1 + a_{ij}B_2$ for all i and j.*

When A is a diagonal matrix, $\mathcal{A}(A)$ is the Poonen-Suprunenko algebra studied in the previous section. An algebra $\mathcal{A}$ is called *local* if it has a unique non-trivial maximal ideal $\mathfrak{m}$. Let $k_i = \dim(\mathfrak{m}^i/\mathfrak{m}^{i+1})$ for each positive integer i. In the above definition, we assume that A is not a scalar multiple of the identity to guarantee that $\mathcal{A}(A)$ is a local algebra satisfying $k_1 = 4$ and $k_2 = 2$.

Our next theorem gives a complete list of all generalized Poonen-Suprunenko algebras up to isomorphism. In order to simplify the notations, we let

$$J_2 = \begin{pmatrix} 1 & i \\ i & -1 \end{pmatrix}, \quad J_3 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & i \\ 1 & i & 0 \end{pmatrix}, \quad J_4 = \begin{pmatrix} 0 & 0 & 1 & -i \\ 0 & 0 & i & 1 \\ 1 & i & 1 & i \\ -i & 1 & i & -1 \end{pmatrix},$$

where $i = \sqrt{-1}$. These matrices play the role of symmetric Jordan canonical forms.

Theorem 3.2. *Let A be a 4×4 symmetric matrix with entries in a field K such that A is not a scalar multiple of the identity matrix.*
(1) If A has 4 distinct eigenvalues, then $\mathcal{A}(A)$ is isomorphic to a Poonen-Suprunenko algebra $A(\alpha_1, \alpha_2, \alpha_3, \alpha_4)$ for distinct $\alpha_1, \alpha_2, \alpha_3$, and α_4, which is classified by Theorem 2.2;
(2) If A has 1 eigenvalue λ, then $\mathcal{A}(A)$ is isomorphic to $\mathcal{A}(B)$ with $B = \mathrm{diag}(J_2, 0, 0)$, or $\mathrm{diag}(J_2, J_2)$, or $\mathrm{diag}(J_3, 0)$, or J_4;
(3) If A has 3 distinct eigenvalues, then $\mathcal{A}(A)$ is isomorphic to $\mathcal{A}(B)$ with $B = \mathrm{diag}(J_2, 1, -1)$ or $\mathrm{diag}(0, 0, 1, -1)$;
(4) If A has 2 distinct eigenvalues with multiplicities 3 and 1, then $\mathcal{A}(A)$ is isomorphic to $\mathcal{A}(B)$ with $B = \mathrm{diag}(0, 0, 0, 1)$, or $\mathrm{diag}(J_3, 1)$, or $\mathrm{diag}(J_2, 0, 1)$;
(5) If A has 2 distinct eigenvalues with multiplicities 2 and 2, then $\mathcal{A}(A)$ is isomorphic to $\mathcal{A}(B)$ with $B = \mathrm{diag}(J_2, 1, 1)$, or $\mathrm{diag}(J_2, I + J_2)$, or $\mathrm{diag}(0, 0, 1, 1)$.

We need some preparations to prove this result.

The following result gives a necessary and sufficient condition for two generalized Poonen-Suprunenko algebras to be isomorphic. This theorem classifies all generalized Poonen-Suprunenko algebras and it plays a key role in the proof of Theorem 3.2.

Theorem 3.3. *Let A and B be two 4×4 symmetric matrices with entries in a field K such that A and B are not a scalar multiples of the identity matrix. The generalized Poonen-Suprunenko algebra $\mathcal{A}(A)$ is isomorphic to $\mathcal{A}(B)$ if and only if there exists an invertible matrix*

$$\begin{pmatrix} q_{11} & q_{12} \\ q_{21} & q_{22} \end{pmatrix}$$

such that B is similar to $\psi(A)$, where

$$\psi(A) = (q_{21} + q_{22}A)(q_{11} + q_{12}A)^{-1}.$$

Next we show that Theorem 2.2 is a consequence of Theorem 3.3.

Proof of Theorem 2.2. By Theorem 3.3, $A(\alpha_1, \alpha_2, \alpha_3, \alpha_4)$ and $A(\beta_1, \beta_2, \beta_3, \beta_4)$ are isomorphic if and only if there exists an invertible matrix

$$\begin{pmatrix} q_{11} & q_{12} \\ q_{21} & q_{22} \end{pmatrix}$$

such that the following two diagonal matrices are similar:

$$\mathrm{diag}(\beta_1, \beta_2, \beta_3, \beta_4),$$

$$\mathrm{diag}(\frac{q_{21} + \alpha_1 q_{22}}{q_{11} + \alpha_1 q_{12}}, \frac{q_{21} + \alpha_2 q_{22}}{q_{11} + \alpha_2 q_{12}}, \frac{q_{21} + \alpha_3 q_{22}}{q_{11} + \alpha_3 q_{12}}, \frac{q_{21} + \alpha_4 q_{22}}{q_{11} + \alpha_4 q_{12}}).$$

Recall that diagonal matrices are similar if and only if they have the same set of eigenvalues. This is equivalent to the existence of a permutation σ of $\{1, 2, 3, 4\}$ such that

$$\beta_i = \frac{q_{21} + q_{22}\alpha_{\sigma(i)}}{q_{11} + q_{12}\alpha_{\sigma(i)}}.$$

□

The "only if" part of Theorem 3.3 follows from the following lemma.

Lemma 3.4. *Let A and B be two 4×4 symmetric matrices with entries in a field K such that A and B are not a scalar multiples of the identity matrix. The generalized Poonen-Suprunenko algebra $\mathcal{A}(A)$ is isomorphic to $\mathcal{A}(B)$ if and only if there exist an invertible matrix $\begin{pmatrix} q_{11} & q_{12} \\ q_{21} & q_{22} \end{pmatrix}$ and an invertible 4×4 matrix T satisfying*

$$T^t T = q_{11} I + q_{12} A, \qquad T^t B T = q_{21} I + q_{22} A.$$

In this case, B is similar to $\psi(A)$, where

$$\psi(A) = (q_{21} + q_{22}A)(q_{11} + q_{12}A)^{-1}.$$

Proof. Let us first prove the "if" part of the lemma. Let $T = (\alpha_{ij})$ and (q_{ij}) be as in the statement of the lemma. We construct an isomorphism ϕ from $\mathcal{A}(A)$ to $\mathcal{A}(B)$ as follows: $\phi(B_i) = q_{1i}C_1 + q_{2i}C_2$ and $\phi(x_i) = \sum_{j=1}^4 \alpha_{ji}y_j$, where $\{I, x_1, x_2, x_3, x_4, B_1, B_2\}$ and $\{I, y_1, y_2, y_3, y_4, C_1, C_2\}$ are respectively generators for $\mathcal{A}(A)$ and $\mathcal{A}(B)$.

Now we prove the "only if" part of the lemma. Let ϕ be the isomorphism from $\mathcal{A}(A)$ to $\mathcal{A}(B)$. We assume that the generators of $\mathcal{A}(A)$ and $\mathcal{A}(B)$ are respectively $\{1, x_1, x_2, x_3, x_4, B_1, B_2\}$ and $\{1, y_1, y_2, y_3, y_4, C_1, C_2\}$.

Step 1. We first reduce the general case to the following special case:

$\phi(B_i) = q_{1i}C_1 + q_{2i}C_2$, where $q_{ji} \in K$; $\phi(x_i) = \sum_{j=1}^4 \alpha_{ji}y_j$, where $\alpha_{ji} \in K$.

In general, there exist α_{ji} $(1 \le i \le 4, \quad 0 \le j \le 6)$ such that

$$\phi(x_i) = \alpha_{0i} + \sum_{j=1}^{4} \alpha_{ji}y_j + \alpha_{5i}C_1 + \alpha_{6i}C_2.$$

We need the following fact: if $a \in A$ is a nilpotent element, i.e., there exist a positive integer n such that $a^n = 0$, then $1 - a$ is invertible. This can be shown as follows.

Let $b = 1 + a + \cdots + a^{n-1}$. We have

$$(1-a)b = (1-a)(1+a+\cdots+a^{n-1}) = (1+a+\cdots+a^{n-1}) - (a+a^2+\cdots+a^{n-1}+a^n) = 1.$$

It follows that b is an inverse of $1 - a$.

If $\alpha_{0i} \neq 0$, then by the above fact, $\phi(x_i)$ is invertible since the third power of $\phi(x_i) - \alpha_{0i}$ is zero. On the other hand, $x_i^3 = 0$. This implies that x_i is not invertible, a contradiction with the above statement that $\phi(x_i)$ is invertible. Hence we have $\alpha_{0i} = 0$ for all $1 \le i \le 4$.

The above fact implies that $\phi(x_ix_j) = \phi(x_i)\phi(x_j) = b_{1ij}C_1 + b_{2ij}C_2$ for some b_{1ij} and b_{2ij} in K. We also have $\phi(x_ix_j) = \delta_{ij}\phi(B_1) + a_{ij}\phi(B_2)$. Hence $\delta_{ij}\phi(B_1) + a_{ij}\phi(B_2) = b_{1ij}C_1 + b_{2ij}C_2$. The fact that A is not a scalar multiple of the identity implies that we can solve $\phi(B_1)$ and $\phi(B_2)$ from the above linear system, i.e., there exist $q_{ij} \in K$ such that $\phi(B_1) = q_{11}C_1 + q_{21}C_2, \quad \phi(B_2) = q_{12}C_1 + q_{22}C_2$. The fact that ϕ is an isomorphism implies that the matrix $\begin{pmatrix} q_{11} & q_{12} \\ q_{21} & q_{22} \end{pmatrix}$ is invertible.

Now we define a homomorphism: $\phi' : \mathcal{A}(A) \to \mathcal{A}(B)$, by: $\phi'(x_i) = \sum_{j=1}^{4} \alpha_{ji} y_j$, and $\phi'(1) = 1$, $\phi'(B_1) = \phi(B_1)$, $\phi'(B_2) = \phi(B_2)$. We can easily verify that ϕ' is an isomorphism satisfying the required special form.

Step 2. We now compute $\phi(x_i x_j)$ in the following two different ways:

(1)

$$\phi(x_i x_j) = \phi(x_i)\phi(x_j) = \sum_{k,l=1}^{4} \alpha_{ki}\alpha_{lj} y_k y_l$$

$$= (\sum_{k=1}^{4} \alpha_{ki}\alpha_{kj})C_1 + (\sum_{k,l=1}^{4} \alpha_{ki}\alpha_{lj} b_{kl})C_2;$$

(2)

$$\phi(x_i x_j) = \phi(\delta_{ij} + a_{ij}B_2) = (\delta_{ij}q_{11} + a_{ij}q_{12})C_1 + (\delta_{ij}q_{21} + a_{ij}q_{22})C_2.$$

Comparing the coefficients of C_1 in the above computations, we obtain

$$\alpha_{1i}\alpha_{1j} + \alpha_{2i}\alpha_{2j} + \alpha_{3i}\alpha_{3j} + \alpha_{4i}\alpha_{4j} = \delta_{ij}q_{11} + a_{ij}q_{12}.$$

Let $T = (\alpha_{ij})$. The above identity implies that $T^tT = q_{11}I + q_{12}A$, where T^t is the transpose of T. Comparing the coefficients of C_2 in the above computations, we have

$$T^tBT = q_{21}I + q_{22}A.$$

The above identities imply $B = BI^{-1} = BT(q_{11}I + q_{12}A)^{-1}T^t = ((T^t)^{-1}(q_{21}I+q_{22}A)T^{-1})(T(q_{11}I+q_{12}A)^{-1}T^t) = (T^t)^{-1}(q_{21}I+q_{22}A)(q_{11}I+q_{12}A)^{-1}T^t$. $\square$

We need a few preparations to prove the "if" part of Theorem 3.3. Recall that a matrix T is called *orthogonal* if $T^tT = I$, where T^t is the transpose of T.

Definition 3.5. *Two matrices A and B in $M_n(K)$ are said to be* orthogonally similar *if there exists an orthogonal matrix $T \in M_n(K)$ satisfying $A = T^{-1}BT$.*

The following result plays an important role in the proof of the "if" part of Theorem 3.3.

Theorem 3.6. *A symmetric matrix S is orthogonally similar to a symmetric matrix T if and only if it is similar to T.*

The above theorem is a consequence of the following result.

Theorem 3.7. *Any invertible $n \times n$ symmetric matrix A has a symmetric square root, i.e., there exists an $n \times n$ symmetric matrix B satisfying $B^2 = A$. Furthermore, if A commutes with another $n \times n$ matrix S, then B commutes with S.*

We note that the invertibility condition is crucial in the above theorem. For example, the symmetric matrix J_2 in Theorem 3.2 does not have a symmetric square root.

Proof of Theorem 3.6 assuming Theorem 3.7. Let T and S be two symmetric matrices such that $C^{-1}TC = S$ for some invertible matrix C. By Theorem 3.7, C^tC has a symmetric square root $(C^tC)^{1/2}$ (we note that such a symmetric square root may not be unique). Let $B = C(C^tC)^{-1/2}$, where $(C^tC)^{-1/2}$ is the inverse of $(C^tC)^{1/2}$. We observe that B is orthogonal.

We have $TC = CS$. By taking the transpose on both sides of this equation and using the fact that T and S are symmetric, we obtain $C^tT = SC^t$. It follows that $S(C^tC) = (C^tC)S$. By Theorem 3.7 we know that $(C^tC)^{1/2}$ commutes with S. Hence we have $B^tTB = (C^tC)^{-1/2}C^tTC(C^tC)^{-1/2} = (C^tC)^{-1/2}SC^tC(C^tC)^{-1/2} = S$. $\square$

Let $K^n = \{(k_1, \ldots, k_n) : k_i \in K\}$. We define the standard inner product on K^n by: $< v, w >= \sum_{i=1}^{n} v_i w_i$ for all v and w in K^n. If X and Y are two linear subspaces of K^n, we say that $X \perp Y$ if $< v, w >= 0$ for all $v \in X$ and $w \in Y$.

We need the following lemmas to prove Theorem 3.7.

Lemma 3.8. *Let A be an $n \times n$ symmetric matrix with entries in a field K. Let λ and μ be two distinct eigenvalues of A with multiplicity l and m. Let $X = \{v \in K^n : (A - \lambda I)^l v = 0\}$ and $Y = \{w \in K^n : (A - \mu I)^m w = 0\}$. We have $X \perp Y$.*

Proof. Without loss generality, we can assume $\lambda = 0$ and $\mu \neq 0$. Let $v \in X$ and $w \in Y$. We shall prove $< A^{l-i}v, (A - \mu I)^{m-j}w >= 0$ for all non-negative integers $i \leq l$ and $j \leq m$ by induction on $i + j$. Our lemma follows from the identity when $i = l$ and $j = m$.

If $i + j = 0$, then the above identity follows from the assumption that $v \in X$ and $w \in Y$. Assume that the above identity is true for all $i + j = k$. If $i + 1 \leq l$, we have $< A^{l-i}v, (A-\mu I)^{m-j}w >=< AA^{l-(i+1)}v, (A-\mu I)^{m-j}w >= < A^{l-(i+1)}v, A(A-\mu I)^{m-j}w >$ $=< A^{l-(i+1)}v, (A - \mu I)^{m-(j-1)}w > +\mu < A^{l-(i+1)}v, (A - \mu I)^{m-j}w >$. By induction hypothesis and the assumption that $\mu \neq 0$, it follows that $< A^{l-(i+1)}v, (A - \mu I)^{m-j}w >= 0$. Similarly we can prove $< A^{l-i}v, (A - \mu I)^{m-(j+1)}w >= 0$ if $j + 1 \leq m$. $\square$

If X is a linear subspace of K^n, we let $X^\perp = \{v \in K^n : \quad < v, w >= 0 \ \ \forall \ \ w \in X\}$. We need the following well known result, which can be easily proved by induction on the dimension of X.

Lemma 3.9. *Let X be a linear subspace of K^n. If $X^\perp \cap X = 0$, then X has an orthonormal basis, i.e., X has a basis $\{v_1, \dots, v_k\}$ satisfying $< v_i, v_j >= \delta_{ij}$ for all i and j.*

Lemma 3.10. *If A is an invertible $n \times n$ symmetric matrix, then A is orthogonally similar to a diagonal block matrix $\mathrm{diag}(A_1, \dots, A_j)$ such that each matrix A_i has only one eigenvalue λ_i satisfying $\lambda_i \neq 0$ for all $1 \leq i \leq j$ and $\lambda_i \neq \lambda_{i'}$ if $i \neq i'$.*

Proof. Let A have distinct eigenvalues $\lambda_1, \dots, \lambda_j$ with multiplicities $l_1, \dots, l_j$. Let $X_i = \{v \in K^n : (A - \lambda_i I)^{l_i} v = 0\}$. By Lemma 3.8, $X_i \perp X_{i'}$ if $i \neq i'$. Observe $dim(X_i) = l_i$ and $\sum_{i=1}^{j} l_i = n$. Hence $X_i \cap X_i^\perp = 0$ for all i. By Lemma 3.9, X_i has an orthonormal basis for each i. We put all these orthonormal bases together to form a matrix $T = (v_1, \dots, v_n)$, where each v_m is a column vector. We have $A(v_1, \dots, v_n) = (v_1, \dots, v_n)\ \mathrm{diag}(A_1, \dots, A_j)$, where each matrix A_i has only one eigenvalue λ_i. It follows that T is orthogonal and $T^t A T = \mathrm{diag}(A_1, \dots, A_j)$. □

Lemma 3.11. *If A is a symmetric matrix with one eigenvalue $\lambda \neq 0$, then there exists a symmetric matrix B such that $B^2 = A$. Furthermore, if A commutes with another matrix S, then B commutes with S.*

Proof. Without of loss of generality, we can assume that the eigenvalue λ is 1. Let $S = A - I$. We have $S^n = 0$. Let $B = I + a_1 S + \cdots + a_{n-1} S^{n-1}$. Comparing the coefficients of powers of S on both sides of the equation $B^2 = S + I$, we obtain the following identities: $2a_1 = 1, \ \ 2a_2 + a_1^2 = 0, \ \ 2a_3 + 2a_1 a_2 = 0, \ \ \dots, \ \ 2a_{n-1} + 2a_1 a_{n-2} + \cdots = 0$. Using the assumption that the field K has characteristic 0, we can solve a_1 from the 1st identity, a_2 from the 2nd identity, ..., and a_{n-1} from the last identity to define B. Obviously B is symmetric and $B^2 = A$. By the construction of B, we know that if A commutes with another matrix S, then B commutes with S. □

We are now ready to prove Theorem 3.7.

Proof of Theorem 3.7. By Lemma 3.10, we can assume that A is a diagonal block matrix $\mathrm{diag}(A_1, \dots, A_j)$ such that each matrix A_k has only one eigenvalue $\lambda_k \neq 0$ for all

$1 \leq k \leq j$ and the eigenvalue of A_k is distinct from the eigenvalue of A_l if $k \neq l$. Assume that another matrix S commutes with A. We can write $S = (S_{kl})$ such that the numbers of rows and columns of S_{kl} are respectively the same as the sizes of A_k and A_l. It follows that $A_k S_{kl} = S_{kl} A_l$. We have $(A_k - \lambda_k I)S_{kl} = S_{kl}(A_l - \lambda_k I)$. Hence $(A_k - \lambda_k I)^m S_{kl} = S_{kl}(A_l - \lambda_k I)^m$. We have $(A_k - \lambda_k I)^n = 0$. It follows that $S_{kl}(A_l - \lambda_k I)^n = 0$. By assumption $A_l - \lambda_k I$ is invertible when $k \neq l$. Hence $S_{kl} = 0$ when $k \neq l$. It follows that S is a diagonal block matrix $\mathrm{diag}(S_1, \ldots, S_j)$ such that each matrix S_i has the same size as A_i. Now Theorem 3.7 is a direct consequence of Lemmas 3.10 and 3.11. $\square$

Lemma 3.12. *Let A and B be two symmetric 4×4 matrices such that A and B are not a scalar multiples of the identity matrix. If A is orthogonally similar to B, then $\mathcal{A}(A)$ is isomorphic to $\mathcal{A}(B)$.*

Proof. This follows from Lemma 3.4 by taking the 2×2 matrix (q_{ij}) in the statement of Lemma 3.4 to be the identity matrix. $\square$

Lemma 3.13. *If A is a 4×4 symmetric matrix with entries in a field K and eigenvalues $\{\lambda_1, \lambda_2, \lambda_3, \lambda_4\}$ such that A is not a scalar multiples of the identity matrix and there exists a Möbius transform $\psi(z) = \frac{q_{21}+q_{22}z}{q_{11}+q_{12}z}$ such that $\mu_i = \psi(\lambda_i)$, then there exists a 4×4 symmetric matrix B with eigenvalues $\{\mu_1, \mu_2, \mu_3, \mu_4\}$ such that $\mathcal{A}(A)$ is isomorphic to $\mathcal{A}(B)$ and B is similar to $\psi(A)$.*

Proof. By Lemma 3.4 it is enough to construct a matrix T and a symmetric matrix B such that

$$T^t T = q_{11} I + q_{12} A, \qquad T^t B T = q_{21} I + q_{22} A \qquad\qquad (*).$$

Let $S = T^{-1}$. The equations $(*)$ are equivalent to

$$I = S^t(q_{11} I + q_{12} A)S, \qquad B = S^t(q_{21} I + q_{22} A)S.$$

If $q_{12} = 0$, then $q_{11} \neq 0$. We let $S = \sqrt{q_{11}} I$.

If $q_{12} \neq 0$, we have $\psi(-\frac{q_{11}}{q_{12}}) = \infty$, which is not an element in $\{\mu_1, \mu_2, \mu_3, \mu_4\}$. Hence $q_{11} I + q_{12} A$ is invertible. This, together with the fact that A is symmetric, implies that there exists S such that $S^t(q_{11} I + q_{12} A)S = I$. We now define $B = S^t(q_{21} I + q_{22} A)S$. By Lemma 3.4 we know that B is similar to $\psi(A)$. $\square$

Now we are ready to prove the "if" part of Theorem 3.3.

Proof of the "if" part of Theorem 3.3. By Lemma 3.13, there exists a symmetric matrix B_1 with eigenvalues $\{\mu_1, \mu_2, \mu_3, \mu_4\}$ such that $\mathcal{A}(A)$ is isomorphic to $\mathcal{A}(B_1)$ and B_1 is similar to $\psi(A)$, where $\mu_i = \psi(\lambda_i)$. It follows that B is similar to B_1. By Theorem 3.6, B_1 is orthogonally similar to B. By Lemma 3.12, $\mathcal{A}(B_1)$ is isomorphic to $\mathcal{A}(B)$. Combining the above facts, we know that $\mathcal{A}(A)$ is isomorphic to $\mathcal{A}(B)$. □

We are now ready to prove Theorem 3.2.

Proof of Theorem 3.2: (1) follows from Theorem 3.3 and Lemma 3.12 since A is similar to a diagonal matrix.

Now we assume that A has at most three distinct eigenvalues. By Theorem 3.3, we can assume that the set of eigenvalues of A is a subset of $\{0, 1, -1\}$ with multiplicities $l_0 \geq l_1 \geq l_{-1}$ satisfying $l_0 + l_1 + l_{-1} = 4$. When the eigenvalues of A satisfy condition (2),or (3), or (4), then A is similar to one of the corresponding matrices B described in (2), or (3) or (4), and the algebra $\mathcal{A}(A)$ is isomorphic to $\mathcal{A}(B)$. When the eigenvalues of A satisfy condition (5), then either A or $I - A$ is similar to one of the matrices B in (5) and the algebra $\mathcal{A}(A)$ is isomorphic to $\mathcal{A}(B)$.

For cases (2) or (4), if ψ is a Möbius transform such that the set of eigenvalues of $\psi(A)$ is a subset of $\{0, 1, -1\}$ with multiplicities $l_0 \geq l_1 \geq l_{-1}$ (in these cases $l_{-1} = 0$), then $\psi(z) = z$ if z is an eigenvalue of A. This implies that $\psi(A)$ is similar to A. For case (3), if ψ is a Möbius transform such that the set of eigenvalues of $\psi(A)$ is a subset of $\{0, 1, -1\}$ with multiplicities $l_0 \geq l_1 \geq l_{-1}$, then $\psi(z) = z$ for all $z \in \mathbb{C}$ or $\psi(z) = -z$ for all $z \in \mathbb{C}$. This implies that $\psi(A) = A$ or $\psi(A) = -A$. Note that each of the two matrices in the list of (3) is similar to its negative. Consequently, any two different algebras in the lists of (2), (3), and (4) are isomorphic to each other if and only if the corresponding matrices are similar. It follows that the algebras in the lists of (2), (3), and (4) are non-isomorphic.

For case (5), if ψ is a Möbius transform such that the set of eigenvalues of $\psi(A)$ is $\{0, 1\}$ with multiplicities 2 and 2, then either either $\psi(z) = z$ for all $z \in \{0, 1\}$ or $\psi(z) = 1 - z$ for all $z \in \{0, 1\}$. This implies that $\psi(A)$ is similar to A or $I - A$. Consequently, any two different algebras on the list of (5) are isomorphic to each other if and only if the corresponding matrices B_1 and B_2 are similar to each other or B_1 is similar to $I - B_2$. We can easily see that none of the three matrices in (5) is similar to B or $I - B$ for another B from (5), so these three algebras are non-isomorphic. □

Let $A = (a_{ij})$ be an $n \times n$ symmetric matrix with entries in a field K such that A is not a scalar multiple of the identity matrix. We define the higher dimensional *generalized Poonen-Suprunenko algebra* $\mathcal{A}(A)$ associated to the matrix A to be the unital commutative algebra over K: $\{c_0 + c_1x_1 + \cdots + c_nx_n + c_{n+1}B_1 + c_{n+2}B_2 : c_i \in K\}$, where $x_1, \ldots, x_n, B_1, B_2$ are the generators satisfying the relations: $x_iB_j = 0$ for all i and j, $B_iB_j = 0$ for all i and j, and $x_ix_j = \delta_{ij}B_1 + a_{ij}B_2$ for all i and j.

The following theorem gives a classification of higher dimensional generalized Poonen-Suprunenko algebras. The proof of this theorem is identical to that of Theorem 3.3.

Theorem 3.14. *Let A and B be two $n \times n$ symmetric matrices with entries in a field K such that A and B are not scalar multiples of the identity matrix. The higher dimensional generalized Poonen-Suprunenko algebra $\mathcal{A}(A)$ is isomorphic to $\mathcal{A}(B)$ if and only if there exists an invertible matrix $\begin{pmatrix} q_{11} & q_{12} \\ q_{21} & q_{22} \end{pmatrix}$ such that B is similar to $\psi(A)$, where $\psi(A) = (q_{21} + q_{22}A)(q_{11} + q_{12}A)^{-1}$.*

4 Classification of 7-dimensional unital commutative local algebras I

An algebra $\mathcal{A}$ is called *local* if it has a unique non-trivial maximal ideal $\mathfrak{m}$. Let $k_i = \dim(\mathfrak{m}^i/\mathfrak{m}^{i+1})$ for each positive integer i. In this section, we classify all 7-dimensional unital commutative local algebras satisfying $k_1 = 4$ and $k_2 = 2$. We reduce the problem of classifying all 7-dimensional unital commutative local algebras satisfying $k_1 = 4$ and $k_2 = 2$ to the classification of 6-dimensional unital commutative local algebras (which is solved in [8]) and generalized Poonen-Suprunenko algebras (which is solved in previous sections).

If $\mathcal{A}$ is a 7-dimensional unital commutative local algebras over K satisfying $k_1 = 4$ and $k_2 = 2$, then the algebra $\mathcal{A}$ has the following form: $\{c_0 + c_1x_1 + c_2x_2 + c_3x_3 + c_4x_4 + c_5B_1 + c_6B_2 : c_i \in K\}$, where $x_1, x_2, x_3, x_4, B_1, B_2$ are the generators satisfying the relations: $x_iB_j = 0$ for all i and j, $B_iB_j = 0$ for all i and j, and $x_ix_j = T_1(x_i, x_j)B_1 + T_2(x_i, x_j)B_2$ for all i and j, where T_1 and T_2 are symmetric bilinear forms over the linear span of $\{x_1, x_2, x_3, x_4\}$ and are not scalar multiple of each other.

For any symmetric bilinear form T over a vector space V, we define its kernel by: $\mathrm{Ker}(T) = \{v \in V : T(v,w) = 0 \quad \forall \quad w \in V\}$. The following result gives a complete classification of all 7-dimensional unital commutative local algebras satisfying $k_1 = 4$ and $k_2 = 2$.

Theorem 4.1. *Let $\mathcal{A}$ be a 7-dimensional unital commutative local algebra as above.*
(1) If $\mathrm{Ker}(T_1) \cap \mathrm{Ker}(T_2) \neq 0$, then $\mathcal{A}$ is isomorphic to $\mathcal{A}' \oplus Kz$, where $z^2 = 0$ and $\mathcal{A}'$ is a 6-dimensional unital commutative local algebra such that $za = 0$ for all $a \in \mathcal{A}'$, where $\mathcal{A}'$ is classified in [8];
(2) If $\mathrm{Ker}(T_1) \cap \mathrm{Ker}(T_2) = 0$ and there exist λ and μ in K such that $\mathrm{Ker}(\lambda T_1 + \mu T_2) = 0$, then $\mathcal{A}$ is isomorphic to a generalized Poonen-Suprunenko algebra, which is classified by Theorem 3.2;
(3) If $\mathrm{Ker}(T_1) \cap \mathrm{Ker}(T_2) = 0$ and $\mathrm{Ker}(\lambda T_1 + \mu T_2) \neq 0$ for all λ and μ in K, then $\mathcal{A}$ is isomorphic to the algebra: $\{c_0 + c_1x_1 + c_2x_2 + c_3x_3 + c_4x_4 + c_5B_1 + c_6B_2 : c_i \in K\}$ with relations $x_1^2 = x_2^2 = x_1x_2 = x_1x_4 = x_2x_4 = x_3x_4 = x_3^2 = 0$, $x_1x_3 = B_2$, and $x_2x_3 = x_4^2 = B_1$.

Proof. (1) Let z be a non-zero element in $\mathrm{Ker}(T_1) \cap \mathrm{Ker}(T_2)$. We have $za = 0$ for all $a \in \mathcal{A}$. Without loss of generality, we can assume that $\{x_1, x_2, x_3, z\}$ is a basis for the linear span of $\{x_1, x_2, x_3, x_4\}$. Let $\mathcal{A}'$ be the 6-dimensional unital commutative local algebra $\{c_0 + c_1x_1 + c_2x_2 + c_3x_3 + c_4B_1 + c_5B_2 : c_i \in K\}$ with relations $x_iB_j = 0$, $B_iB_j = 0$, and $x_ix_j = T_1(x_i, x_j)B_1 + T_2(x_i, x_j)B_2$. We have $\mathcal{A} \cong \mathcal{A}' \oplus Kz$, where $z^2 = 0$ and $za = 0$ for all $a \in \mathcal{A}'$.

(2) Assume that there exist λ and μ in K such that $\mathrm{Ker}(\lambda T_1 + \mu T_2) = 0$. By a change of basis for the linear span of $\{B_1, B_2\}$. We can assume that $\mathrm{Ker}(T_1) = 0$. Using the assumption that K is algebraically closed and has characteristic 0, by a change of basis for the linear span of $\{x_1, x_2, x_3, x_3\}$, we can assume that $T_1(x_i, x_j) = \delta_{ij}$. It follows that $\mathcal{A}$ is a generalized Poonen-Suprunenko algebra with respect to this new basis.

(3) By assumption, we have $\dim \ \mathrm{Ker}(T_1) \neq 0$ and $\dim \ \mathrm{Ker}(T_2) \neq 0$.

Claim: $\dim \ \mathrm{Ker}(T_1) + \dim \ \mathrm{Ker}(T_2) \leq 3$.

If $\dim \ \mathrm{Ker}(T_1) + \dim \ \mathrm{Ker}(T_2) > 4$, it is a contradiction with $\mathrm{Ker}(T_1) \cap \mathrm{Ker}(T_2) = 0$.

If $\dim \ \mathrm{Ker}(T_1) + \dim \ \mathrm{Ker}(T_2) = 4$, by $\mathrm{Ker}(T_1) \cap \mathrm{Ker}(T_2) = 0$, we know that span $\{\mathrm{Ker}(T_1), \mathrm{Ker}(T_2)\} =$ span $\{x_1, x_2, x_3, x_4\}$. It follows that $\mathrm{Ker}(T_1 + T_2) = 0$. This is a contradiction with the assumption. We have proved the claim.

Without loss of generality, we can assume that dim $\ \mathrm{Ker}(T_1) = 1$ and dim $\ \mathrm{Ker}(T_2) \leq 2$. The assumption that K is algebraically closed implies that it has infinitely many elements. As a consequence we know that there exists a nonzero element ϵ in K such that dim $\ \mathrm{Ker}(T_1 + \epsilon T_2) \leq 1$. But dim $\ \mathrm{Ker}(T_1 + \epsilon T_2) \neq 0$. Hence dim $\ \mathrm{Ker}(T_1 + \epsilon T_2) = 1$.

We now change basis for the linear span of $\{B_1, B_2\}$: $\quad B_1 = C_1 + C_2, \quad B_2 = \epsilon C_2$.

We have $x_i x_j = T_1(x_i, x_j)C_1 + (T_1 + \epsilon T_2)(x_i, x_j)C_2$.

This implies that we can assume that dim $\ \mathrm{Ker}(T_1) =$ dim $\ \mathrm{Ker}(T_2) = 1$. After a change of basis for the linear span of $\{x_1, x_2, x_3, x_4\}$, we can assume $x_1 \in \mathrm{Ker}(T_1)$ and $x_2 \in \mathrm{Ker}(T_2)$. It follows that, if $A = (T_1(x_i, x_j))$ and $B = (T_2(x_i, x_j))$, then

$$A = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & a_{22} & a_{23} & a_{24} \\ 0 & a_{23} & a_{33} & a_{34} \\ 0 & a_{24} & a_{34} & a_{44} \end{pmatrix}, \quad B = \begin{pmatrix} b_{11} & 0 & b_{13} & b_{14} \\ 0 & 0 & 0 & 0 \\ b_{13} & 0 & b_{33} & b_{34} \\ b_{14} & 0 & b_{34} & b_{44} \end{pmatrix}.$$

Let A' and B' be respectively the 3×3 matrices (a_{ij}) and (b_{kl}), where $2 \leq i,\ j \leq 4$ and $k,\ l = 1, 3, 4$. We have $\det(A') \neq 0$ and $\det(B') \neq 0$.

By assumption, $\det(\lambda A + \mu B) = 0$. The coefficient of $\mu\lambda^3$ in $\det(\lambda A + \mu B)$ is $b_{11}\det(A')$. This implies $b_{11} = 0$. Similarly, we obtain $a_{22} = 0$ by considering the coefficient of $\mu^3\lambda$.

Note that b_{13} and b_{14} can not be 0 simultaneously. Without loss of generality, we can assume $b_{13} \neq 0$. After a change of basis for the linear span of $\{x_1, x_2, x_3, x_4\}$ using the product of two elementary matrices and the assumption $b_{13} \neq 0$, we can assume $b_{14} = 0$ and keep the 0 entries in A and B unchanged.

We have $\det(A + B) = b_{13}^2 a_{24}^2$. This implies that $a_{24} = 0$. By another change of basis, we can assume $a_{23} = 1$ and $b_{13} = 1$, and keep the 0 entries in A and B unchanged. Again by a change of basis, we can assume that $a_{33} = a_{34} = 0$, $a_{44} = 1$, and $b_{33} = 0$, and keep the 0 entries in A and B unchanged.

Let $a = b_{44}$ and $b = b_{34}$. We have $x_1^2 = x_2^2 = x_1x_2 = x_1x_4 = x_2x_4 = x_3^2 = 0$, $x_2x_3 = C_1$, $x_1x_3 = C_2$, $x_3x_4 = bC_2$, $x_4^2 = C_1 + aC_2$. We let $x_2' = x_2 + ax_1$ and $x_4' = x_4 - bx_1$. We have $x_1^2 = (x_2')^2 = x_1x_2' = x_1x_4' = x_2'x_4' = x_3x_4' = x_3^2 = 0$, $x_1x_3 = C_2$, and $x_2'x_3 = (x_4')^2 = C_1 + aC_2$. Let $B_1' = C_1 + aC_2$ and $B_2' = C_2$. We know that the algebra $\mathcal{A}$ is linearly spanned by $\{I, x_1, x_2', x_3, x_4', B_1', B_2'\}$ satisfying relations $x_1^2 = (x_2')^2 = x_1x_2' = x_1x_4' = x_2'x_4' = x_3x_4' = x_3^2 = 0$, $x_1x_3 = B_2'$, and $x_2'x_3 = (x_4')^2 = B_1'$. $\square$

5 Classification of 7-dimensional unital commutative local algebras II

In this section, we classify all 7-dimensional unital commutative local algebras when (1) $k_1 = 2, \quad k_2 = 3, \quad k_3 = 1$; (2) $k_1 = 2, \quad k_2 = 2, \quad k_3 = 2$; (3) $k_1 = 2, \quad k_2 = 2, \quad k_3 = 1$ and $k_4 = 1$; (4) $k_1 = 3, \quad k_2 = 2, \quad k_3 = 1$. Our classification results imply that in each of these cases there are finitely many algebras up to isomorphism. The classification problem for $k_2 = 1$ will be solved in the next section. The only remaining unresolved case is when $k_1 = 3$ and $k_2 = 3$.

Theorem 5.1. *If $\mathcal{A}$ is a 7-dimensional unital commutative local algebras satisfying $k_1 = 2, \quad k_2 = 3, \quad k_3 = 1$, then $\mathcal{A}$ is isomorphic to the algebra linearly spanned by: $\{1, x, y, x^2, y^2, xy, B\}$ with relations: (1) $x^3 = y^3 = x^2y = 0$ and $xy^2 = B$; or (2) $x^3 = y^3 = 0$ and $x^2y = xy^2 = B$; or (3) $x^3 = B$ and $x^2y = xy^2 = y^3 = 0$.*

Proof. By assumption, we know that the algebra $\mathcal{A}$ is linearly spanned by

$$\{1, x, y, x^2, y^2, xy, B\}$$

with relations: $x^3 = \alpha B$, $x^2y = \beta B$, $xy^2 = \gamma B$, and $y^3 = \delta B$ for some α, β, γ and δ in K which are not identically 0.

First we consider the case which can be reduced to $\alpha = \delta = 0$ by a linear transformation. Note that the positions of α and δ are symmetric to each other. Hence without loss of generality, we only need to consider the case $\delta \neq 0$.

Let $\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$. If $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is invertible, then $\mathcal{A}$ is linearly spanned by: $\{1, x', y', (x')^2, (y')^2, x'y', B\}$ with relations: $x^3 = \alpha' B$, $x^2y = \beta' B$, $xy^2 = \gamma' B$, and $y^3 = \delta' B$.

By straightforward computation, we have

$$\begin{pmatrix} \alpha' \\ \beta' \\ \gamma' \\ \delta' \end{pmatrix} = \begin{pmatrix} a^3 & 3a^2b & 3ab^2 & b^3 \\ a^2c & a^2d + 2abc & 2abd + b^2c & b^2d \\ ac^2 & bc^2 + 2acd & 2bcd + ad^2 & bd^2 \\ c^3 & 3c^2d & 3cd^2 & d^3 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \\ \gamma \\ \delta \end{pmatrix}.$$

Consider the equation $\qquad \delta x^3 + 3\gamma x^2 + 3\beta x + \alpha = 0 \qquad (*)$.

If this equation has two distinct solutions, we let b and d be its two solutions. We also let $a = c = 1$. In this case, we have $\alpha' = \delta' = 0$. It follows that $\mathcal{A}$ is isomorphic to the algebra linearly spanned by $\{1, x, y, x^2, y^2, xy, B\}$ with relations $x^3 = y^3 = 0$, $x^2y = \beta B$, and $xy^2 = \gamma B$.

If $\beta = \gamma = 0$, then $k_3 = 0$. This is a contradiction with our assumption.

If $\beta = 0$ and $\gamma \neq 0$, by using a substitution $x' = \frac{x}{\gamma}$, we can assume $\gamma = 1$. In this case, $\mathcal{A}$ is isomorphic to the algebra linearly spanned by $\{1, x, y, x^2, y^2, xy, B\}$ with relations: $x^3 = y^3 = x^2y = 0$ and $xy^2 = B$.

If $\beta \neq 0$ and $\gamma \neq 0$, by using substitutions $x' = \frac{(\beta\gamma)^{\frac{1}{3}}}{\beta} x$ and $y' = \frac{(\beta\gamma)^{\frac{1}{3}}}{\gamma} y$, we can assume $\beta = \gamma = 1$. Hence $\mathcal{A}$ is isomorphic to the algebra linearly spanned by $\{1, x, y, x^2, y^2, xy, B\}$ with relations: $x^3 = y^3 = 0$ and $x^2y = xy^2 = B$.

Now we consider the case when the equation $(*)$ has a unique solution. This case cannot be reduced to $\alpha = \delta = 0$ by any linear transformation. By using a substitution $B' = \delta B$, we can assume that $\delta = 1$. The equation $(*)$ has the form $(x + \xi)^3 = x^3 + 3\xi x^2 + 3\xi^2 x + \xi^3$ for some $\xi \in K$. It follows that $\alpha = \xi^3$, $\beta = \xi^2$, $\gamma = \xi$ and $\delta = 1$. By interchanging x with y, we can assume that $\alpha = 1$, $\beta = \xi$, $\gamma = \xi^2$ and $\delta = \xi^3$. By using a substitution $\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -\xi & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$, we can assume $\alpha = 1$ and $\beta = \gamma = \delta = 0$. Hence $\mathcal{A}$ is isomorphic to the algebra linearly spanned by $\{1, x, y, x^2, y^2, xy, B\}$ with relations $x^3 = B$ and $x^2y = xy^2 = y^3 = 0$.

Let $\mathfrak{m}$ be the unique maximal ideal of $\mathcal{A}$, i.e., $\mathfrak{m}$ is the linear span of $\{x, y, x^2, y^2, xy, B\}$. Let $\mathrm{ann}(\mathfrak{m}) = \{x \in \mathcal{A} : x\mathfrak{m} = 0\}$. Then for the 1st algebra, $\mathrm{ann}(\mathfrak{m})$ is the linear span of $\{x^2, B\}$ has dimension 2. For the 2nd algebra, $\mathrm{ann}(\mathfrak{m})$ is the linear span of $\{B\}$ and has dimension 1. For the 3rd algebra, $\mathrm{ann}(\mathfrak{m})$ is the linear span of $\{xy, y^2, B\}$ and has dimension 3. It follows that the three algebras are non-isomorphic. $\square$

Theorem 5.2. *If $\mathcal{A}$ is a 7-dimensional unital commutative local algebras satisfying $k_1 = 2, \quad k_2 = 2, \quad k_3 = 2$, then $\mathcal{A}$ is isomorphic to the algebra linearly spanned by:*
(1) $\{1, x, y, x^2, y^2, x^3, y^3\}$ with relations: $xy = x^4 = y^4 = 0$;
(2) $\{1, x, y, xy, y^2, xy^2, y^3\}$ with relations: (a) $y^4 = 0$, $xy^3 = 0$ and $x^2 = 0$; or (b) $y^4 = 0$, $xy^3 = 0$ and $x^2 = y^3$.

Proof. Let $\mathfrak{m}$ be the unique maximal ideal of $\mathcal{A}$. $\mathcal{A}/\mathfrak{m}^3$ is a 5-dimensional local algebra. Poonen completely classifies $\mathcal{A}/\mathfrak{m}^3$ in [8]. According to [8], there are two such $\mathcal{A}/\mathfrak{m}^3$ as described in Cases (1) and (2) below.

Case (1): $\mathcal{A}/\mathfrak{m}^3 = \text{Span } \{1, x, y, x^2, y^2\}$ with relations $xy = 0, x^3 = 0, y^3 = 0$. Hence $xy \in \mathfrak{m}^3$, here we are using the same notation for elements in $\mathcal{A}$ whose equivalence classes are x and y. By assumption, we have $\mathfrak{m}^4 = 0$. Hence $x^2y = xy^2 = 0$. This implies that $\mathfrak{m}^3 = \text{Span } \{x^3, y^3\}$. It follows that $\mathcal{A} = \text{Span } \{1, x, y, x^2, y^2, x^3, y^3\}$ with $xy = ax^3 + by^3$ for some a and b in K.

Let $x' = x - by^2$ and $y' = y - ax^2$. We have $x'y' = xy - ax^3 - by^3 = 0$. In summary, $\mathcal{A} = \text{Span } \{1, x', y', (x')^2, (y')^2, (x')^3, (y')^3\}$ with relations $x'y' = 0, (x')^4 = 0, (y')^4 = 0$.

Case (2): $\mathcal{A}/\mathfrak{m}^3 = \text{Span } \{1, x, y, xy, y^2\}$ with relations $x^2 = 0, xy^2 = 0, y^3 = 0$. We have $x^2 \in \mathfrak{m}^3$. Hence $x^3 = 0, x^2y = 0$. It follows that $\mathfrak{m}^3 = \text{Span } \{xy^2, y^3\}$. This implies that $\mathcal{A} = \text{Span } \{1, x, y, xy, y^2, xy^2, y^3\}$ with $x^2 = axy^2 + by^3$.

Let $x' = x - \frac{a}{2}y^2$. We have $(x')^2 = by^3$. By rescaling, we can assume $b = 0$ or 1. In conclusion, we can see that $\mathcal{A}$ is isomorphic to the algebra linearly spanned by $\{1, x', y, x'y, y^2, x'y^2, y^3\}$ with relations: (a) $y^4 = 0$, $x'y^3 = 0$ and $(x')^2 = 0$; or (b) $y^4 = 0$, $x'y^3 = 0$ and $(x')^2 = y^3$.

It is easy to verify that, in case (b), any element in $\mathfrak{m} - \mathfrak{m}^2$ has nonzero square. But in case (a), $x \in \mathfrak{m} - \mathfrak{m}^2$ satisfies $x^2 = 0$. Hence these two algebras are non-isomorphic. $\square$

Theorem 5.3. *If $\mathcal{A}$ is a 7-dimensional unital commutative local algebras satisfying $k_1 = 2, \quad k_2 = 2, \quad k_3 = 1$ and $k_4 = 1$, then $\mathcal{A}$ is isomorphic to the algebra linearly spanned by*
(1) $\{1, x, y, x^2, y^2, x^3, x^4\}$ with relations: $x^5 = 0, \quad xy = 0, \quad y^3 = x^4$;
or (2) $\{1, x, y, x^2, y^2, x^3, x^4\}$ with relations: $x^5 = 0, \quad xy = 0, \quad y^3 = 0$;
or (3) $\{1, x, y, xy, y^2, y^3, y^4\}$ with relations $x^2 = 0, \quad xy^2 = 0, \quad y^5 = 0$;
or (4) $\{1, x, y, xy, y^2, y^3, y^4\}$ with relations: $y^5 = 0, \quad xy^2 = 0, \quad x^2 - y^4 = 0$;
or (5) $\{1, x, y, xy, y^2, y^3, y^4\}$ with relations: $y^5 = 0, \quad xy^2 = 0, \quad x^2 - y^3 = 0$.

Proof. Let $\mathfrak{m}$ be the unique maximal ideal of $\mathcal{A}$. $\mathcal{A}/\mathfrak{m}^4$ is a unital commutative local algebra with dimension 6. By the work of Poonen [8] and the assumption that K has characteristic 0, there exist 5 such algebras up to isomorphism. After carefully inspecting this list, three of them give rise to 7-dimensional unital commutative local algebras satisfying $k_1 = 2,\ \ k_2 = 2,\ \ k_3 = 1$ and $k_4 = 1$. We will only study the case when $\mathcal{A}/\mathfrak{m}^4$ is the algebra linearly spanned by $\{1, x, x^2, x^3, y, y^2\}$ with the relations $x^4 = 0$, $xy = 0$, $y^3 = 0$. The other cases can be handled in a similar way. Using the fact that $xy, y^3 \in \mathfrak{m}^4$, we know that all monomials with degree 4 is 0 in $\mathcal{A}$ except x^4. This implies that $\mathfrak{m}^4$ is equal to Kx^4.

As a consequence, $\mathcal{A}$ is linearly spanned by $\{1, x, y, x^2, y^2, x^3, x^4\}$ with relation $x^5 = 0,\ \ xy = ax^4,\ \ y^3 = bx^4$ for some constants a and b in K. Now let $y' = y - ax^3$. A straightforward computation shows $(y')^3 = y^3 = bx^4$ and $xy' = 0$.

If $b \neq 0$, by a substitution, $\mathcal{A}$ is linearly spanned by $\{1, x, y', x^2, (y')^2, x^3, x^4\}$ with relations: $x^5 = 0,\ \ xy' = 0,\ \ (y')^3 = x^4$.

If $b = 0$, $\mathcal{A}$ is linearly spanned by $\{1, x, y', x^2, (y')^2, x^3, x^4\}$ with relations: $x^5 = 0,\ \ xy' = 0,\ \ (y')^3 = 0$.

In the case $y^3 = 0$, $\mathrm{ann}(\mathfrak{m}) = \{x \in \mathcal{A} : x\mathfrak{m} = 0\}$ is the linear span of $\{y^2, x^4\}$ and hence has dimension 2; and in the case $y^3 = x^4$, $\mathrm{ann}(\mathfrak{m})$ is the linear span of $\{x^4\}$ and has dimension 1. It follows that these two algebras are non-isomorphic. $\square$

We can use similar methods to prove the following result.

Theorem 5.4. *If $\mathcal{A}$ is a 7-dimensional unital commutative local algebras satisfying $k_1 = 3,\ \ k_2 = 2,\ \ k_3 = 1$, then $\mathcal{A}$ is isomorphic to the algebra generated by $\{1, x, y, z\}$ satisfying the relations: (1) $xy = y^2 - xz = z^2 = yz = x^4 = 0$; (2) $xy = y^2 - xz = z^2 = yz - x^3 = x^4 = 0$; (3) $xy = y^2 - xz = yz = z^2 - x^3 = x^4 = 0$; (4) $x^2 = y^2 = xz = yz^2 = xy - z^3 = z^4 = 0$; (5) $x^2 = y^2 = xz = yz^2 = xy = z^4 = 0$; (6) $y^2 = xz = yz^2 = xy = x^2 - z^3 = z^4 = 0$; (7) $x^2 = xy = xz = y^2 - z^3 = yz^2 = z^4 = 0$; (8) $xy = xz = y^2 = z^3 = x^2 - yz^2 = 0$; (9) $x^2 = xy = xz = y^2 = z^3 = 0$; (10) $xy = xz = yz = y^3 = x^2 - z^3 = 0$; (11) $x^2 = xz = yz = y^3 = xy - z^3 = 0$; (12) $x^2 = xy = xz = yz = y^3 = z^4 = 0$; (13) $xy = xz = yz = y^3 - z^3 = x^2 - z^3 = 0$; (14) $x^2 = xy = xz = yz = y^3 - z^3 = 0$; (15) $xy = xz = y^2 = z^2 = x^4 = 0$; (16) $xy = xz = y^2 = z^2 - x^3 = x^4 = 0$; (17) $xy = xz = y^2 - x^3 = z^2 - x^3 = x^4 = 0$; (18) $x^2 = yz = xz - z^2 + y^2 = xy - z^3 = 0$; (19) $x^2 = yz = xz - z^2 + y^2 = xy = 0$.*

6 Classification of finite dimensional unital commutative local algebras satisfying $k_2 = 1$

In this section, we classify all finite dimensional unital commutative local algebras satisfying $k_2 = 1$.

Theorem 6.1. *If $\mathcal{A}$ is finite dimensional unital commutative local algebra satisfying $k_2 = 1$, then $\mathcal{A}$ is isomorphic to*
(1) the unital commutative algebra generated by $\{1, x_1, \dots, x_k\}$ with relations $x_1^2 = \cdots = x_r^2$, $x_i^3 = 0$ for all i, and $x_ix_j = 0$ for $i \neq j$ or $i > r$ or $j > r$, where r is a natural number less than or equal to $k = k_1$;
(2) the unital commutative algebra generated by $\{1, x, y_1, \dots, y_{k-1}\}$ with relations $x^{m+1} = 0$, $y_1^2 = \cdots = y_r^2 = x^m$, $y_{r+1}^2 = 0$, $\dots$, $y_{k-1}^2 = 0$, $y_iy_j = 0$ for $i \neq j$ or $i > r$ or $j > r$, and $xy_i = 0$ for all i, where $k = k_1$ and m is a non-negative integer and r is a natural number less than or equal to $k - 1$.

Proof. Let $\mathfrak{m}$ be the unique maximal ideal of $\mathcal{A}$. Let $V = \mathfrak{m}/\mathfrak{m}^2$. Let $\{x_1, \dots, x_{k_1}\}$ be a collection of vectors in $\mathfrak{m} - \mathfrak{m}^2$ such that it gives a basis for V. In this proof, we will use the same notation x_i for its corresponding element in V.

By assumption, $\mathfrak{m}^2/\mathfrak{m}^3$ is linearly spanned by a vector B_1. The product operation gives a symmetric bilinear form on V: $xy = T(x, y)\ B_1$ for all x and y in V. By the condition that $\mathcal{A}$ is local, we know rank $(T) \geq 1$.

Case (1): rank $(T) = r \geq 2$. There exists a basis $\{x_1, \dots, x_k\}$ of $\mathfrak{m}/\mathfrak{m}^2$ such that $T(x_i, x_j)$ equals to δ_{ij} if $1 \leq i, j \leq r$ and 0 otherwise, where $k = k_1$. Hence $x_1^2 = \cdots = x_r^2 = B_1$ in $\mathfrak{m}^2/\mathfrak{m}^3$, and $x_ix_j = 0$ in $\mathfrak{m}^2/\mathfrak{m}^3$ for $i \neq j$ or $i > r$ or $j > r$. It follows that $x_i^2 - x_j^2 \in \mathfrak{m}^3$ if $1 \leq i,\ j \leq r$ and $x_i^2 \in \mathfrak{m}^3$ for $i > r$ and $x_ix_j \in \mathfrak{m}^3$ for all $i \neq j$. This implies that $x_ix_jx_l \in \mathfrak{m}^4$ if $i \neq j$. We also have $x_i^3 - x_ix_j^2 = x_i(x_i^2 - x_j^2) \in x_i\mathfrak{m}^3 \subseteq \mathfrak{m}^4$. Hence $x_i^3 \in \mathfrak{m}^4$. Therefore $\mathfrak{m}^3 = \mathfrak{m}^4$. This implies that $k_3 = 0$. By the condition that $\mathcal{A}$ is local, we have $\mathfrak{m}^3 = 0$. Summarizing the above discussions, we conclude that $\mathcal{A}$ is isomorphic to the unital commutative algebra generated by $\{1, x_1, \dots, x_k\}$ with relations $x_1^2 = x_2^2 = \cdots = x_r^2$, $\ x_ix_j = 0$ for $i \neq j$ or $i > r$ or $j > r$, and $x_i^3 = 0$ for all i.

Case (2): rank $(T) = 1$. There exists a basis $\{x, x_1, \dots, x_{k-1}\}$ for V such that $T(x, x) = 1$, $T(x, x_i) = 0$, and $T(x_i, x_j) = 0$, where $k = k_1$. It follows that x^2 spans

$\mathfrak{m}^2/\mathfrak{m}^3$ and $x_i x_j \in \mathfrak{m}^3$, $xx_i \in \mathfrak{m}^3$. Hence $\mathfrak{m}^l/\mathfrak{m}^{l+1}$ is spanned by x^l for each $l \geq 2$. This implies that $\mathfrak{m}^2$ is linearly spanned by $\{x^2, \dots, x^m\}$, where m is a natural number such that $x^m \neq 0$ but $x^{m+1} = 0$. As a consequence, $\mathfrak{m}^3$ is spanned by $\{x^3, \dots, x^m\}$.

By the fact $xx_i \in \mathfrak{m}^3$, we have $xx_i = \sum_{j=3}^m a_{ij} x^j$ for some $a_{ij} \in K$. Let $y_i' = x_i - \sum_{j=3}^m a_{ij} x^{j-1}$. We have $xy_i' = 0$ for all i. Let $W = \{z \in \mathfrak{m} : zx = 0\}$. We have $\mathfrak{m}^3 \cap W = Kx^m \subseteq \mathfrak{m}^3 \cap W$. Let V' be the linear span of $\{y_1', \dots, y_{k-1}'\}$. Define a symmetric bilinear form S over V' by: $vw = S(v, w)x^m \in \mathfrak{m}^3 \cap W$. Assume that rank $(S) = r$. Choose a basis $\{y_1, \dots, y_r\}$ such that $S(y_i, y_j)$ is δ_{ij} if $1 \leq i,\ j \leq r$ and 0 otherwise. We conclude that $\mathcal{A}$ is isomorphic to the unital commutative algebra generated by $\{1, x, y_1, \dots, y_{k-1}\}$ with relations $x^{m+1} = 0,\ \ y_1^2 = \dots = y_r^2 = x^m,\ \ y_{r+1}^2 = 0,\ \ \dots,\ \ y_k^2 = 0,\ \ y_i y_j = 0$ for $i \neq j$ or $i > r$ or $j > r$, and $xy_i = 0$ for all i. $\square$

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