Avoidance in (2+2)-Free Posets

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Partially Ordered Set $(P, \prec)$

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  $i \prec i$ for all $i \in P$
- *Antisymmetry*
  If $i \prec j$ and $j \prec i$, then $i = j$ for $i, j \in P$
- *Transitivity*
  If $i \prec j$ and $j \prec k$, then $i \prec k$ for $i, j, k \in P$

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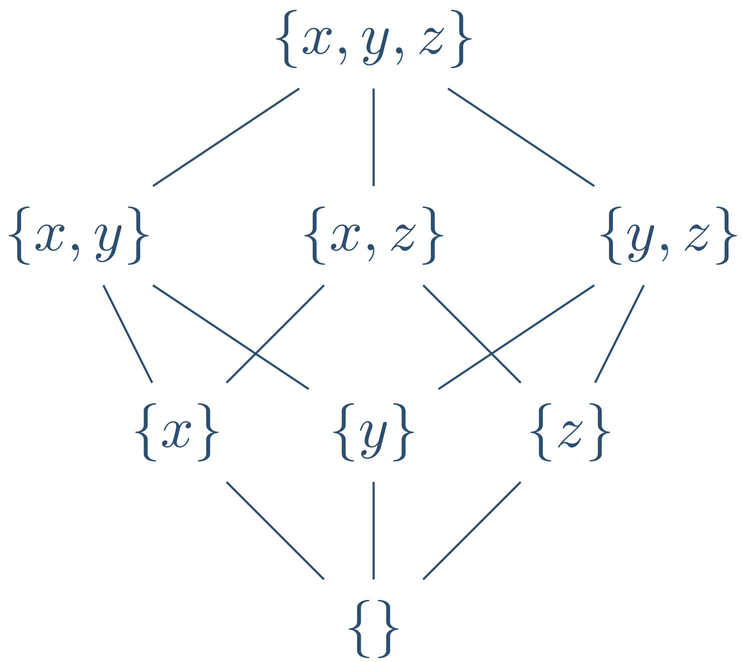
Call $i, j \in P$ *comparable* if $i \prec j$ or $j \prec i$

Hasse Diagrams

$(\mathcal{P}(\{x, y, z\}), \subseteq)$



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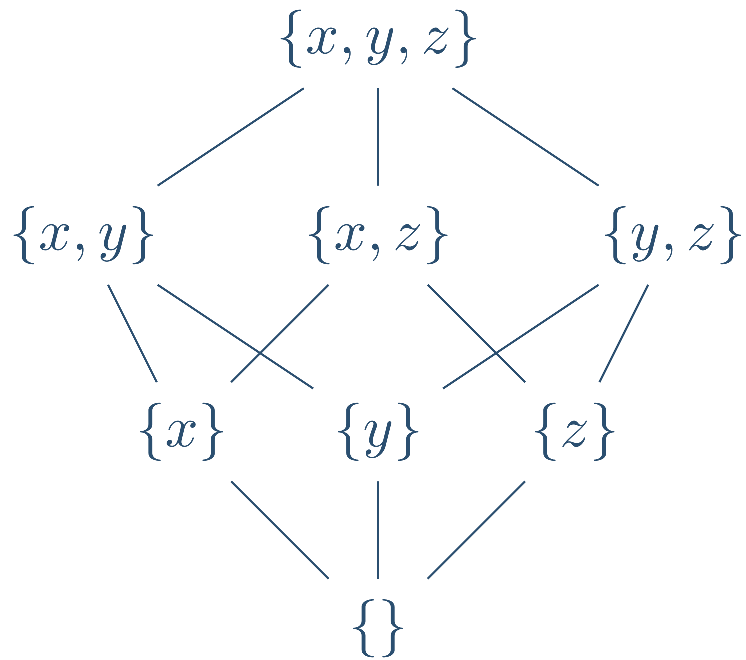

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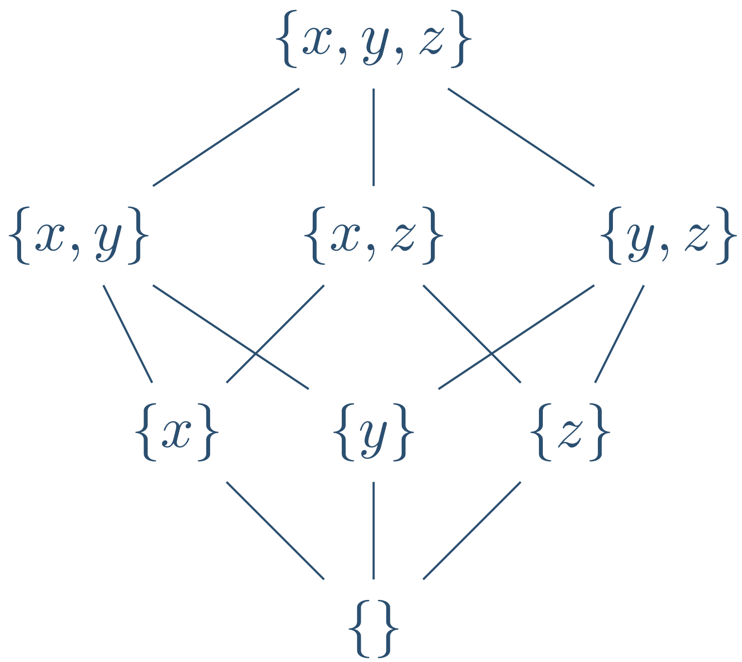

- $\{\}$ and $\{x, z\}$ are comparable
- $\{y\}$ and $\{x, z\}$ are *incomparable*
- $\{\}$, $\{x\}$, $\{x, z\}$, and $\{x, y, z\}$ form a *chain* of length 4.

Avoidance in Posets

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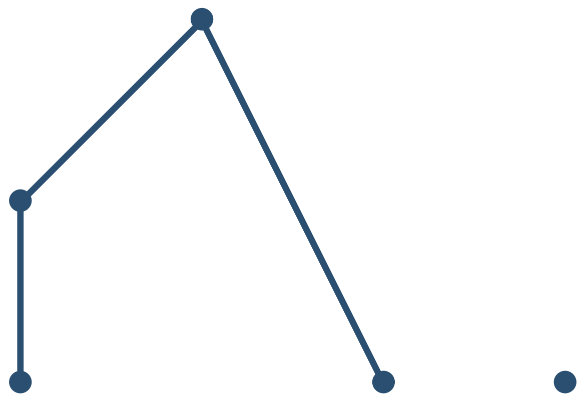

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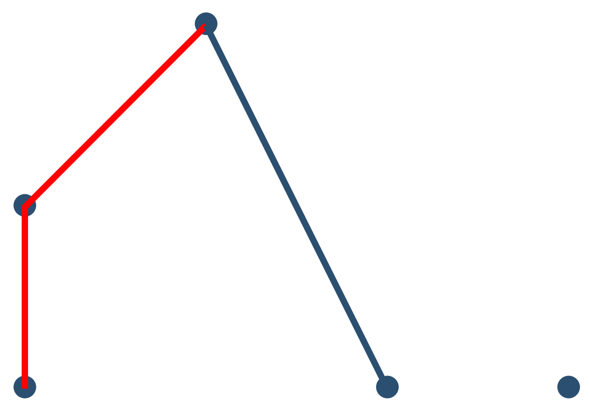
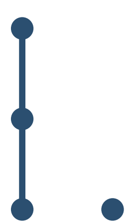
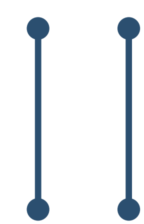


Contains (3+1)



Avoids (2+2)



Why (2+2)-Free Posets?

- Interval Orders
  - A poset is an interval order if it is isomorphic to some set of intervals on the real line ordered by left-to-right precedence.
  - Interval orders are important in mathematics, computer science, and engineering (Ex. task distributions in complex manufacturing processes).
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- Ascent Sequences
  - An ascent sequence is a sequence $x_1 x_2 \cdots x_n$ satisfies $x_1 = 0$ and, for all i with $1 < i \leq n$, $x_i \leq asc(x_1 x_2 \cdots x_{i-1}) + 1$.
  - (Bousquet-Mélou et al 2009) The ascent sequences are in bijection with the (2+2)-free posets.

Previous Results with (2+2)-Free posets

- Let $P_n(x)$ refer to the set of posets of size n that avoid the poset x.
- Define a function $a(x)$ to return the ascent sequence associated with a poset x.
- Let $A_n(y)$ refer to the set of posets of size n whose ascent sequences avoid the ascent sequence y.

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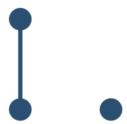
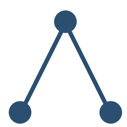
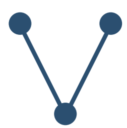
- **Question 1**: Can we explicitly compute $|P_n(2+2, p)|$ for other posets p?
- **Question 2**: For what posets p is it true that $P_n(2+2, p) = A_n(a(p))$?

(2+2)-Free Posets and Ascent Sequences

(2+2)-Free Poset p	Ascent Sequence $a(p)$	$\lvert P_n(2+2, p)\rvert$
	012	2^{n-1}
	010	2^{n-1}
	001	2^{n-1}
	011	2^{n-1}

(2+2, ∨)-Free and (2+2, ∧)-Free Posets

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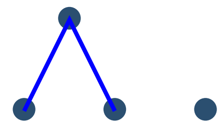
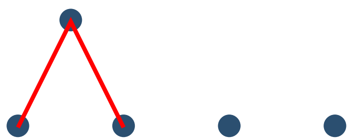
  - ❖ Add a maximal node.





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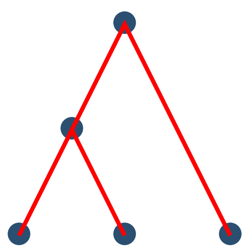

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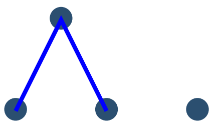

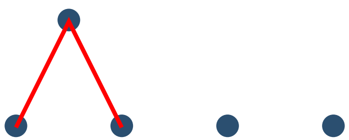

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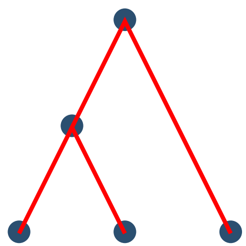

- $|P_n(2+2,\vee)| = |P_n(2+2,\wedge)| = 2^{n-1}$

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- Define $S_i = \{y_j | y_j \prec x_i\}$.
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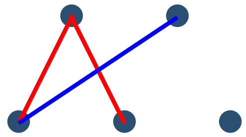

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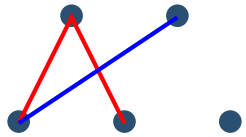

- $\{|S_1|, |S_2|, \cdots, |S_a|\} \rightarrow \left(\!\binom{b}{a}\!\right) = \binom{a+b-1}{a}$.
- $|P_n(2+2,3)| = \sum_{a+b=n} \binom{a+b-1}{a} = \sum_{a=0}^{n-1} \binom{n-1}{a} = 2^{n-1}$

Bijection between (2+2, 3) and (2+2, $\wedge$)-Free Posets

- Poset $P \to (A, B)$.
  - $A = \{$Maximal Nodes in $P\}$.
  - $B = P \setminus A$.

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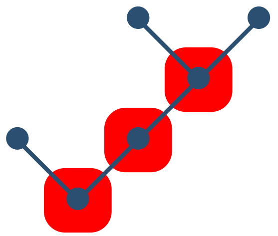

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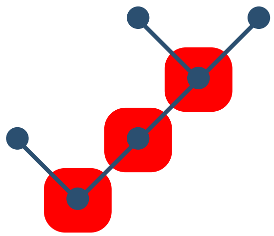
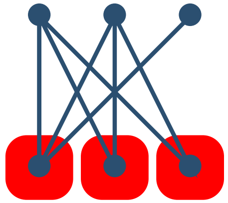


- In (2+2, 3)-Free, B forms the lower level.



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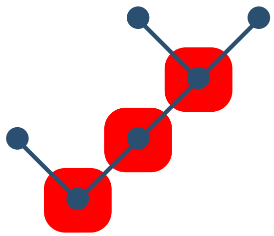

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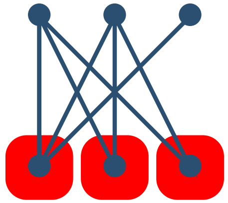

- Maintains all order relations between A and B.

(2+2, 4) and (2+2, Y)-Free Posets

- $P_n(2+2, Y) \leftrightarrow P_n(2+2, 4)$.

  Theorem. $|P_n(2+2, Y)| = |P_n(2+2, 4)| = 1 + \sum_{r+m<n} \binom{n+mr+1}{n-m-r} - \binom{n+m(r-1)+1}{n-m-r} - \binom{n+r(m-1)}{n-m-r} + \binom{n+(r-1)(m-1)}{n-m-r}$, *where* $r \geq 0$ *and* $m > 0$.

Future Directions of Research

- $P_n(2+2,\vee) \leftrightarrow P_n(2+2,3)$.
- $P_n(2+2,Y) \leftrightarrow P_n(2+2,4)$.

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Conjecture. *Define a function $Y(n)$, $n \geq 3$ as follows.*

- $Y(3) = \vee$.
- $Y(n)$ *is the result of adding a minimal node to* $Y(n-1)$.

Then, $P_n(2+2, Y(k)) \leftrightarrow P_n(2+2, k)$.

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- $|P_n(2+2, 3+1)| = |P_n(2+2, N)|$.
- $|P_n(2+2, Y)| = |P_n(2+2, 4)|$

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- $|P_n(2+2, 3+1)| = |P_n(2+2, N)|$.
- $|P_n(2+2, Y)| = |P_n(2+2, 4)|$

Query. *Do there exist other nontrivial Wilf-Equivalences in (2+2)-Free Posets? What other posets* p, q *exist such that* $|P_n(2+2, p)| = |P_n(2+2, q)|$ *for all* $n \in \mathbb{N}$?

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- The PRIMES program for making this experience possible.
- My parents for their support.

Thanks to all of you for listening.